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QUANTUM WEYL ALGEBRAS AND DEFORMATIONS OF $U(\mathbf{G})$

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We construct new deformations of the universal enveloping algebras from the quantum Weyl algebras for any R-matrix. Our new algebra (in the case of $\mathbf{g} = \mathbf{sl}_2$) is a noncommutative and noncocommutative bialgebra (i.e. quantum semigroup) with its localization being a Hopf algebra (i.e. quantum group). The ring structure and representation theory of our algebra are studied in the case of $\mathbf{sl}_2$.

Introduction.

Quantum groups are usually considered to be examples of q-deformations of universal enveloping algebras of simple Lie algebras or their restricted dual algebras. The simplest example is the Drinfeld-Jimbo quantum group $U_q(\mathbf{sl}_2)$, which is an associative algebra over $\mathbb{C}(q)$ generated by e, f, k, k^{-1} subject to the following relations:

$$kk^{-1} = k^{-1}k = 1,$$
$$ke = q^2ek, \qquad kf = q^{-2}fk,$$
$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.$$

One sees that the element k does not stay in the same level as the elements e and f do, since k is actually an exponential of the original element h in the Cartan subalgebra. There do exist deformations to deform all generators into the same level, for example, Sklyanin algebras. However it is still uncertain if Sklyanin algebra is a bialgebra or not.

Using the quantum Weyl algebras discussed in [**WZ**, **GZ**], we will construct new deformations for the enveloping algebras which have such a nice property. Our algebra in the case of $\mathbf{sl}_2$ is an associative algebra generated by e, f, h with the relations:

$$qhe - eh = 2e,$$
$$hf - qfh = -2f,$$
$$ef - qfe = h + \frac{1 - q}{4}h^2.$$

Moreover the algebra is a bialgebra with the comultiplication

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \frac{1}{2}(1-q)h \otimes x$$

with x being any of the basic generators e, f, h. Its localization is a Hopf algebra. Clearly our algebra is noncommutative and noncocommutative or a quantum group according to Drinfeld.

Our new deformations will enjoy all the nice properties carried by the enveloping algebras such as having the same global, Krull, Gelfand-Kirillov dimensions as well as the same Hilbert series for the associated graded rings. Our deformation is similar to Witten's algebra ([**W**], see also [**L**]) in one aspect that the Casimir element in our algebra skew-commutes with other generators. We also remark that the homogenization of both Witten's algebra and our algebra belong to a single family of algebras studied in a recent work [**LSV**] from a different perspective.

The idea behind our construction is very simple and natural. We start from the derivatives and multiplication operators on a skew polynomial ring associated to a Hecke R-matrix. They form a quantum Weyl algebra as discussed by [**GZ**]. Then we go on to consider the subalgebra generated by the quadratic elements to obtain a new deformation.

We study the ring structure for our new algebras in the case of $\mathbf{sl}_2$. The representation theory is also worked out for both generic and singular cases of q. It is not surprising that the representation theory of the new algebra is quite similar to that of the Drinfeld-Jimbo quantum algebra $U_q(\mathbf{g})$.

Our approach is general in the sense that our method works for a wide class of R-matrices. In other words, we just reverified the profound principle that a meaningful R-matrix can give a deformation to the simple Lie algebras [**FRT**].

Recently Ding and Frenkel [**DF**] have introduced another kind of quantum Weyl and Clifford algebras from the L-operator approach of the quantum inverse scattering method and use them to give representations of the quantum algebras. Their work differs from ours in forming a different quadratic expression for the basic generators to give realization of the usual Drinfeld-Jimbo quantum algebras, while our result is to focus on obtaining a new quantum algebra structure.

We will use the usual symbol $U_q(\mathbf{g})$ to denote our new deformed algebra unless indicated otherwise.

1. Quantum Weyl Algebras.

Let $R = (r_{ij}^{st})$ be a Hecke R-matrix. Namely, R is a n^2 by n^2 matrix satisfying the Hecke relation and braid relation:

$$(R-q)(R+q^{-1}) = 0 \quad \text{or} \quad R^2 = qR - q^{-1}R + 1; \tag{1.1}$$

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}. \tag{1.2}$$

The braid relation (1.2) means that for all i, j, g and all u, k, f

$$\sum_{l,s,t} r_{lt}^{kf} r_{is}^{ul} r_{jg}^{st} = \sum_{l,s,t} r_{ij}^{st} r_{sl}^{uk} r_{tg}^{lf} \tag{1.3}$$

summed over the repeated indices l, s, t and here $R = (r_{ij}^{st})$.

Given such an R-matrix, we can define the quantum Weyl algebra $A_n(R)$ as follows. Quantum Weyl algebras were first considered in [**WZ**]. Here we follow [**GZ**] to define the quantum Weyl algebra $A_n(R)$. The generators are $x_1, \cdots, x_n$ and $\partial^1, \cdots, \partial^n$, and each ∂^i corresponds to the partial derivative in the ordinary Weyl algebra. The relations are

$$\sum_{s,t} r_{ij}^{st} x_s x_t = q x_i x_j, \tag{1.4.1}$$

$$\partial^i x_j = \delta_j^i + q \sum_{k,l} r_{jl}^{ik} x_k \partial^l, \tag{1.4.2}$$

$$\sum_{s,t} r_{ts}^{ji} \partial^s \partial^t = q \partial^i \partial^j \tag{1.4.3}$$

for all i and j. We also assume that R has a skew inverse matrix in the following sense: there is $P = (p_{ij}^{st})$ such that $\sum p_{ij}^{st} r_{jk}^{tu} = \delta_{ik}\delta_{su} = \sum r_{ij}^{st} p_{jk}^{tu}$, which is always true in the considered cases. If such P did not exist then we could not write $x_k\partial^l$ in terms of $\partial^i x_j$ as we see from (1.4). We refer the reader to [**GZ**] for a detailed explanation why the relations (1.4) are natural analog of the usual defining relations for the classical Weyl algebra realized on the polynomial algebra in n variables.

In the classical case, there is a well-known homomorphism from the enveloping algebra $U(\mathbf{gl}_n)$ to the n-th Weyl algebra A_n by sending e_{ij} to $x_i\partial^j$. (This homomorphism is not injective.) Motivated by this classical fact we are going to construct a quantum enveloping algebra generated by the quantum elements $x_i\partial^j$. Let's denote the element $x_i\delta^j$ by e_i^j, then we have the following commutation relations for e_i^j.

Proposition 1.1. *For all u, z, l, k*

(a) $(\sum r_{zj}^{ui} e_i^j)\cdot e_k^l = \sum r_{zk}^{ui} e_i^l + q^{-1}\sum(\sum r_{ji}^{uv} e_v^i)(\sum r_{zk}^{js} e_s^l) - q^{-1}\sum r_{zk}^{js}(\sum r_{js}^{uv} e_v^l)$,

(b)

$$e_k^l \cdot \left(\sum r_{zj}^{ui} e_i^j\right) = \sum r_{zj}^{ul} e_k^j + q^{-1} \sum \left(\sum r_{ji}^{ul} e_k^i\right) \left(\sum r_{zw}^{jt} e_t^w\right) - q^{-1} \sum r_{jt}^{ul} \left(\sum r_{zw}^{jt} e_k^w\right).$$

Proof. We only give a proof for (a) (and (b) can be handled similarly). It follows from (1.4.2) that

$$x_i \partial^j \cdot x_k \partial^l = x_i \left(\delta_k^j + q \sum r_{kt}^{js} x_s \partial^t\right) \partial^l = \delta_k^j x_i \partial^l + q \sum r_{kt}^{js} x_i x_s \partial^t \partial^l.$$

Then we have

$$\begin{aligned}
\left(\sum r_{zj}^{ui} e_i^j\right) \cdot e_k^l &= \sum r_{zj}^{ui} \delta_k^j x_i \partial^l + q \sum r_{zj}^{ui} r_{kt}^{js} x_i x_s \partial^t \partial^l \\
&= \sum r_{zk}^{ui} x_i \partial^l + \sum r_{zj}^{ui} r_{kt}^{js} \left(q x_i x_s\right) \partial^t \partial^l \\
&= \sum r_{zk}^{ui} x_i \partial^l + \sum r_{zj}^{ui} r_{kt}^{js} r_{is}^{vw} x_v x_w \partial^t \partial^l \qquad \text{by (1.4.1)} \\
&= \sum r_{zk}^{ui} x_i \partial^l + \sum r_{zk}^{js} r_{ji}^{uv} r_{st}^{iw} x_v x_w \partial^t \partial^l \qquad \text{by (1.2)} \\
&= \sum r_{zk}^{ui} x_i \partial^l + \sum r_{zk}^{js} r_{ji}^{uv} x_v \left(\sum r_{st}^{iw} x_w \partial^t\right) \partial^l \\
&= \sum r_{zk}^{ui} x_i \partial^l + \sum r_{zk}^{js} r_{ji}^{uv} x_v q^{-1} \left(\partial^i x_s - \delta_s^i\right) \partial^l \qquad \text{by (1.4.2)} \\
&= \sum r_{zk}^{ui} x_i \partial^l + q^{-1} \sum r_{zk}^{js} r_{ji}^{uv} x_v \partial^i x_s \partial^l - q^{-1} \sum r_{zk}^{js} r_{js}^{uv} x_v \partial^l \\
&= \sum r_{zk}^{ui} e_i^l + q^{-1} \sum \left(\sum r_{ji}^{uv} e_v^i\right) \left(\sum r_{zk}^{js} e_s^l\right) \\
&\qquad - q^{-1} \sum r_{zk}^{js} \left(\sum r_{js}^{uv} e_v^l\right).
\end{aligned}$$

□

Remark. We conjecture that the relations in (a) are equivalent to the relations in (b).

For simplicity we introduce some notations:

$$b_z^u = \sum r_{zj}^{ui} e_i^j = \sum r_{zj}^{ui} x_i \partial^j.$$

Of course, by (1.4.2), $b_z^u = q^{-1}(\partial^u x_z - \delta_z^u)$. Then the matrix $(b_z^u)_{n\times n}$ is denoted by B.

$$c_{zk}^{ul} = \sum r_{zk}^{ui} e_i^l = \sum r_{zk}^{ui} x_i \partial^l$$

and the matrix $(c_{z\cdot}^{u\cdot})_{n\times n}$ is denoted by $C_{\cdot}^{\cdot}$. It is easy to see that $\sum C_k^k = B$.

$$d_{zk}^{ul} = \sum r_{zi}^{ul} e_k^i = \sum r_{zi}^{ul} x_k \partial^i$$

and the matrix $(d^{u\cdot}_{z\cdot})_{n\times n}$ is denoted by $D^{\cdot}_{\cdot}$. It is easy to see that $\sum D^k_k = B$.

$$T = \sum x_k \partial^k = \sum e^k_k.$$

By using these notations we can re-write the relations in (a) and (b) as follows:

(a) $b^u_z e^l_k = q^{-2} c^{ul}_{zk} - q^{-1}\delta^u_z e^l_k + q^{-1}\sum c^{jl}_{zk} b^u_j$

for which we use $R^2 = (q - q^{-1})R + 1$. In the matrix form,

$$Be^l_k = q^{-1}C^l_k B + q^{-2}C^l_k - q^{-1}Ie^l_k$$

where I is the identity matrix.

(b) $e^j_i b^u_z = q^{-2} d^{uj}_{zi} - q^{-1}\delta^u_z e^j_i + q^{-1}\sum d^{uj}_{li} b^l_z$,

or,

$$e^j_i B = q^{-1} B D^j_i + q^{-2} D^j_i - q^{-1} I e^j_i.$$

By (a)

$$\begin{aligned} B \cdot T &= \sum Be^k_k \\ &= q^{-1}\sum C^k_k B + q^{-2}\sum C^k_k - q^{-1} I \sum e^k_k \\ &= q^{-1}BB + q^{-2}B - q^{-1} I \cdot T. \end{aligned}$$

By (b)

$$\begin{aligned} T \cdot B &= \sum e^i_i B \\ &= q^{-1}\sum BD^i_i + q^{-2}\sum D^i_i - q^{-1} I \sum e^i_i \\ &= q^{-1}BB + q^{-2}B - q^{-1} I \cdot T. \end{aligned}$$

As a consequence $B \cdot T = T \cdot B$ and $(B + q^{-1}T)(B - qT) = 0$.

Corollary 1.2. *The element $T = \sum e^k_k$ commutes with elements $x_k\partial_l$ for all k, l in the quantum Weyl algebra $A_n(R)$.*

Proof. This follows from the equality $TB = BT$ and the fact that $e^l_k = \sum p^{ku}_{lz} b^u_z$. □

2. Definition of $U_q(\mathbf{gl}_n)$.

We are going to construct a deformation of the enveloping algebra $U(\mathbf{gl}_n)$ out of the subalgebra generated by the quadratic elements $x_i\partial^j$. We can not use all the relations in Proposition 1.1 (a), since in the classical case, the map from $U(\mathbf{gl}_n)$ or $U(\mathbf{sl}_n)$ for $n > 2$ to the n-th Weyl algebra A_n by

sending e_{ij} to $x_i\partial^j$ is not injective. We need to find out what are the proper relations for $U_q(\mathbf{gl}_n)$.

Let R be the R-matrix for the standard quantum group $GL_n(q)$. As a tensor, we may write R as

$$R = q\sum_i e_i^i \otimes e_i^i + \sum_{i<j}(e_i^j \otimes e_j^i + e_j^i \otimes e_i^i) + (q - q^{-1})\sum_{i<j} e_i^i \otimes e_j^j.$$

As a matrix R is given by $R = (r_{ij}^{st})$ and $r_{ii}^{ii} = q$, $r_{ij}^{ji} = 1$ for all i, j, and $r_{ij}^{ij} = q - q^{-1}$ for all $i < j$. The quantum Weyl algebra constructed from this particular R-matrix is denoted by $A_n(q)$.

It follows from (1.4) in Section 1 that the relations of $A_n(q)$ are the following:

$$x_i x_j = q x_j x_i, \forall i < j; \tag{2.1.1}$$

$$\partial^i \partial^j = q^{-1}\partial^j \partial^i, \forall i < j; \tag{2.1.2}$$

$$\partial^i x_j = q x_j \partial^i, \forall i \neq j; \tag{2.1.3}$$

$$\partial^i x_i = 1 + q^2 x_i \partial^i + (q^2 - 1)\sum_{j>i} x_j \partial^j, \forall i. \tag{2.1.4}$$

If we denote $|ij| = 1$ for $i < j$, $|ij| = 0$ for $i = j$, and $|ij| = -1$ for $i > j$, then (2.1.1) and (2.1.2) can be re-written as

$$x_i x_j = q^{|ij|} x_j x_i, \forall i, j, \tag{2.1.1'}$$

$$\partial^i \partial^j = q^{-|ij|}\partial^j \partial^i, \forall i, j. \tag{2.1.2'}$$

Let us find the relations between $e_i^j e_k^l = x_i\partial^j \cdot x_k \partial^l$ and $e_k^l e_i^j = x_k\partial^l \cdot x_i\partial^j$. They should be the relations for $U_q(\mathbf{gl}_n)$.

Proposition 2.2. *The quadratic elements e_i^j have the following relations:*

$$e_i^j e_k^l = q^{|ik|+|lj|} e_k^l e_i^j,$$

$$e_i^j e_j^l = e_i^l + q^{1+|ij|+|lj|} e_j^l e_i^j + (q - q^{-1}) q^{|it|+|lt|} \sum_{t>j} e_t^l e_i^t,$$

$$e_i^j e_k^i = q^{|ik|+|ij|}\left(q^{-1} e_k^i e_i^j - q^{-1} e_k^j + (q^{-2} - 1)\sum_{t>i} q^{|kt|+|jt|} e_t^j e_k^t\right),$$

$$e_i^j e_j^i - q^2 e_j^i e_i^j = e_i^i - e_j^j + (q^2 - 1)\sum_{t>j} e_t^t e_i^i - (q^2 - 1)\sum_{s>i} e_s^s e_j^j.$$

Proof. We only show the last one. From the relations (2.1) it follows that

$$
\begin{aligned}
&x_i\partial^j \cdot x_j\partial^i - q^2 x_j\partial^i \cdot x_i\partial^j \\
&\quad = x_i\partial^i - q^2 x_j\partial^j + (q^2-1)\sum_{t>j} x_i x_t \partial^t\partial^i - q^2(q^2-1)\sum_{s>i} x_j x_s \partial^s\partial^j \\
&\quad = x_i\partial^i - q^2 x_j\partial^j + (q^2-1)\sum_{t>j} x_t\partial^t x_i\partial^i \\
&\qquad - q^2(q^2-1)\left\{\sum_{j>s>i} q^{-2} x_s\partial^s x_j\partial^j + \sum_{s>j} x_s\partial^s x_j\partial^j + x_j\partial^j x_j\partial^j\right\} \\
&\quad = x_i\partial^i - q^2 x_j\partial^j + (q^2-1)\sum_{t>j} x_t\partial^t x_i\partial^i \\
&\qquad - (q^2-1)\left\{\sum_{j>s>i} x_s\partial^s x_j\partial^j + q^2\sum_{s>j} x_s\partial^s x_j\partial^j + \right. \\
&\qquad \left. + q^2\cdot q^{-2} x_j\partial^j \cdot x_j\partial^j - x_j\partial^j - (q^2-1)\sum_{t>j} x_t\partial^t x_j\partial^j\right\} \\
&\quad = x_i\partial^i - q^2 x_j\partial^j + (q^2-1)x_j\partial^j + (q^2-1)\sum_{t>j} x_t\partial^t x_i\partial^i \\
&\qquad - (q^2-1)\left\{\sum_{j>s>i} x_s\partial^s x_j\partial^j + x_j\partial^j\cdot x_j\partial^j + \sum_{t>j} x_t\partial^t x_j\partial^j\right\} \\
&\quad = x_i\partial^i - x_j\partial^j + (q^2-1)\sum_{t>j} x_t\partial^t x_i\partial^i - (q^2-1)\sum_{s>i} x_s\partial^s x_j\partial^j.
\end{aligned}
$$

□

We will take this as our definition of new deformation of $U(\mathbf{gl}_n)$ and denote it also by $U_q(\mathbf{gl}_n)$. The elements e_i^j are the analog of the Weyl generators for $U(\mathbf{gl}_n)$. However the element $T = \sum e_i^i$ is not central when $n > 2$ though it commutes with all the basic elements e_j^i, $|i-j| \leq \pm 1$. If we add all relations in Proposition 1.1, then T will be central, but it is not natural to do so. This phenomenon is not strange: we know that there is no unique way to define all root elements (elements like e_j^i) in the Drinfeld-Jimbo quantum algebras, while in our situation we have the analog of the Weyl generators but we do not have the commutativity of T in $U_q(\mathbf{gl}_n)$.

3. Quantum algebras $U_q(\mathbf{gl}_2)$ and $U_q(\mathbf{sl}_2)$.

We will study our algebra in detail for the case $n = 2$ in this section. As a direct consequence of the relations (2.2) we have

Proposition 3.1. *The algebra $U_q(\mathbf{gl}(2))$ generated by e_j^i, $i,j=1,2$ has the following complete relations:*

$$\begin{aligned}
e_1^1e_1^2 - e_1^2e_1^1 &= e_1^2 + (q^2-1)e_2^2e_1^2,\\
e_2^1e_1^1 - e_1^1e_2^1 &= e_2^1 + (q^2-1)e_2^1e_2^2,\\
e_1^1e_2^2 - e_2^2e_1^1 &= 0,\\
e_1^2e_2^1 - q^2e_2^1e_1^2 &= e_1^1 - e_2^2 + (1-q^2)(e_2^2)^2,\\
e_1^2e_2^2 - q^2e_2^2e_1^2 &= e_1^2,\\
e_2^2e_2^1 - q^2e_2^1e_2^2 &= e_2^1.
\end{aligned}$$

Notice that the element $T = e_1^1 + e_2^2$ belongs to the center. Replacing q^2 by q and using the standard notations: $e = e_1^2, f = e_2^1, h = e_1^1 - e_2^2$ and $a = e_1^1 + e_2^2$, we obtain

Corollary 3.2. *The algebra $U_q(\mathbf{gl}_2)$ is generated by e,f,a,h subject to the following relations:*

$$\begin{aligned}
[a,e] = [a,f] = [a,h] &= 0,\\
qhe - eh &= 2e,\\
hf - qfh &= -2f,\\
ef - qfe &= a + h + \frac{1-q}{4}h^2.
\end{aligned}$$

We define the algebra $U_q(\mathbf{sl}_2)$ to be the quotient algebra of the above $U_q(\mathbf{gl}_2)$ modulo the central element a. We will still use the same symbols for the generators for $U_q(\mathbf{sl}_2)$, namely, the algebra is generated by the elements e,f,h subject to relations (3.2) with $a=0$. It is clear that they specialize to the usual enveloping algebras $U(\mathbf{gl}_2)$ or $U(\mathbf{sl}_2)$ when q goes to 1.

Our algebra is a bialgebra with the following comultiplication:

$$\Delta(x) = x\otimes 1 + 1\otimes x + \frac{1}{2}(1-q)h\otimes x$$

where x is the generator h,e, and f, and Δ is extended to the whole algebra by linearity and multiplicativity. The counit is the morphism given by

$$\epsilon(h) = \epsilon(e) = \epsilon(f) = 0, \qquad \epsilon(1) = 1.$$

In fact it is easy to check that the mappings Δ and ϵ are well-defined and satisfy all the properties of a bialgebra. We can also enlarge our algebra into

a Hopf algebra via its localization by the element $1 + \frac{1}{2}(1-q)h$ using the antipode determined by

$$S(x) = -\left(1 + \frac{1}{2}(1-q)h\right)^{-1} x$$

for x being any of the basic generators h, e, f. It is easy to check that $1 + \frac{1}{2}(1-q)h$ is a normal and group-like element, and it is also a regular element because $U_q(\mathbf{sl}_2)$ is a domain (see Theorem 3.7). The noncommutativity and noncocommutativity of the bialgebra structure means that our algebra is a quantum (semi)group in the sense of Drinfeld.

The algebra $U_q(\mathbf{sl}_2)$ has a Casimir element given by

$$\begin{aligned} C &= ef + fe + \frac{1+q}{4}h^2 \\ &= 2qfe + h + \frac{1}{2}h^2 \\ &= 2ef - h + \frac{q}{2}h^2. \end{aligned} \tag{3.3}$$

The Casimir element q-commutes with the generators in the following sense:

Lemma 3.4.

$$\begin{aligned} eC &= qCe, \\ fC &= q^{-1}Cf, \\ hC &= Ch. \end{aligned}$$

Proof. Let's check one of them, say the first one:

$$\begin{aligned} eC - qCe &= e\left(2qfe + h + \frac{1}{2}h^2\right) - q\left(2ef - h + \frac{q}{2}h^2\right)e \\ &= (eh + qhe) + \frac{1}{2}(eh^2 - q^2h^2e) = 0. \end{aligned}$$

□

Remark. The Casimir element also plays an important role in the following associated homogenization ring $H_q(\mathbf{sl}_2)$:

$$\begin{aligned} qhe - eh &= 2et, \qquad \text{t is central,} \\ hf - qfh &= -2ft, \\ ef - qfe &= ht + \frac{1-q}{4}h^2. \end{aligned}$$

When $q = 1$, this ring is studied by [**LS**]. It is easy to check that $H_q(\mathbf{sl}_2)$ is an Ore extension $k[k,t][e,\sigma_1][f,\sigma_2,\delta_2]$ for some proper automorphisms σ_1, σ_2 and derivative δ_2, and that the elements $h - 2t/(q-1)$ and C are normal. In [**LSV**] similar type of algebras are considered and they also found such properties for their algebras.

To study the ring structure of our new algebra we have the following result.

Proposition 3.5. (1) *The algebra $U_q(\mathbf{gl}_2)$ is an Ore extension*

$$k[h,a][e,\sigma_1][f,\sigma_2,\delta_2]$$

for some proper automorphisms σ_1, σ_2 and σ_2-derivation δ_2. The ideal generated by elements h, a, e, f is a maximal ideal of co-dimension 1;
(2) *The algebra $U_q(\mathbf{sl}_2)$ is an Ore extension $k[h][e,\sigma_1'][f,\sigma_2',\delta_2']$ for some proper automorphisms σ_1', σ_2' and σ_2'-derivation δ_2'. The ideal generated by elements h, e, f is a maximal ideal of co-dimension 1.*

Proof. It is routine to verify the statements from the relations (3.2). For example, $\sigma_1(a) = a$ and $\sigma_1(h) = qh - 2$. □

The following result is clear by Proposition 3.5.

Corollary 3.6. *The algebra $U_q(\mathbf{sl}_2)$ (resp. $U_q(\mathbf{gl}_2)$) has a PBW-type basis consisting of elements $f^i h^j e^k$ with $i,j,k \in \mathbb{Z}^+$ (resp. $f^i h^j e^k a^l$ with $i,j,k,l \in \mathbb{Z}^+$). Consequently, the associated graded ring $gr(U_q(\mathbf{sl}_2))$ (resp. $gr(U_q(\mathbf{gl}_2))$) has the same Hilbert series as its classical counterpart. In fact, both $gr(U_q(\mathbf{sl}_2))$ and $gr(U_q(\mathbf{gl}_2))$ are skew polynomial rings.*

As a consequence of Proposition 3.5 and Corollary 3.6, we get

Theorem 3.7. *The algebra $U_q(\mathbf{sl}_2)$ (resp. $U_q(\mathbf{gl}_2)$) is a noetherian domain of Gelfand-Kirillov dimension, Krull dimension, and global dimension* 3 (*resp.* 4).

Proof. By Proposition 3.5 and [**MR**, Thm 1.2.9], both algebras are noetherian domains. The assertion about Gelfand-Kirillov dimension is clear because the associated graded rings are skew polynomial rings. For simplicity the dimension in this proof will mean either Krull dimension or global dimension. By Proposition 3.5 and [**MR**, Prop 6.5.4, Thm 7.5.3] it follows that the dimension of $U_q(\mathbf{sl}_2)$ (resp. $U_q(\mathbf{gl}_2)$) is at most 3 (resp. 4). Note that a, $1 + \frac{1}{2}(1-q)h$, $ef - fe$ is a regular sequence of $U_q(\mathbf{gl}_2)$ with the factor ring isomorphic to $k[x, x^{-1}]$. By [**MR**, Lemma 6.3.10, Thm 7.3.5], the dimension of $U_q(\mathbf{sl}_2)$ (resp. $U_q(\mathbf{gl}_2)$) is at least 3 (resp. 4). □

Remark. Note that $H_q(\mathbf{sl}_2)/(t-1) \cong U_q(\mathbf{sl}_2)$ and $H_q(\mathbf{sl}_2)/(t) \cong gr(U_q(\mathbf{sl}_2))$. Thus $H_q(\mathbf{sl}_2)$ has Hilbert series $1/(1-t)^4$, and it is a Noetherian domain, a maximal order, Auslander-regular ring of dimension 4 with the Cohen-Macaulay property (see [**LS**]).

4. Representation theory of $U_q(\mathbf{sl}_2)$.

Our algebra $U_q(\mathbf{gl}_n)$ is realized on the infinite dimensional quantum Weyl algebra $A_n(R)$, which serves as a natural defining module. In this section we will study finite dimensional modules for our algebra $U_q(\mathbf{sl}_2)$.

Assume q is a complex number. For $\mu \in \mathbb{C}$, we define the μ-weight space for a U_q-module V as

$$V_\mu = \{v \in V | h.v = \mu v\}$$

whose elements are called weight vectors with weight μ. It is easy to check that $fV_\mu \subseteq V_{q\mu-2}, eV_\mu \subseteq V_{q^{-1}\mu+2q^{-1}}$.

As in the classical case we define the Borel subalgebra $U(b)$ as the subalgebra generated by the element e and h. A λ-weight vector is called a highest weight vector if it is annihilated by the element e and $dim\, V_\lambda = 1$. A $U_q(\mathbf{sl}_2)$-module V is called a highest weight module if it is generated by a highest weight vector.

Note that when $\lambda = \frac{2}{q-1}$, the weight space V_λ is stablized under the actions of e and f, which is not the case we want to consider. *From now on we assume any considered weight is not equal to* $\frac{2}{q-1}$ *except in some remarks.*

Let $\mathbb{C}_\lambda = \mathbb{C} \otimes v_\lambda$ be the 1-dimensional module for the Borel subalgebra $U(b)$ generated by e, h. Here

$$h.v_\lambda = \lambda v_\lambda, \qquad e.v_\lambda = 0.$$

The Verma module $V(\lambda) = U_q \otimes_{U(b)} \mathbb{C}_\lambda$ is a highest weight module for U_q with the highest weight λ.

Lemma 4.1. *Fix $\lambda \in \mathbb{C}$. The submodules of $V(\lambda)$ are generated by vectors $f^j v_\lambda$ with some j such that $\lambda = -2\frac{1 \pm q^{-(j-1)/2}}{1-q}$.*

Proof. We will use the following q-integers:

$$[j] = 1 + q + \cdots + q^{j-1}.$$

As a direct computation we have that

$$hf^j = q^j f^j h - 2[j]f^j,$$
$$he^j = q^{-j} e^j h + 2q^{-j}[j]e^j,$$
$$ef^j - q^j f^j e = -[j][j-1]f^{j-1} + q^{j-1}[j]f^{j-1}h + \frac{1}{4}q^{j-1}(1-q^j)f^{j-1}h^2.$$

Set $v = 1 \otimes v_\lambda$. From the above computations It is clear that a submodule of $V(\lambda)$ has a linear basis consisting of elements of the form $f^j v$. Suppose j is the minimum integer for a non-zero $f^j v$ inside the submodule, then e kills the element $f^j v$. Thus

$$ef^j v = [j]\left(-[j-1] + \lambda q^{j-1} + \frac{1}{4}\lambda^2 q^{j-1}(1-q)\right) f^{j-1} v = 0,$$

which implies that $\frac{1}{4}\lambda^2 q^{j-1}(1-q) + \lambda q^{j-1} - [j-1] = 0$. □

The above result implies that the Verma module has a unique maximal submodule $U_q f^j v$ for some $j \in \mathbb{N}$ (or $\{0\}$ if no such j exists). The quotient module is a simple irreducible U_q-module with the highest weight λ, denoted as $L(\lambda)$.

Theorem 4.2. *Suppose q is not a root of 1. For each half positive integer $s \in \mathbb{N}/2$, there exist exactly two irreducible highest weight U_q-modules of dimension $2s+1$, namely $V = L(\lambda)$ where*

(1) *The highest weights are $\lambda = -2\frac{1 \mp q^{-s}}{1-q}$;*

(2) *Relative to h, V is the direct sum of weight spaces*

$$V_\mu, \mu = -2\frac{1 \mp q^{-s}}{1-q}, -2\frac{1 \mp q^{-(s-1)}}{1-q}, \cdots, -2\frac{1-q^s}{1 \mp q},$$

and $\dim V_\mu = 1$;

(3) *The action of U_q is given by the following formulae:*

$$\begin{aligned} f v_j &= [j+1] v_{j+1}, \\ e v_j &= -[-2s+j-1] v_{j-1}, \\ h v_j &= -2\frac{1 \mp q^{j-s}}{1-q} v_j \end{aligned}$$

where $v_j = f^j v_\lambda / [j]!$, and $j = 0, 1, \cdots, 2s$.

Proof. Let V be an irreducible U_q-module of dimension $2s+1$. Pick an eigenvector or weight vector v for h. The irreducibility implies that V is spanned by the vectors of the form $\{e^i f^j . v\}$. It follows from the finite dimensionality that there exists a highest weight vector v_λ for V. Thus $V = U_q v_\lambda$. Set $v_j = f^j v_\lambda / [j]!, j = 0, 1, \cdots$. Since v_j has weight $q^j \lambda - 2[j], j = 0, 1, \cdots$, all the weights $q^j \lambda - 2[j]$ are different due to $\lambda \neq \frac{2}{q-1}$. Thus the vectors v_j are linearly independent. Therefore it must happen that $v_{2s+1} = 0$.

Applying e to v_{2s+1}, we have that

$$-[2s] + \lambda q^{2s} + \frac{1}{4}\lambda^2 q^{2s}(1-q) = 0,$$

thus $\lambda = -2\frac{1\mp q^{-s}}{1-q}$.

For $\lambda = -2\frac{1\mp q^{s}}{1-q}$ it is easy to check that the weights for v_j are actually $q^j\lambda - 2[j] = -2\frac{1\mp q^{-s+j}}{1-q}$ for $0 \le j \le 2s$. Then

$$ev_j = \left(-[j-1] + \lambda q^{j-1} + \frac{1}{4}\lambda^2 q^{j-1}(1-q)\right) v_{j-1} = -[-2s+j-1]v_{j-1}.$$

Finally it is easy to verify that the formulae in (3) do give a representation for U_q. □

Remark. (1) When $q \to 1$, the module $L(\lambda)$ with $\lambda = -2\frac{1-q^{-s}}{1-q}$ specializes to the standard $(2s+1)$-dimensional module for $U(\mathbf{sl}_2)$, while the other one does not have a specialization.

(2) The 1-dimensional modules are easily seen to be determined by

$$(a) \quad h = \frac{2}{q-1}, ef = -\frac{1}{(q-1)^2};$$

$$(b) \quad e = f = 0, \quad \text{either} \quad h = 0, \text{ or } \quad h = \frac{4}{q-1}.$$

Proposition 4.3. *For each $n > 0$, there is an n-dimensional right $U_q(\mathbf{sl}_2)$-module W_n satisfying*

(1) $0 \subset W_1 \subset W_2 \subset \cdots \subset W_{n-1} \subset W_n$ *and all factor modules W_n/W_{n-1} (which are 1-dimensional) are isomorphic to W_1, and*

(2) *W_n is not isomorphic to a direct sum of two submodules, in particular W_n is not completely reducible. Hence $U_q(\mathbf{sl}_2)$ is not "semi-simple".*

Proof. For any $n > 0$, let $W_n = \mathbb{C}^n$. The $n \times n$ matrix ring acts on W_n naturally. Denote

$$e_n := \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}_{n\times n}$$

$$f_n = -\frac{1}{(q-1)^2}e_n^{-1}, \quad \text{and} \quad h_n = \frac{2}{q-1}I_n,$$

where I_n is the identity $n \times n$-matrix. One can easily check that e_n, f_n, h_n satisfy the relations of $U_q(\mathbf{sl}_2)$. Hence there is a unique ring homomorphism

from $U_q(\mathbf{sl}_2)$ to $\mathrm{End}_k(W_n)$ mapping e to e_n, f to f_n and h to h_n, and W_n is right $U_q(\mathbf{sl}_2)$-module. For example we have

$$x_i \cdot e = x_i \cdot e_n = x_i + x_{i-1}, \quad \text{and} \quad x_i \cdot h = x_i \cdot h_n = \frac{2}{q-1} x_i$$

where $x_1, \cdots, x_n$ is a basis of W_n. From this we see that $e_n|_{W_m} = e_m$, $e_n|_{W_m} = f_m$ and $h_n|_{W_m} = h_m$ where W_m is identified with the subspace $\sum_{i=1}^{m} kx_i$ of W_n for all $m < n$. Hence W_m is a submodule of W_n. It is easy to check the the factor module W_n/W_m is isomorphic to W_{n-m}. By the definition of e_n, the vector space W_n can not be written as a direct sum of two e_n-invariant subspaces. Hence W_n is not a direct sum of two submodules. □

Now let us focus on the case when the parameter q is a root of unity. In the remaining part of this section we assume that q is an n-th primitive root of 1, i.e., $q = e^{2\pi m/n}$ for $n \geq 2$ and $(m, n) = 1$ for positive integers m, n.

Lemma 4.4. *The elements e^n and f^n are in the center of the algebra U_q.*

Proof. It is a consequence of the commutation relations and $[n] = 0$. (Note that h^n does not belong to the center.) □

We are going to consider the representations of the quotient algebra $\overline{U}_q$ of the algebra U_q by the ideal generated by e^n, f^n.

Theorem 4.5. *Every finite dimensional simple $\overline{U}_q$-module is of dimension $\leq n$.*

(1) *For each $2 \leq d < n$, there are exactly two simple d-dimensional $\overline{U}_q$-modules which are precisely $L(\lambda)$ with $\lambda = -2\frac{1 \mp q^{-(d-1)/2}}{1-q}$;*

(2) *There are infinitely many simple n-dimensional $\overline{U}_q$-modules which are $L(\lambda)$, where λ is not a zero of the polynomials*

$$\frac{1}{4}\lambda^2 q^{j-1}(1-q) + \lambda q^{j-1} - [j-1], \qquad j = 0, \cdots, n-1.$$

Proof. Let V be a simple module of dimension d. By a similar argument as at the beginning of the proof for Theorem (4.2) (for generic q) it follows that V is a highest weight module. Say V is of highest weight λ with the highest weight vector v_λ, then V is determined by

$$e.v_\lambda = 0, \qquad h.v_\lambda = \lambda v_\lambda$$
$$e.f^j v_\lambda \neq 0, \qquad j = 1, \cdots, d-1, \text{ and } \qquad e.f^d v_\lambda = 0.$$

Then the set of vectors $f^j v_\lambda$, $j = 0, \cdots, d-1$ form a basis for V, and $d \leq n$ due to $[n] = 0$.

If $dim\, V \leq n$, then the last condition in the above says that

$$ef^d v_\lambda = -[d]\left(-[d-1] + \lambda^{d-1} + \frac{1}{4}\lambda^2 q^{d-1}(1-q)\right) f^{d-1} v_\lambda = 0,$$

which means that $\lambda = -2\frac{1-q^{-(d-1)/2}}{1-q}$. Then

$$ef^j = -[j][-d+j]f^{j-1} \neq 0.$$

Therefore $V \backsimeq L(\lambda)$ and the action is described in Theorem 4.2.

Suppose the dimension of V is n. The condition $e.f^n v_\lambda = 0$ is satisfied automatically due to $[n] = 0$. We only worry about the first set of requirements:

$$\frac{1}{4}\lambda^2(1-q) + \lambda + [-j+1] \neq 0,$$

which completes the proof. □

In the end we will demonstrate that our algebra has (reducible) indecomposable modules of dimension $2n$ when q is an nth primitive root, which is similar to the observation made by [**K**] for the Drinfeld-Jimbo algebra $U_q(\mathbf{sl}_2)$. However, we only have finite number of indecomposable modules for each case versus the situation for the Drinfeld-Jimbo algebra $U_q(\mathbf{sl}_2)$.

Theorem 4.6. *For each $1 \leq s \leq n-1$ there exist indecomposable $\overline{U}_q$-modules V of dimension $2n$ such that*

(1) *The module V is a direct sum of its weight spaces V_μ with*

$$\mu = -2\frac{1 \mp q^{-(n+s-1)/2}}{1-q}, -2\frac{1 \mp q^{-(n+s-3)/2}}{1-q}, \cdots, -2\frac{1 \mp q^{(n+s-1)/2}}{1-q}$$

with the highest weight $-2\frac{1\mp q^{-(n+s-1)/2}}{1-q}$;

(2) *The dimensions of the weight spaces are 2 for the weight string*

$$\mu = -2\frac{1 \mp q^{(-n+s-1)/2}}{1-q}, -2\frac{1 \mp q^{(-n+s+1)/2}}{1-q}, \cdots, -2\frac{1 \mp q^{(n-s+1)/2}}{1-q},$$

and 1 otherwise;

(3) *We can choose v to be a highest weight vector and w to be a weight vector in V_μ with $\mu = -2\frac{1\mp q^{-(n-s+1)/2}}{1-q}$ independent from v such that the*

action is given by

$$
\begin{aligned}
f.f^j v &= f^{j+1}v, \qquad j = 0, \cdots, n-2, \\
e.f^j v &= -[j][-(n+s-j)]f^{j-1}v, \qquad j = 0, \cdots, n-1, \\
h.f^j v &= -2\frac{1 \mp q^{j-(n+s-1)/2}}{1-q} f^j v; \qquad j = 0, \cdots, \cdots, n-1, \\
e.w &= f^{s-1}v, \\
f.f^j w &= f^{j+1}w, \qquad j = 0, \cdots, n-s, \\
e.f^j w &= q^j f^{j+s-1}v - [j][-(n-s-j)]f^{j-1}w, \qquad j = 1, \cdots, n-1, \\
h.f^j w &= -2\frac{1 \mp q^{j-(n-s-1)/2}}{1-q} f^j w; \qquad j = 0. \cdots n-1.
\end{aligned}
$$

Proof. Let V be a module satisfying conditions (1) and (2). We want to show that the module structure is given by formulae in (3). Take a weight vector v with weight $\lambda = -2\frac{1 \mp q^{-(n+s-1)/2}}{1-q}$. Then

$$e.f^j v = -[j][-(n+s-j)]f^{j-1}v \neq 0 \qquad \text{for} \quad j \neq s.$$

Since $dim\, V_\mu = 2$ for $\mu = -2\frac{1 \mp q^{-(n-s+1)/2}}{1-q}$, there exists a weight vector $w \in V_\mu$ not in the kernel of e such that

$$e.w = f^{s-1}v,$$

which implies that $f^s v$ and w form a basis for the V_μ.

Then we have

$$e.f^j w = q^j f^{j+s-1}v - [j][-(n-s-j)]f^{j-1}w, \qquad j = 1, \cdots, n-1$$

which implies that $e.f^{n-s}w = q^{-s}f^{n-1}v$, so vectors $f^j w, j = 0, \cdots, n-1$ are independent (nonzero). It is clear that vectors $f^j w$ and $f^{s+j}v$ generate the weight space V_μ for $\mu = -2\frac{1 \mp q^{j-(n-s+1)/2}}{1-q}, j = 0, \cdots, j = n-s+1$. So vectors $f^j v, f^j w, j = 0, \cdots, n-1$ form a basis for V. Finally it is easy to check that the formulae in (3) do define a representation of $\overline{U}_q$. The indecomposability of the module is clear from the defining formulae. □

5. Connection to Drinfeld-Jimbo quantum groups.

The Drinfeld-Jimbo quantum algebra $U_q(\mathbf{sl}_2)$ was first introduced by Kulish-Reshetikhin and Sklyanin in early 80's as an associative algebra generated

by e, f, h over the ring of power series in t with the relations:

$$[h, e] = 2e, \qquad [h, f] = -2f, \tag{5.1}$$

$$[e, f] = \frac{sh(th)}{t}, \tag{5.2}$$

where the function $sh(x)$ is $\frac{exp(x)-exp(-x)}{2}$. In this case all the finite dimensional irreducible modules are in one to one correspondence to those for the simple Lie algebra $\mathbf{sl}_2$ instead of two to one for the Jimbo case with element k.

We can approximate the quantum group by a family of associative algebras $U(\mathbf{sl}_2)_k$, $k = 0, \cdots, \in \mathbb{Z}^+$, which are the associative algebras generated by e, f, h subject the relations (5.1) and

$$[e, f] = \sum_{i=0}^{k} \frac{t^{2i} h^{2i+1}}{(2i+1)!}. \tag{5.3}$$

The first of the family is the usual enveloping algebra $U(\mathbf{sl}_2)$. Note that the commutator $[e, f]$ is a polynomial in h of degree $2k+1$. This type of algebra is studied in general in [**S**].

Our new algebra would have belonged to this type of algebras if there were pure commutators in the defining relations. Another feature is that our algebra is a degree 2 approximation or perturbation to the enveloping algebra. To see the behavior of our deformation one could consider another family of algebras deviated from $U(\mathbf{sl}_2)$ by changing the commutators in (5.1) and (5.3) to q-commutators as in our deformation of $U(\mathbf{sl}_2)$. Thus one will have to generalize or q-deform the work of [**S**].

References

[DF] J. Ding and I. Frenkel, *Quantum Clifford algebra, quantum Weyl algebra and quantum L-matrix*, Yale preprint, 1992.

[FRT] L. Faddeev, N. Reshetikhin and L. Takhtajan, *Quantization of Lie groups and Lie algebras*, Leningrad Math J., **1** (1989), 193-225.

[GZ] T. Giaquinto and J. Zhang, *Quantizations and deformations of Weyl algebras*, Jour. Alg., **176** (1995), 861-881.

[K] G. Keller, *Fusion rules of* $U_q(sl(2, \mathbb{C}), q^m = 1$, Lett. Math. Phys., **21** (1991), 273-286.

[L] L. Le Bruyn, *Two remarks on Witten's quantum enveloping algebra*, U. Antwerp (UIA) preprint, 1993.

[LS] L. Le Bruyn and S. P. Smith, *Homogenized* sl_2, Proc. Amer. Math. Soc., **118** (1993), 725-730.

[LSV] L. Le Bruyn, S. P. Smith and M. Van den Bergh, *Central extensions of 3-dimensional Artin-Schelter regular algebras*, Math. Zeit., to appear.

[MR] J.C. McConnell and J.C. Robson, *Noncommutative Noetherian rings*, Wiley-Interscience, Chichester, 1987.

[S] S. P. Smith, *A class of algebra similar to the enveloping algebra of* $\mathbf{sl}(2)$, Trans. Amer. Math. Soc., to appear.

[WZ] J. Wess and B. Zumino, *Covariant differential calculus on the quantum hyperplanes*, Nucl. Phys. B (Proc. Suppl.), **18** (1990), 302-312.

[W] E. Witten, *Gauge theory, vertex models, and quantum groups*, Nucl. Phys., **B330** (1990), 285-346.

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