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FACTORIZATION OF P-COMPLETELY BOUNDED MULTILINEAR MAPS

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Given Banach spaces $X_1, \dots, X_N$, $Y_1, \dots, Y_N, X, Y$ and subspaces $S_i \subset B(X_i, Y_i)$ $(1 \le i \le N)$, we study p-completely bounded multilinear maps $A: S_N \times \cdots \times S_1 \to B(X,Y)$. We obtain a factorization theorem for such A which is entirely similar to the Christensen-Sinclair representation theorem for completely bounded multilinear maps on operator spaces. Our main tool is a generalisation of Ruan's representation theorem for operator spaces in the Banach space setting. As a consequence, we are able to compute the norms of adapted multilinear Schur product maps on $B(\ell_p^n)$.

1. Introduction and preliminaries.

1.1. Introduction. In a recent paper, Pisier [**Pi1**] proved that the Wittstock factorization theorem for completely bounded maps (cf. [**Ha**], [**Pa1**], [**Pa2**], [**W**]) has a natural generalization to the more general framework of p-completely bounded maps defined on sets of Banach space operators. The main goal of this paper is to prove a version of the Christensen-Sinclair theorem (cf. [**CS**, **PS**]) in this extensive setting.

Let us first recall the definition of p-complete boundedness as introduced (or suggested) in [**Pi**]. Let $1 \le p < +\infty$ be a number. Let X, Y be Banach spaces. We denote by $B(X,Y)$ the space of all bounded operators from X into Y. Let $S \subset B(X,Y)$ be a subspace. We denote by $M_{n,m}(S)$ the vector space of all $n \times m$ matrices with entries from S. Any $s = [s_{ij}] \in M_{n,m}(S)$ may be canonically identified with a bounded operator from $\ell_p^m(X)$ into $\ell_p^n(Y)$. Under this identification, s has the following norm:

$$(1.1)\quad \|[s_{ij}]\|_{M_{n,m}(S)} = \sup\left\{\left(\sum_i \left\|\sum_j s_{ij}(x_j)\right\|^p\right)^{\frac{1}{p}} / x_1, \dots, x_m \in X, \sum_j \|x_j\|^p \le 1\right\}.$$

Then the usual concept of complete boundedness has the following natural extension.

Definition 1.1. Let $X_1, \dots, X_N$, $Y_1, \dots, Y_N, X, Y$ be Banach spaces. For each $1 \leq i \leq N$, let $S_i \subset B(X_i, Y_i)$ be a subspace. Let $A: S_N \times \cdots \times S_1 \to B(X, Y)$ be a N-linear map. We will say that A is p-completely bounded if there is a constant $C > 0$ for which the following holds.

For any

$$s_N \in M_{n,k_{N-1}}(S_N), s_{N-1} \in M_{k_{N-1},k_{N-2}}(S_{N-1}), \dots, s_2 \in M_{k_2,k_1}(S_2), s_1 \in M_{k_1,m}(S_1),$$

we have:

$$\left\| \left[\sum_{\substack{1 \leq \ell \leq N-1 \\ 1 \leq r_\ell \leq k_\ell}} A\big(s_{N(i,r_{N-1})}, s_{(N-1)(r_{N-1},r_{N-2})}, \dots, s_{2(r_2,r_1)}, s_{1(r_1,j)} \big) \right]_{(i,j)} \right\|_{M_{n,m}(B(X,Y))}$$

$$\leq C \|s_N\|_{M_{n,k_{N-1}}(S_N)} \|s_{N-1}\|_{M_{k_{N-1},k_{N-2}}(S_{N-1})} \cdots\cdots \|s_1\|_{M_{k_1,m}(S_1)}.$$

Moreover, we denote by $\|A\|_{pcb}$ the least constant $C > 0$ for which this holds.

We will prove that whenever $p \in]1, +\infty[$, a p-completely bounded multilinear map A as above factors as a product of p-completely bounded linear maps defined on each S_i (see Theorem 5.1 for a precise statement). Thus using Pisier's generalization of the Wittstock theorem, we obtain a representation of A which is quite similar to the Christensen-Sinclair representation for a completely bounded multilinear map on operator spaces. This answers the question raised by Pisier in the Final Remark of [**Pi1**]. Note that our result is new only for $N \geq 3$. However, even in the case $N = 2$, we feel that our proof is simpler than Pisier's one.

The recently developped theory of operator spaces (see [**B**, **BP**, **BS**, **ER1**, **ER2**]) has emphasized the role of the Haagerup tensor product in the study of completely bounded multilinear maps. It is now well-known to specialists (see [**B**, Theorem 2.4] for example) that the Christensen-Sinclair theorem may be viewed as a combination of the factorization theorem for completely bounded bilinear forms (which goes back to [**EK**]), Ruan's representation theorem for operator spaces (see [**R**, **ER3**]) and simple properties of the Haagerup tensor product. In this approach, the crucial point is that given two operator spaces, their Haagerup tensor product is again an operator space. This essentialy follows from Ruan's theorem. In order to prove our main Theorem 5.1, we will follow the above scheme. We will especially prove a generalization of Ruan's theorem (see our Theorem 4.1) which is of independant interest.

Let us now explain the organization of the paper. In the two following subsections, we recall Pisier's result about p-completely bounded linear maps and introduce necessary definitions about matrix normed spaces. In Section 2, we define a generalized Haagerup tensor product $\otimes_h$ adapted to our definition of p-complete boundedness and prove elementary properties which will be needed later. In Section 3, we combine ideas from [**E**], [**ER3**] and [**Pi1**] in order to prove an abstract factorization theorem which is used in the two following sections. Section 4 is devoted to our generalization of Ruan's theorem. We follow the same line of attack as Effros and Ruan [**ER3**]. Our main result explained above is proved in Section 5. In the last Section 6, we investigate some of the properties of our new tensor product $\otimes_h$. We then prove a theorem about multilinear Schur products on $B(\ell_p^n, \ell_p^n)$ which generalizes previous works on this subject (see [**ER4, Gr, Ha, S**] for example).

1.2. Pisier's theorem. We wish to recall Pisier's theorem as stated in [**Pi1**]. It will be formulated in the language of ultraproducts. We first introduce a notation which will be frequently used in this paper.

Definition 1.2. Let E and X be Banach spaces. Let $1 \leq p < +\infty$ be a number. We will write $E \in SQ_p(X)$ provided that E is (isometric to) a quotient of a subspace of an ultraproduct of spaces of the form $L_p(\mu; X)$.

Let X_1, Y_1 be Banach spaces and $S \subset B(X_1, Y_1)$. Consider a number $1 \leq p < +\infty$. Let $(\Omega_j, \mu_j)_{j \in J}$ be a family of measure spaces and let $\mathcal{U}$ be an ultrafilter on the index set J. Let us denote by $\widehat{X_1}$ and $\widehat{Y_1}$ respectively the ultraproducts relative to $\mathcal{U}$ of the families $(L_p(\mu_j\,;\, X_1))_{j \in J}$ and $(L_p(\mu_j\,;\, Y_1))_{j \in J}$. For any $a \in B(X_1, Y_1)$, we may define $\widehat{\pi_j}(a) : \; L_p(\mu_j\,;\, X_1) \to L_p(\mu_j\,;\, Y_1)$ by setting $(\pi_j(a) f)(w) = a.\, f(w)$. We denote by $\widehat{\pi}(a) : \widehat{X_1} \to \widehat{Y_1}$ the map associated to the family $(\widehat{\pi}_j(a))_{j \in J}$. Let $N \subset M \subset \widehat{X_1}$ and $N' \subset M' \subset \widehat{Y_1}$ be closed subspaces such that for any $s \in S$, $\widehat{\pi}(s)(M) \subset M'$ and $\widehat{\pi}(s)(N) \subset N'$. Then letting $G = \dfrac{M}{N}$ and $G' = \dfrac{M'}{N'}$, we obtain that $\widehat{\pi}_{/S}$ canonically induces a map $\pi : S \to B(G, G')$. Namely we may set $\pi(s)(m+N) = \widehat{\pi}(s)(m) + N'$ for any $(s, m) \in S \times M$. Such a map will be called a p-representation from S into $B(G, G')$. More precisely we state the following:

Definition 1.3. Let $G \in SQ_p(X_1)$ and $G' \in SQ_p(Y_1)$ be two Banach spaces. Let $\pi : S \to B(G, G')$ be a bounded linear map. We will say that π is a p-representation provided that it may be constructed as above.

Theorem 1.4 ([**Pi1**, Theorem 2.1]). *Let $S \subset B(X_1, Y_1)$, let $A : S \to B(X, Y)$ be a linear map and let C be a constant. The following assertions* *are equivalent:*

(i) *A is p-completely bounded and $\|A\|_{pcb} \leq C$.*

(ii) *There are two Banach spaces $G \in SQ_p(X_1), G' \in SQ_p(Y_1)$ and a p-representation $\pi : S \to B(G, G')$ as well as operators $V : X \to G$ and $W : G' \to Y$ with $\|V\| \, \|W\| \leq C$ such that:*

$$\forall\, s \in S, \quad A(s) = W\pi(s)V.$$

1.3. Matrix normed spaces. Let S be a complex normed space. Let us denote by $M_{n,m}(S)$ the vector space of $n \times m$ matrices over S. As usual, we just denote by $M_n(S)$ the space $M_{n,n}(S)$. In the case when $S = \mathbb{C}$, we will simply write $M_{n,m}$ or M_n for these spaces. We will say that S is a matrix normed space provided that we are given norms $\|\ \|_{n,m}$ on each $M_{n,m}(S)$ satisfying $M_1(S) = S$ and:

(i) For any $s \in M_{n,m}(S), s' \in M_{n,k}(S)$,

$$\max \left\{ \|s\|_{n,m}, \|s'\|_{n,k} \right\} \leq \|(s, s')\|_{n,m+k}.$$

(ii) For any $s \in M_{n,m}(S), 0 \in M_{n,k}(S)$,

$$\|s\|_{n,m} = \|(s, 0)\|_{n,m+k} = \|(0, s)\|_{n,m+k}.$$

(iii) For any $s \in M_{n,m}(S), s' \in M_{k,m}(S)$,

$$\max \left\{ \|s\|_{n,m}, \|s'\|_{k,m} \right\} \leq \left\| \begin{pmatrix} s \\ s' \end{pmatrix} \right\|_{n+k,m}.$$

(iv) For any $s \in M_{n,m}(S), 0 \in M_{k,m}(S)$,

$$\|s\|_{n,m} = \left\| \begin{pmatrix} s \\ 0 \end{pmatrix} \right\|_{n+k,m} = \left\| \begin{pmatrix} 0 \\ s \end{pmatrix} \right\|_{n+k,m}.$$

Actually, these are very weak conditions. They are chosen to ensure two reasonnable properties. First, for any $n, m \leq k$, the canonical embedding of $M_{n,m}(S)$ in $M_k(S)$ is isometric. Secondly, for any $s = [s_{ij}] \in M_{n,m}(S)$,

$$\sup_{i,j} \|s_{ij}\| \leq \|s\|_{n,m} \leq \sum_{i,j} \|s_{ij}\|. \tag{1.2}$$

Thus $M_{n,m}(S)$ and S^{nm} are isomorphic as normed spaces. From now on, we leave the notation $\|\ \|_{n,m}$ and merely denote by $\|\ \|$ the norm on all the spaces $M_{n,m}(S)$. We will have to distinguish a possible property of a matrix normed space S. For any $s \in M_{n,m}(S)$, $s' \in M_{n',m'}(S)$, we set $s \oplus s' = \begin{pmatrix} s & 0 \\ 0 & s' \end{pmatrix} \in M_{n+n',m+m'}(S)$. We will say that S satisfies $\mathcal{D}_\infty$ whenever the following condition is fulfilled:

$\mathcal{D}_\infty$: For any $s \in M_{n,m}(S), s' \in M_{n',m'}(S)$, $\|s \oplus s'\| = \max\ \{\|s\|, \|s'\|\}$.

The latter property is one of the characteristic conditions in Ruan's representation theorem for operator spaces. It will play a similar role in our Theorem 4.1.

Let us now introduce some standard definitions and traditional notation. Let S, T be two matrix normed spaces and let $u \in B(S,T)$. We define $u^{(n)} : M_n(S) \to M_n(T)$ by $u^{(n)}([s_{ij}]) = [u(s_{ij})]$. We let $\|u\|_{cb} = \sup\limits_{n \geq 1} \left\|u^{(n)}\right\|$. We say that u is completely bounded (in short c.b.) provided that $\|u\|_{cb} < +\infty$. We denote by $CB(S,T)$ the resulting normed space. We say that u is completely contractive (in short c.c.) when $\|u\|_{cb} \leq 1$ and u is completely isometric provided that for any $n \geq 1$, $u^{(n)}$ is isometric.

2. p-matrix normed spaces.

We introduce a special kind of matrix normed spaces.

Definition 2.1. Let $p, q \in]1, +\infty[$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let S be a matrix normed space. We will say that S is a p-matrix normed space if it satisfies the condition $\mathcal{D}_\infty$ above and the following:

(2.1) For any $s \in M_{n,m}(S), s' \in M_{n',m}(S)$, $\left\| \begin{pmatrix} s \\ s' \end{pmatrix} \right\|^p \leq \|s\|^p + \|s'\|^p$.

(2.2) For any $s \in M_{n,m}(S)$, $s' \in M_{n,m'}(S)$, $\|(s, s')\|^q \leq \|s\|^q + \|s'\|^q$.

(2.3) For any $s \in M_{n,m}(S), \alpha \in M_{m,1}$, $\|s\alpha\| \leq \|s\| \left(\sum_{j=1}^m |\alpha_j|^p\right)^{1/p}$.

(2.4) For any $s \in M_{n,m}(S), \beta \in M_{1,n}$, $\|\beta s\| \leq \|s\| \left(\sum_{i=1}^n |\beta_i|^q\right)^{1/q}$.

Example 2.2. Let X, Y be Banach spaces. Let $S \subset B(X,Y)$ be a subspace. Let us equip each $M_{n,m}(S)$ with the norm defined by (1.1). Recall that this yields an isometric embedding $M_{n,m}(S) \subset B(\ell_p^m(X), \ell_p^n(Y))$. Then it is not hard to check that S becomes a p-matrix normed space. Let us emphasize for further that given a finite family $(s_j)_{1 \leq j \leq n}$ in S, the corresponding column and row matrices have the following norms:

$$(2.5) \qquad \left\| \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} \right\| = \sup \left\{ \left(\sum_{j=1}^n \|s_j(x)\|^p \right)^{1/p} \ / \ x \in X,\ \|x\| \leq 1 \right\}$$

$$(2.6) \qquad \|(s_1, \cdots, s_n)\| = \sup \left\{ \left(\sum_{i=1}^{n} \|s_i^*(y^*)\|^q \right)^{1/q} \ / \ y^* \in Y^*, \|y^*\| \leq 1 \right\}.$$

Throughout the rest of the paper, we fix a number $p \in]1, +\infty[$ and let $q = \dfrac{p}{p-1}$ (i.e.: $\frac{1}{p} + \frac{1}{q} = 1$). Given a subspace S of some $B(X, Y)$, we will always assume that it is endowed with its p-matrix normed space structure as defined in Example 2.2. As announced in the introduction, our purpose is to define an adapted variant of the Haagerup tensor product. Although we will be mainly concerned by matrix normed spaces $S \subset B(X, Y)$ as above, it is convenient to work in the slighly more general setting of p-matrix normed spaces. We will only give short proofs of the results listed below since they are all variants of known results of the classical theory of operator spaces. We will use the following well-know fact :

$$(2.7) \qquad \forall \ (a, b) \in \mathbb{R}_+^2, \ ab = \inf \left\{ \frac{\theta^p a^p}{p} + \frac{\theta^{-q} b^q}{q} \ / \ \theta > 0 \right\}.$$

Let S, T be two p-matrix normed spaces. Given $s = [s_{ir}] \in M_{n,k}(S)$ and $t = [t_{rj}] \in M_{k,m}(T)$, we define $s \odot t = \Big[\sum_{r=1}^{k} s_{ir} \otimes t_{rj}\Big] \in M_{n,m}(S \otimes T)$. For any $z \in M_{n,m}(S \otimes T)$ we set:

$$(2.8) \qquad \|z\|_h = \inf \ \{\|s\| \ \|t\| \ / \ s \in M_{n,k}(S), t \in M_{k,m}(T), z = s \odot t\}.$$

Proposition-Definition 2.3. The function $\| \ \|_h$ is a norm on each space $M_{n,m}(S \otimes T)$. Endowed with these norms, $S \otimes T$ becomes a p-matrix normed space.

We will denote by $S \otimes_h T$ this p-matrix normed space.

Proof. Let $z = s \odot t$ and $z' = s' \odot t' \in M_{n,m}(S \otimes T)$. Then $z + z' = (s, s') \odot \begin{pmatrix} t \\ t' \end{pmatrix}$. Therefore, applying (2.2) to (s, s'), (2.1) to $\begin{pmatrix} t \\ t' \end{pmatrix}$ and (2.8), we deduce that $\| \ \|_h$ is a semi-norm on $M_{n,m}(S \otimes T)$. It is clear that these semi-norms satisfy all the conditions (i), (ii), (iii), (iv) required in the definition of a matrix normed space. Hence by (1.2), in order to prove that $\| \ \|_h$ is a norm on each $M_{n,m}(S \otimes T)$, it is enough to show that $\| \ \|_h$ is non-degenerate on $S \otimes T$. Let $z = \sum_{r=1}^{N} s_r \otimes t_r \in S \otimes T$. Let $s^* \in S^*, t^* \in T^*$. Since the space S satisfies (2.3), we have: $\left(\sum_{r=1}^{N} |\langle s^*, s_r \rangle|^q \right)^{1/q} \leq \|s^*\| \ \|(s_1, \ldots, s_N)\|$.

Analogoulsy,

$$\left(\sum_{r=1}^{N}|\langle t^*, t_r\rangle|^p\right)^{1/p} \leq \|t^*\| \left\|\begin{pmatrix} t_1 \\ \vdots \\ t_N \end{pmatrix}\right\|.$$

Now,

$$|\langle z, s^* \otimes t^*\rangle| \leq \left(\sum_{r=1}^{N}|\langle s^*, s_r\rangle|^q\right)^{1/q} \left(\sum_{r=1}^{N}|\langle t^*, t_r\rangle|^p\right)^{1/p},$$

hence we obtain

$$|\langle z, s^* \otimes t^*\rangle| \leq \|s^*\| \ \|t^*\| \ \|(s_1, \dots, s_N)\| \left\|\begin{pmatrix} t_1 \\ \vdots \\ t_N \end{pmatrix}\right\|.$$

Therefore, $\|z\|_h = 0$ implies $z = 0$ and we are done.

It remains to check the condition $\mathcal{D}_\infty$ and the four properties (2.1) - (2.4). Let $z = s \odot t \in M_{n,m}(S \otimes T)$ and $z' = s' \odot t' \in M_{n',m'}(S \otimes T)$. Then $\begin{pmatrix} z \\ z' \end{pmatrix} = (s \oplus s') \odot \begin{pmatrix} t \\ t' \end{pmatrix}$. Hence, applying $\mathcal{D}_\infty$ to $s \oplus s'$ and (2.1) to $\begin{pmatrix} t \\ t' \end{pmatrix}$ we obtain that $S \otimes T$ satisfies (2.1). The proofs of (2.2), (2.3), (2.4) and $\mathcal{D}_\infty$ are similar, we omit them. □

Remark 2.4. In the case when S and T are operator spaces, the space $S \otimes_h T$ defined above is the usual Haagerup tensor product of S and T.

Remark 2.5. The tensor product $\otimes_h$ is associative. Thus given p-matrix normed spaces $S_1, \dots, S_N$, we may define unambigously the space $S_N \otimes_h \cdots \otimes_h S_1$. Let us now come back to Example 2.2. Let $N \geq 2$ and $X_1, \dots, X_N, Y_1, \dots, Y_N$ be Banach spaces. For any $1 \leq i \leq N$, we give ourselves $S_i \subset B(X_i, Y_i)$. From above, we may consider the p-matrix normed space $S = S_N \otimes_h \cdots \otimes_h S_1$. Let X, Y be two Banach spaces and let $A : S_N \times \cdots \times S_1 \to B(X, Y)$ be a multilinear map. It may be viewed as a linear map $\widehat{A} : S \to B(X, Y)$ as well. Now it is easy to see that A is p-completely bounded in the sense of Definition 1.1 if and only if $\widehat{A}$ is completely bounded. Moreover, $\left\|\widehat{A}\right\|_{cb} = \|A\|_{pcb}$.

Let E be a Banach space. The identification $E = B(\mathbb{C}, E)$ allows us to define a p-matrix normed space structure on E. To conform with the notation used in the operator space theory, we denote by E_c the p-matrix normed space above. Similarly, we denote by E_r^* the p-matrix normed space structure on E^* defined by the identification $E^* = B(E, \mathbb{C})$. Two simple facts should be noticed:

(2.9) For any $x_1, \dots, x_n \in E$, $$\left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\|_{E_c} = \left(\sum_{j=1}^{n} \|x_j\|^p \right)^{1/p}.$$

(2.10) For any $x_1^*, \dots, x_n^* \in E^*$, $$\|(x_1^*, \dots, x_n^*)\|_{E_r^*} = \left(\sum_{i=1}^{n} \|x_i^*\|^q \right).$$

We end this section by two simple lemmas about these p-matrix normed spaces.

Lemma 2.6. *Let S be a p-matrix normed space and let E, F be Banach spaces.*

(a) *For any $u : S \to E_c$,*

$$\|u\|_{cb} = \sup \left\{ \left(\sum_{j=1}^{n} \|u(s_j)\|^p \right)^{1/p} \ / \ \left\| \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} \right\| \leq 1 \right\}.$$

(b) *For any $v : S \to F_r^*$,*

$$\|v\|_{cb} = \sup \left\{ \left(\sum_{i=1}^{n} \|v(s_i)\|^q \right)^{1/q} \ / \ \|(s_1, \dots, s_n)\| \leq 1 \right\}.$$

Proof. Apply (2.9), (2.3) to show (a) and apply (2.10), (2.4) to show (b). □

Let S be a p-matrix normed space and let E, F be Banach spaces. Let $u : S \to B(E, F^{**})$ be a linear map. We can regard u as a trilinear form $\hat{u}$ on $F^* \times S \times E$ by setting:

$$\hat{u}(b^*, s, e) = \langle u(s)(e), b^* \rangle$$

Lemma 2.7. *The map $u \mapsto \hat{u}$ gives rise to the isometric identification*

$$CB(S, B(E, F^{**})) = (F_r^* \otimes_h S \otimes_h E_c)^*.$$

Proof. Apply (2.9) to E and (2.10) to F. □

Remark 2.8. It should be noticed that the one-dimensional vector space $\mathbb{C}$ may be endowed with several different p-matrix structures. Very natural examples may be obtained as follows. We give oursleves a Banach space X. Let us denote by I_X the identity map on X. Then we set

(2.11) $$\mathbb{C}^X = \text{Span } \{I_X\} \subset B(X, X)$$

and this provides us a p-matrix structure on $\mathbb{C}$. We refer to Section 3 below for more about $\mathbb{C}^X$. In the sequel, we keep the notation $\mathbb{C}$ to refer to the p-matrix normed space $\mathbb{C}^{\mathbb{C}}$. Now let S be a p-matrix normed space. We wish to point out two simple facts.

(a) For any linear form $\xi : S \to \mathbb{C}$, $\|\xi\| = \|\xi\|_{cb}$. This is a straightfoward consequence of the assertions (2.3) and (2.4).

(b) Let X, Y be Banach spaces. Let J (resp. J_1, J_2) be the canonical identification map from S onto $\mathbb{C}^Y \otimes_h S \otimes_h \mathbb{C}^X$ (respectively $S \otimes_h \mathbb{C}^X$, $\mathbb{C}^Y \otimes_h S$). Then it follows from (2.3) and (2.4) again that J, J_1, J_2 are isometric. Moreover they are obviously c.c. maps. However, in general, they are not completely isometric. We will come back to this problem in Remark 4.3.

3. An abstract factorization theorem.

Let X be a Banach space. Given $a = [a_{ij}] \in M_{n,m}$, we let

$$\|a\|_{p,X} = \sup \left\{ \left(\sum_{i=1}^{n} \left\| \sum_{j=1}^{m} a_{ij} x_j \right\|^p \right)^{1/p} \right\} \tag{3.1}$$

where the supremum runs over all the $x_1, \ldots, x_m$ in X which satisfy

$$\sum_j \|x_j\|^p \leq 1.$$

To understand the relation between this definition and preceding ones, consider the subspace $S = \mathbb{C}^X \subset B(X, X)$ defined by (2.11). Let $s = a \otimes I_X \in M_{n,m}(S)$. Then the definitions (1.1) and (3.1) obviously give $\|s\| = \|a\|_{p,X}$.

The following criterion of Hernandez will be used several times.

Theorem 3.2 [**He1, He2**]. *Let E and X be Banach spaces. Then $E \in SQ_p(X)$ if and only if:*

$$\forall\ a \in M_n,\ \|a\|_{p,E} \leq \|a\|_{p,X}.$$

Proof. We follow [**Pi1**, Section 3] and refer to this for more informations. Let $A : \mathbb{C}^X \to B(E, E)$ be defined by $A(I_X) = I_E$. Then A is c.c. iff $\forall\ a \in M_n$, $\|a\|_{p,E} \leq \|a\|_{p,X}$. Hence the result follows from Pisier's theorem 1.4. Finally we should mention that in the particular case $X = \mathbb{C}$, this result goes back to Kwapien [**K**]. □

In order to prove our Theorem 3.4 below, we will need techniques used by Pisier in the proof of Theorem 1.4. As in [**Pi1**], the following form of the Hahn-Banach theorem will prove useful.

Lemma 3.3. *Let $\wedge$ be a real vector space equipped with a cone $\wedge_+$. Let $\lambda : \wedge \to \mathbb{R}$ be sublinear and let $\mu : \wedge_+ \to \mathbb{R}_+$ be superlinear. Assume that $\mu \leq \lambda$ on $\wedge_+$. Then there is a positive linear form $f : \wedge \to \mathbb{R}$ such that $\mu \leq f$ on $\wedge_+$ and $f \leq \lambda$ on $\wedge$.*

We are now ready to prove the main result of this section.

Theorem 3.4. *Let X_1, X_2, Y_1, Y_2 be Banach spaces and let $T \subset B(X_1, Y_1)$, $Z \subset B(X_2, Y_2)$ be subspaces. Let S be a matrix normed space and $\sigma : Z \times S \times T \to \mathbb{C}$ be a trilinear map. Assume that S satisfies the condition $\mathcal{D}_\infty$ and that for any $z_1, \dots, z_m \in Z$, $s = [s_{ij}] \in M_m(S)$, $t_1, \dots, t_m \in T$:*

$$\left| \sum_{1 \leq i,j \leq m} \sigma(z_i, s_{ij}, t_j) \right| \leq \|s\| \; \|(z_1, \dots, z_m)\| \; \left\| \begin{pmatrix} t_1 \\ \vdots \\ t_m \end{pmatrix} \right\| . \tag{3.2}$$

Then there exist Banach spaces $E \in SQ_p(Y_1), F \in SQ_p(X_2)$ and three completely contractive maps $\varphi : S \to B(E, F)$, $u : T \to E_c$ and $v : Z \to F_r^*$ such that:

$$\forall \; (z, s, t) \in Z \times S \times T, \quad \sigma(z, s, t) = \langle \varphi(s) u(t), v(z) \rangle.$$

Proof. Let $\wedge$ be the set of all functions $\phi : X_1 \times Y_2^* \to \mathbb{R}$ for which there exist $\alpha > 0, \beta > 0$ such that

$$\forall \; (x_1, y_2^*) \in X_1 \times Y_2^*, \; |\phi(x_1, y_2^*)| \leq \alpha^p \|x_1\|^p + \beta^q \|y_2^*\|^q . \tag{3.3}$$

Then $\wedge$ is a real vector space and the subset $\wedge_+$ of non-negative functions in $\wedge$ is a cone. We will apply Lemma 3.3 in this space. For any $\phi \in \wedge$, we let:

$$\lambda(\phi) = \inf \left\{ \frac{\alpha^p}{p} + \frac{\beta^q}{q} \right\}$$

where the infimum runs over all $(\alpha, \beta) \in \mathbb{R}_+^{*2}$ such that (3.3) holds. This clearly defines a sublinear map $\lambda : \wedge \to \mathbb{R}$. For any $\phi \in \wedge_+$, we let:

$$\mu(\phi) = \sup \left\{ \sum_{1 \leq i,j \leq m} Re\, \sigma(z_i, s_{ij}, t_j) \right\}$$

where the supremum runs over all $m \geq 1$, $z_1, \dots, z_m \in Z$, $s = [s_{ij}] \in M_m(S)$, $t_1, \dots, t_m \in T$ such that $\|s\| \leq 1$ and

$$\forall \; (x_1, y_2^*) \in X_1 \times Y_2^*, \; \phi(x_1, y_2^*) \geq \sum_{j=1}^m \|t_j(x_1)\|^p + \sum_{i=1}^m \|z_i^*(y_2^*)\|^q .$$

We claim that μ is superadditive on $\wedge_+$. To check this, consider $\phi, \phi' \in \wedge_+, (z_i)_{1\le i\le n}, (z'_i)_{1\le i\le m}$ in Z, $(t_j)_{1\le j\le n}, (t'_j)_{1\le j\le m}$ in T such that for any $(x_1, y_2^*) \in X_1 \times Y_2^*$:

$$\phi(x_1,y_2^*) \ge \sum_{j=1}^{n} \|t_j(x_1)\|^p + \sum_{i=1}^{n} \|z_i^*(y_2^*)\|^q \qquad \text{and}$$

$$\phi'(x_1,y_2^*) \ge \sum_{j=1}^{m} \left\|t'_j(x_1)\right\|^p + \sum_{i=1}^{m} \left\|z_i'^*(y_2^*)\right\|^q .$$

Then letting

$$(z''_i)_{i\le n+m} = (z_1,\dots,z_n,z'_1,\dots,z'_m)$$

and

$$(t''_j)_{j\le n+m} = (t_1,\dots,t_n,t'_1,\dots,t'_m),$$

we obtain for all x_1, y_2^* :

$$(\phi+\phi')(x_1,y_2^*) \ge \sum_{j=1}^{n+m} \left\|t''_j(x_1)\right\|^p + \sum_{i=1}^{n+m} \left\|z_i''^*(y_2^*)\right\|^q .$$

Now the point is that if we consider $s = [s_{ij}] \in M_n(S)$ and $s' = [s'_{ij}] \in M_m(S)$ with norms less than one and let $s'' = s \oplus s' = [s''_{ij}] \in M_{n+m}(S)$, we have $\|s''\| \le 1$ (by our assumption on S) and

$$\sum_{i,j} Re\,\sigma(z''_i, s''_{ij}, t''_j) = \sum_{i,j} Re\,\sigma(z_i, s_{ij}, t_j) + \sum_{i,j} Re\,\sigma(z'_i, s'_{ij}, t'_j).$$

We thus obtain $\mu(\phi+\phi') \ge \mu(\phi)+\mu(\phi')$ as claimed above. Hence $\mu : \wedge_+ \to \mathbb{R}$ is a non-negative superlinear map. Let us now prove that:

$$\forall\, \phi \in \wedge_+, \quad \mu(\phi) \le \lambda(\phi). \tag{3.4}$$

We give ourselves $(\alpha,\beta) \in \mathbb{R}_+^{*2}$, $(z_i)_{1\le i\le m}$ in Z and $(t_j)_{1\le j\le m}$ in T such that for any $(x_1, y_2^*) \in X_1 \times Y_2^*$:

$$\sum_j \|t_j(x_1)\|^p + \sum_i \|z_i^*(y_2^*)\|^q \le \phi(x_1,y_2^*) \le \alpha^p \|x_1\|^p + \beta^q \|y_2^*\|^q .$$

Then $\left\| \begin{pmatrix} t_1 \\ \vdots \\ t_m \end{pmatrix} \right\| \le \alpha$ and $\|(z_1,\dots,z_m)\| \le \beta$ by (2.5) and (2.6). Hence for any $s = [s_{ij}] \in M_m(S)$ of norm less than one, we have by (3.2):

$$\left| \sum_{i,j} \sigma(z_i, s_{ij}, t_j) \right| \le \alpha\beta.$$

Therefore $\sum_{i,j} Re\,\sigma(z_i, s_{ij}, t_j) \leq \dfrac{\alpha^p}{p} + \dfrac{\beta^q}{q}$ whence (3.4).

By Lemma 3.3, we thus obtain a positive linear form $f : \wedge \to \mathbb{R}$ such that:

$$\forall\, \phi \in \wedge,\ f(\phi) \leq \lambda(\phi) \tag{3.5}$$

$$\forall\, \phi \in \wedge_+,\ \mu(\phi) \leq f(\phi). \tag{3.6}$$

We now come to the definitions of E, F, u, v. We proceed with similar constructions as in [**Pi1**].

Let $\mathcal{G}_1$ be the set of all the functions $\psi : X_1 \to Y_1$ for which there exists $\alpha > 0$ such that for any $x_1 \in X_1$, $\|\psi(x_1)\| \leq \alpha\,\|x_1\|$. Clearly $\mathcal{G}_1$ is a complex vector space. Moreover, for any $\psi \in \mathcal{G}_1$, the function $\widetilde{\psi} : X_1 \times Y_2^* \to \mathbb{R}$ defined by $\widetilde{\psi}(x_1, y_2^*) = \|\psi(x_1)\|^p$ belongs to $\wedge$, hence we may define:

$$N_1(\psi) = f(\widetilde{\psi})^{1/p}.$$

The function N_1 is a semi-norm on $\mathcal{G}_1$. We denote by G_1 the Banach space obtained after passing to the quotient by the kernel of N_1 and completing the resulting normed space.

For any $t \in T$, let us denote by $\psi_t \in \mathcal{G}_1$ the function defined by $\psi_t(x_1) = t(x_1)$. We may define a linear map $u : T \to G_1$ by setting (up to equivalence classes): $u(t) = p^{1/p}\psi_t$.

Let us regard u as a map from T into $(G_1)_c$. Then $\|u\|_{cb} \leq 1$. Indeed for any finite family $(t_1, \ldots, t_n)$ in T:

$$\begin{aligned}
\tfrac{1}{p}\textstyle\sum_{j=1}^n \|u(t_j)\|^p &= \textstyle\sum_{j=1}^n N_1(\psi_{t_j})^p = f\left(\sum_{j=1}^n \widetilde{\psi}_{t_j}\right)\\
&\leq \left\| \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} \right\|^p f\left((x_1, y_2^*) \mapsto \|x_1\|^p\right) \quad \text{by} \quad (2.5)\\
&\leq \left\| \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} \right\|^p \lambda\left((x_1, y_2^*) \mapsto \|x_1\|^p\right) \quad \text{by} \quad (3.5)\\
&\leq \tfrac{1}{p} \left\| \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} \right\|^p .
\end{aligned}$$

Hence the result follows from Lemma 2.6 (a).

In the same manner, we can introduce the vector space $\mathcal{G}_2$ of all the functions $\psi : Y_2^* \to X_2^*$ for which there exists $\beta > 0$ such that for any $y_2^* \in$

Y_2^*, $\|\psi(y_2^*)\| \leq \beta \|y_2^*\|$. Letting $\widetilde{\psi}(x_1, y_2^*) = \|\psi(y_2^*)\|^q$ and $N_2(\psi) = f(\widetilde{\psi})^{1/q}$, we can similarly define a Banach space G_2 from $(\mathcal{G}_2, N_2)$. We then define a map $v : Z \to G_2$ by letting (up to equivalence classes) $v(z)(y_2^*) = q^{1/q} z^*(y_2^*)$. Then using (2.6) and Lemma 2.6 (b), we obtain that $v : Z \to (G_2^*)_r^*$ satisfies $\|v\|_{cb} \leq 1$.

Finally, we set $E = \overline{u(T)}, F = v(Z)^*$ and can consider that we actually have $u : T \to E_c$ and $v : Z \to F_r^*$ with $\|u\|_{cb} \leq 1$ and $\|v\|_{cb} \leq 1$.

In view of Theorem 3.2, we clearly have $G_1 \in SQ_p(Y_1)$ and therefore $E \in SQ_p(Y_1)$. Similarly, we obtain that $G_2 \in SQ_q(X_2^*)$. Thus by a simple duality argument, we deduce that $F \in SQ_p(X_2)$.

In order to complete the proof of Theorem 3.4, it remains to show that for any $z_1, \dots, z_m \in Z, s = [s_{ij}] \in M_m(S)$, $t_1, \dots, t_m \in T$, we have:

$$\left| \sum_{i,j} \sigma(z_i, s_{ij}, t_j) \right| \leq \|s\| \left(\sum_j \|u(t_j)\|^p \right)^{1/p} \left(\sum_i \|v(z_i)\|^q \right)^{1/q}. \tag{3.7}$$

Indeed, such an inequality allows to define $\varphi : S \to B(E, F)$ by letting $\langle \varphi(s)u(t), v(z) \rangle = \sigma(z, s, t)$ and proves that φ is c.c.. Let us now check (3.7). By trivial scaling, we may assume that $\|s\| = 1$ and $\sum\limits_{i,j} \sigma(z_i, s_{ij}, t_j) \in \mathbb{R}^+$. We define $\phi \in \wedge_+$ by setting $\phi(x_1, y_2^*) = \sum\limits_j \|t_j(x_1)\|^p + \sum\limits_i \|z_i^*(y_2^*)\|^q$. Then we have:

$$\begin{aligned} \left| \sum_{i,j} \sigma(z_i, s_{ij}, t_j) \right| &\leq \mu(\phi) \leq f(\phi) \quad \text{by} \quad (3.6) \\ &\leq \frac{1}{p} \sum_j \|u(t_j)\|^p + \frac{1}{q} \sum_i \|v(z_i)\|^q. \end{aligned}$$

Since we have $\sum\limits_{i,j} \sigma(z_i, s_{ij}, t_j) = \sum\limits_{i,j} \sigma(\theta^{-1} z_i, s_{ij}, \theta t_j)$ for any $\theta > 0$, the preceding inequality implies for all $\theta > 0$:

$$\left| \sum_{i,j} \sigma(z_i, s_{ij}, t_j) \right| \leq \frac{\theta^p}{p} \sum_j \|u(t_j)\|^p + \frac{\theta^{-q}}{q} \sum_i \|v(z_i)\|^q.$$

From (2.7), we deduce that (3.7) holds. □

4. A generalization of Ruan's representation theorem.

Let X and Y be Banach spaces and let $S \subset B(X, Y)$ be a subspace. For any $n, m \geq 1$, we may define (unambigously) a p-matrix structure on $M_{n,m}(S)$

by letting $M_{k,l}(M_{n,m}(S)) = M_{kn,lm}(S)$ for all $k, l \geq 1$. In other words, this p-matrix structure is given by the canonical embedding $M_{n,m}(S) \subset B(\ell_p^m(X), \ell_p^n(Y))$. Now let X be a Banach space and let $n \geq 1$ be an integer. Recall the definition (2.11). We set :

$$R_n^X = M_{1,n}(\mathbb{C}^X). \tag{4.1}$$

Actually, R_n^X is a p-matrix structure on the Banach space ℓ_q^n.

Indeed for any $t = (t(\ell))_{1\leq \ell \leq n} \in \ell_q^n$, let $\widehat{t} : \ell_p^n(X) \to X$ be defined by $\widehat{t}((x_\ell)_{\ell\leq n}) = \sum_{\ell=1}^{n} t(\ell)x_\ell$. Then $\left\|\widehat{t}\right\| = \left(\sum_{\ell=1}^{n} |t(\ell)|^q\right)^{1/q} = \|t\|$ and we clearly have:

$$R_n^X = \left\{\widehat{t} \ / \ t \in \ell_q^n\right\} \subset B(\ell_p^n(X), X).$$

In the same manner, given a Banach space Y, we set for any $n \geq 1$:

$$C_n^Y = M_{n,1}(\mathbb{C}^Y). \tag{4.2}$$

C_n^Y is a p-matrix structure on ℓ_p^n. For any $z = (z(k))_{1\leq k\leq n} \in \ell_p^n$, we may let $\widehat{z}(y) = (z(k)y)_{k\leq n} \in \ell_p^n(Y)$ for all $y \in Y$ and:

$$C_n^Y = \left\{\widehat{z} \ / \ z \in \ell_p^n\right\} \subset B(Y, \ell_p^n(Y)).$$

The spaces R_n^X and C_n^Y will be used in the proof of Proposition 4.2.

The following is the main result of this section:

Theorem 4.1. *Let X, Y be Banach spaces and let S be a matrix normed space. The following assertions are equivalent:*

(i) *S satisfies the two following conditions:*

$\mathcal{D}_\infty$: *For any $s \in M_{n,m}(S)$, $s' \in M_{n',m'}(S)$,*

$$\|s \oplus s'\| = \max\ \{\|s\|, \|s'\|\}.$$

$\mathcal{M}_{p,Y,X}$: *For any $a \in M_{n,m}$, $s \in M_m(S), b \in M_{m,n}$,*

$$\|asb\| \leq \|a\|_{p,Y}\ \|s\|\ \|b\|_{p,X}.$$

(ii) *There exist Banach spaces $E \in SQ_p(X), F \in SQ_p(Y)$ and a completely isometric map $J : S \to B(E, F)$.*

This statement will allow us to consider any matrix normed space S which satisfy $\mathcal{D}_\infty$ and $\mathcal{M}_{p,Y,X}$ as a subspace of $B(E, F)$ for some suitable Banach spaces E, F. In the particular case when $p = 2$ and $X = Y = \mathbb{C}$, we recover

Ruan's representation theorem [**R**, **ER3**]. However for $1 < p \neq 2 < +\infty$, the particular case $X = Y = \mathbb{C}$ is already new. We will come back to this in the last Section 6. We do not know whether Theorem 4.1 can be extended to the case $p = 1$. Before coming into the proof of Theorem 4.1, note that a matrix normed space S satisfying the condition (i) above for some Banach spaces X and Y is obviously a p-matrix normed space as defined in Section 2. Although we could not find any convincing example, it seems unlikely that the converse is true. The problem arising here is the following: given a p-matrix normed space S, does there exist a couple of Banach spaces X and Y for which $\mathcal{M}_{p,Y,X}$ holds ?

In order to prove Theorem 4.1, we will follow the approach of [**ER3**]. More precisely, we will deduce the non-trivial implication (i) $\Rightarrow$ (ii) from a convenient factorization of the linear forms $\xi \in M_n(S)^*$.

Proposition 4.2. *Let S be a matrix normed space satisfying the assumptions $\mathcal{D}_\infty$ and $\mathcal{M}_{p,Y,X}$. Let $n \geq 1$ and $\xi \in M_n(S)^*$ with $\|\xi\| = 1$.*

Then there exist Banach spaces $E \in SQ_p(X), F \in SQ_p(Y)$ and a completely contractive map $\varphi : S \to B(E,F)$ such that: $\forall\ s \in M_n(S),\ |\xi(s)| \leq \left\|\varphi^{(n)}(s)\right\|$.

Proof. We denote by $T = R_n^X$ and $Z = C_n^Y$ the two p-matrix normed spaces defined in (4.1) and (4.2). Fix $\xi \in M_n(S)^*$ with $\|\xi\| = 1$.

Given $z = (z(k))_{k \leq n} \in Z$ and $t = (t(\ell))_{\ell \leq n} \in T$, we denote by $zt \in M_n$ the matrix obtained by the product of the column matrix $\begin{pmatrix} z(1) \\ \vdots \\ z(n) \end{pmatrix}$ with the row matrix $(t(1), \ldots, t(n))$. Namely, we have $zt = [z(k)t(\ell)]$. With the above notation, we define $\sigma = Z \times S \times T \to \mathbb{C}$ by letting $\sigma(z,s,t) = \xi(zt \otimes s)$.

We claim that σ satisfies the assumption (3.2) of Theorem 3.4. In order to show that, consider $z_1, \ldots, z_m \in Z,\ t_1, \ldots, t_m \in T$ and $s = [s_{ij}] \in M_m(S)$. Let $a = [a_{ki}] \in M_{n,m}$ and $b = [b_{j\ell}] \in M_{m,n}$ be defined by $a_{ki} = z_i(k)$ and $b_{j\ell} = t_j(\ell)$. Clearly we have $\sum_{i,j} \sigma(z_i, s_{ij}, t_j) = \xi(asb)$ hence $\left|\sum_{i,j} \sigma(z_i, s_{ij}, t_j)\right| \leq \|asb\|$. Note that from the definitions (4.1) and (4.2), we have $\|a\|_{p,Y} = \|(z_1, \ldots, z_m)\|$ and $\|b\|_{p,X} = \left\|\begin{pmatrix} t_1 \\ \vdots \\ t_m \end{pmatrix}\right\|$. Therefore, the assumption $\mathcal{M}_{p,Y,X}$ implies that (3.2) holds.

Moreover we assumed that S satisfies $\mathcal{D}_\infty$. Hence we may apply Theorem 3.4 to the trilinear map σ and this yields two Banach spaces $E \in$

$SQ_p(X), F \in SQ_p(Y)$ and three c.c maps $u : T \to E_c, v : Z \to F_r^*$ and $\varphi : S \to B(E, F)$ such that $\sigma(z, s, t) = \langle \varphi(s)u(t), v(z)\rangle$ for all $(z, s, t) \in Z \times S \times T$. Let us denote by $(\eta_j)_{1\leq j\leq n}$ and $(\nu_i)_{1\leq i\leq n}$ the canonical bases of T and Z respectively. Then for any $s = [s_{ij}] \in M_n(S)$,

$$\xi(s) = \sum_{1\leq i,j\leq n} \sigma(\nu_i, s_{ij}, \eta_j) = \sum_{i,j} \langle \varphi(s_{ij})u(\eta_j), v(\nu_i)\rangle$$

hence

$$|\xi(s)| \leq \left\|\varphi^{(n)}(s)\right\| \left(\sum_{j=1}^{n} \|u(\eta_j)\|^p\right)^{1/p} \left(\sum_{i=1}^{n} \|v(\nu_i)\|^q\right)^{1/q}.$$

Now $\sum_{j=1}^{n} \|u(\eta_j)\|^p \leq 1$ and $\sum_{i=1}^{n} \|v(\nu_i)\|^q \leq 1$ by Lemma 2.6. Hence $|\xi(s)| \leq \left\|\varphi^{(n)}(s)\right\|$. This achieves the proof. □

Proof of Theorem 4.1. We assume (i). Let I_n be the unit sphere of $M_n(S)^*$ and let $I = \bigcup_{n\geq 1} I_n$. For any $\xi \in I_n$, we may apply Proposition 4.2 and thus obtain $E_\xi \in SQ_p(X), F_\xi \in SQ_p(Y)$ and a c.c. map $\varphi_\xi : S \to B(E_\xi, F_\xi)$ such that for any $s \in M_n(S)$, $|\xi(s)| \leq \left\|\varphi_\xi^{(n)}(s)\right\|$.

Let $E = \overset{p}{\underset{\xi\in I}{\oplus}} E_\xi$ and $F = \overset{p}{\underset{\xi\in I}{\oplus}} F_\xi$. Of course we have $E \in SQ_p(X), F \in SQ_p(Y)$. We now define $J : S \to B(E, F)$ by setting

$$J(s)\big((e_\xi)_{\xi\in I}\big) = \big((\varphi_\xi(s)(e_\xi)\big)_{\xi\in I}\big).$$

Since each $J(s)$ acts diagonally we have for any $s \in M_n(S)$:

$$\left\|J^{(n)}(s)\right\| = \sup_{\xi\in I} \left\|\varphi_\xi^{(n)}(s)\right\|.$$

Therefore, J is a completely isometric map. This proves (i) $\Rightarrow$ (ii). The converse implication is obvious. □

Remark 4.3. Let S be a p-matrix normed space. Note that S satisfies $\mathcal{D}_\infty$. Therefore an obvious reformulation of Theorem 4.1 is that the two following are equivalent:

(i) The canonical identification $\mathbb{C}^Y \otimes_h S \otimes_h \mathbb{C}^X = S$ is completely isometric.

(ii) There exist Banach spaces $E \in SQ_p(X), F \in SQ_p(Y)$ and a completely isometric embedding $S \subset B(E, F)$. This complements Remark 2.8 (b).

Remark 4.4. It is not hard to modify the proofs of Proposition 4.2 and Theorem 3.4 in order to settle an isomorphic variant of Theorem 4.1. Consider the three following properties depending on some constants C_1, C_2, C_3.
(a) For any $s_1 \in M_{n_1,m_1}(S_1), s_2 \in M_{n_2,m_2}(S_2), \dots , s_k \in M_{n_k,m_k}(S_k)$,

$$\|s_1 \oplus \dots \oplus s_k\| \leq C_1 \max \{\|s_1\|, \dots , \|s_k\|\}.$$

(b) For any $a \in M_{n,m}, s \in M_m(S), b \in M_{m,n}$,

$$\|asb\| \leq C_2 \|a\|_{p,Y} \|s\| \|b\|_{p,X}.$$

(c) There exist Banach spaces $E \in SQ_p(X), F \in SQ_p(Y)$ and a complete C_3-isomorphic embedding $J : S \to B(E, F)$.

Then, the assertion (c) implies that (a) and (b) hold with $C_1 = C_2 = C_3$. The converse (and more significant result) is that if (a) and (b) hold, then condition (c) is fulfilled with $C_3 = C_1C_2$.

5. Representation of p-completely bounded multilinear maps.

In this section we show how to deduce a representation theorem for p-c.b. multilinear maps from our previous work. We will give two formulations of this result. Here is the first one:

Theorem 5.1. *Let $X_1, \dots , X_N, Y_1, \dots , Y_N, X, Y$ be Banach spaces. For each $1 \leq i \leq N$, let $S_i \subset B(X_i, Y_i)$ be a subspace. Let $S = S_N \otimes_h \cdots \otimes_h S_1$ (see Remark 2.5 for the definition) and let $A : S \to B(X, Y)$ be a c.b. map.*
Then there are Banach spaces K_i $(1 \leq i \leq N-1)$ and c.b. maps $A_1 : S_1 \to B(X, K_1)$, $A_j : S_j \to B(K_{j-1}, K_j)$ $(2 \leq j \leq N-1)$, $A_N : S_N \to B(K_{N-1}, Y)$ such that

$$\|A_1\|_{cb} \cdots \|A_N\|_{cb} \leq \|A\|_{cb}$$

and:

$$\forall\ (s_N, \dots , s_1) \in S_N \times \cdots \times S_1,$$
$$A(s_N, \dots , s_1) = A_N(s_N) \circ \cdots \circ A_2(s_2) \circ A_1(s_1).$$

The proof of Theorem 5.1 will rely upon two lemmas which are now simple corollaries of Section 3 and 4.

Lemma 5.2. *Let X_1, Y_1, X_2, Y_2 be Banach spaces and let $T \subset B(X_1, Y_1)$ and $Z \subset B(X_2, Y_2)$ be subspaces. Then there are Banach spaces $E \in SQ_p(X_1)$, $F \in SQ_p(Y_2)$ and a completely isometric map $J : Z \otimes_h T \to B(E, F)$.*

Proof. Let $z \in M_{m,k}(Z), t \in M_{k,m}(T), a \in M_{n,m}, b \in M_{m,n}$. Then $a(z \odot t)b = az \odot tb$ and $\|az\| \leq \|a\|_{p,Y_2} \|z\|$, $\|tb\| \leq \|t\| \|b\|_{p,X_1}$. Hence we may apply Theorem 4.1 with $S = Z \otimes_h T$, $X = X_1$, $Y = Y_2$. □

Lemma 5.3. *The statement of Theorem* 5.1 *holds in the case* $N = 2$, $X = Y = \mathbb{C}$.

Proof. We consider a c.c. map $A : S_2 \otimes_h S_1 \to \mathbb{C}$. Let $S = \mathbb{C}^{Y_1} \subset B(Y_1, Y_1)$ and let $\sigma : S_2 \times S \times S_1 \to \mathbb{C}$ be defined by $\sigma(s_2, I_{Y_1}, s_1) = A(s_2, s_1)$. Then we may clearly apply Theorem 3.4 with $T = S_1$ and $Z = S_2$ and this yields the result. □

Proof of Theorem 5.1. We follow the approach of [**B**, Theorem 2.4]. Since Lemma 5.2 allows us to use induction, we only need to consider the case $N = 2$. We thus consider a c.c. map $A : S_2 \otimes_h S_1 \to B(X, Y)$. Let us define $\widetilde{A} : (Y_r^* \otimes_h S_2) \otimes_h (S_1 \otimes_h X_c) \to \mathbb{C}$ by setting:
(5.1)
$$\forall\ (y^*, s_2, s_1, x) \in Y^* \times S_2 \times S_1 \times X, \quad \widetilde{A}(y^* \otimes s_2, s_1 \otimes x) = \langle A(s_2, s_1)x, y^* \rangle.$$

From the associativity of $\otimes_h$ (see Remark 2.5) and Lemma 2.7, we have $\left\|\widetilde{A}\right\|_{cb} \leq 1$. Apply Lemma 5.2 to $S_1 \otimes_h X_c$ and $Y_r^* \otimes_h S_2$ together with Lemma 5.3. This yields a Banach space K and two completely contractive maps $\widetilde{A_1} : S_1 \otimes_h X_c \to K_c$ and $\widetilde{A_2} : Y_r^* \otimes_h S_2 \to K_r^*$ such that:

$$\forall\ z \in Y_r^* \otimes_h S_2, \forall\ t \in S_1 \otimes_h X_c, \quad \widetilde{A}(z, t) = \langle \widetilde{A}_1(t), \widetilde{A}_2(z) \rangle. \tag{5.2}$$

We now proceed with converse identifications. We define $A_1 : S_1 \to B(X, K)$ and $A_2 : S_2 \to B(K, Y^{**})$ by setting

$$\forall\ (s_1, x) \in S_1 \times X, \quad A_1(s_1)(x) = \widetilde{A_1}(s_1 \otimes x). \tag{5.3}$$

$$\forall\ (s_2, y^*) \in S_2 \times Y^*, \quad (A_2(s_2))^*(y^*) = \widetilde{A_2}(y^* \otimes s_2). \tag{5.4}$$

Clearly, (5.1), (5.2), (5.3), (5.4) imply that for any $(s_2, s_1) \in S_2 \times S_1$,

$$A(s_2, s_1) = A_2(s_2) \circ A_1(s_1).$$

Now it is easy to see that we may as well assume that $K = \overline{A_1(S_1)(X)}$ and then, A_2 is actually a c.c. map from S_2 into $B(K, Y)$. This concludes the proof. □

Remark 5.4. The converse of Theorem 5.1 obviously holds. Namely, given c.c. maps $A_1 : S_1 \to B(X, K_1)$, $A_N : S_N \to B(K_{N-1}, Y)$ and $A_j : S_j \to B(K_{j-1}, K_j)$ $(2 \leq j \leq N-1)$, the map $A : S_N \times \cdots \times S_1 \to B(X, Y)$ defined by $A(s_N, \ldots, s_1) = A_N(s_N) \circ \cdots \circ A_1(s_1)$ provides a c.c. map from $S_N \otimes_h \cdots \otimes_h S_1$ into $B(X, Y)$.

Remark 5.5. In view of Lemma 5.2, we could have been more precise in the statement of Theorem 5.1. For example we may write that for any $1 \leq j \leq N-1$, $K_j \in SQ_p(Y_j)$. However, we shall see in Theorem 5.6 that such an information is not really an improvement.

We now turn back to the terminology of p-completely bounded maps defined in the introduction (see Definition 1.1). Recall that given $S \subset B(X_1, Y_1)$ and two Banach spaces $G \in SQ_p(X_1), G' \in SQ_p(Y_1)$, it made sense to define a notion of p-representation from S into $B(G, G')$ (see Definition 1.3).

Then by an obvious combination of Theorem 5.1, Remark 5.4, Remark 2.5 and Theorem 1.4, we obtain:

Theorem 5.6. *Let $X_1, \ldots, X_N, Y_1, \ldots, Y_N, X, Y$ be Banach spaces. For each $1 \leq i \leq N$, let $S_i \subset B(X_i, Y_i)$ be a subspace. Let $A : S_N \times \cdots \times S_1 \to B(X, Y)$ be a N-linear map and let C be a constant. The following assertions are equivalent:*

(i) *A is p-completely bounded and $\|A\|_{pcb} \leq C$.*

(ii) *There exist Banach spaces*

$$G_j \in SQ_p(X_j)(1 \leq j \leq N), G'_j \in SQ_p(Y_j)(1 \leq j \leq N),$$

p-representations $\pi_j : S_j \to B(G_j, G'_j)$ $(1 \leq j \leq N)$ and operators $V_0 : X \to G_1, V_N : G'_N \to Y$ and $V_j : G'_j \to G_{j+1}$ $(1 \leq j \leq N-1)$ such that $\|V_0\| \ldots \|V_N\| \leq C$ and $\forall\ (s_N, \ldots, s_1) \in S_N \times \cdots \times S_1$,

$$A(s_N, \ldots, s_1) = V_N \pi_N(s_N) V_{N-1} \ldots V_2 \pi_2(s_2) V_1 \pi_1(s_1) V_0.$$

6. Complements.

6.1. Some remarks about $\otimes_h$. The Haagerup tensor product of operator spaces has been extensively studied recently (see [**B**, **BP**, **BS**, **ER2**, **PS**]). A main feature of this tensor product is that it is both injective and projective in the category of operator spaces. It is then natural to study similar properties in our more general framework. We will easily obtain that our tensor product $\otimes_h$ is projective and is not injective. Let us make these statements precise.

Let S be a matrix normed space and let $T \subset S$ be a closed subspace. We may define a norm on each $M_{n,m}\left(\frac{S}{T}\right)$ by setting $M_{n,m}\left(\frac{S}{T}\right) = \frac{M_{n,m}(S)}{M_{n,m}(T)}$. Endowed with these norms, $\frac{S}{T}$ becomes a matrix normed space. Moreover, if we assume that S is a p-matrix normed space, then $\frac{S}{T}$ is also a p-matrix normed space.

The announced surjectivity of $\otimes_h$ is:

Proposition 6.1. *Let S_1, S_2 be two p-matrix normed spaces. For $i = 1, 2$, let $T_i \subset S_i$ be a closed subspace and let $q_i : S_i \to \frac{S_i}{T_i}$ be the associated quotient map. Consider $Q = q_2 \otimes q_1 : S_2 \otimes_h S_1 \to \frac{S_2}{T_2} \otimes_h \frac{S_1}{T_1}$.*

Then Q is a complete quotient map, i.e. for any $n \geq 1$, $Q^{(n)}$ is a quotient map.

Proof. Mimic the proof of [**ER2**, Proposition 3.1]. $\square$

Remark 6.2. The tensor product $\otimes_h$ is not injective. Indeed let E, F, G be Banach spaces such that $E \subset F$. Let $j : G_r^* \otimes_h E_c \to G_r^* \otimes_h F_c$ be the canonical embedding. We wish to prove that j is not isometric in general. Assume for simplicity that G is reflexive. Then $(G_r^* \otimes_h E_c)^* = B(E, G)$, $(G_r^* \otimes_h F_c)^* = B(F, G)$ and $j^* : B(F, G) \to B(E, G)$ is the restriction map. Therefore, j^* is onto if and only if any bounded linear map from E into G has a bounded linear extension to F. This fails in general and then, j is not even isomorphic in general.

We now fix two Banach spaces X, Y. Let us denote by $\mathcal{C}_{X,Y}$ the class of all p-matrix normed spaces S defined by a completely isometric embedding $S \subset B(E, F)$ for some $E \in SQ_p(X)$ and $F \in SQ_p(Y)$. Note for further the following straightforward consequence of our Theorem 4.1:

$$\frac{S}{T} \in \mathcal{C}_{X,Y} \quad \text{whenever} \quad S \in \mathcal{C}_{X,Y}. \tag{6.1}$$

The end of this subsection is devoted to a convenient identification result about $\mathcal{C}_{X,Y}$. Let S be a p-matrix normed space. Recall from Section 4 that given $z \in C_n^Y$ and $t \in R_m^X$, we may define $zt \in M_{n,m}$ as a matrix product. Thus we can introduce a canonical map

$$J : C_n^Y \otimes_h S \otimes_h R_m^X \to M_{n,m}(S)$$

by letting $J(z \otimes s \otimes t) = zt \otimes s$.

Proposition 6.3. *Assume that $S \in \mathcal{C}_{X,Y}$. Then the above map J induces a completely isometric identification*

$$C_n^Y \otimes_h S \otimes_h R_m^X = M_{n,m}(S). \tag{6.2}$$

Proof. 1*st step.* Under our assumption, it is clear from the proof of Proposition 4.2 that the map J is isometric (see also Remark 4.3).

2nd step. We claim that for any $k, N, n \geq, 1$, we have canonical isometric identifications :

$$M_{1,k}\left(C_N^Y \otimes_h C_n^Y\right) = M_{1,k}\left(C_{Nn}^Y\right) \tag{6.3}$$

$$M_{k,1}\left(R_n^X \otimes_h R_N^X\right) = M_{k,1}\left(R_{nN}^X\right). \tag{6.4}$$

Let us check (6.3). We have the following isometric identifications

$$\begin{aligned} M_{1,k}\left(C_N^Y \otimes_h C_n^Y\right) &= C_N^Y \otimes_h C_n^Y \otimes_h R_k^Y \quad \text{by the first step} \\ &= M_{N,k}\left(C_n^Y\right) \quad \text{by the first step} \\ &= M_{1,k}\left(C_{nN}^Y\right) \quad \text{by (4.2)} \end{aligned}$$

whence (6.3). The proof of (6.4) is similar.

3rd step. We now prove that (6.2) is indeed a completely isometric identification. Fix $N \geq 1$. Then we have (isometrically):

$$\begin{aligned} M_N\left(C_n^Y \otimes_h S \otimes_h R_m^X\right) &= C_N^Y \otimes_h C_n^Y \otimes_h S \otimes_h R_m^X \otimes_h R_N^X \quad \text{by the first step} \\ &= C_{Nn}^Y \otimes_h S \otimes_h R_{mN}^X \quad \text{by (6.3) and (6.4)} \\ &= M_{Nn,Nm}(S) \quad \text{by the first step} \end{aligned}$$

and thus $M_N\left(C_n^Y \otimes_h S \otimes_h R_m^X\right) = M_N(M_{n,m}(S))$. □

6.2. Multilinear Schur products on $B(\ell_p^n)$. Although Schur products have been studied for a long time (see [**Gr**, **Be**]), Haagerup [**Ha**] was the first to realize the link between Schur products and the theory of completely bounded maps. Namely he proved that for any Schur product map $\phi : B(\ell_2^n) \to B(\ell_2^n)$, we have $\|\phi\| = \|\phi\|_{cb}$. This approach was lately exploited in [**PPS**]. We refer to this paper for further information. Recently, Effros and Ruan [**ER4**] proved that multilinear Schur products may be naturally defined on $B(\ell_2^n)$ and that their c.b. norms may be easily computed from the Christensen-Sinclair theorem. Moreover, it is not hard to deduce from [**S**] that for such a multilinear Schur product map $\phi : B(\ell_n^2) \times \cdots \times B(\ell_n^2) \to B(\ell_n^2)$, we have $\|\phi\| = \|\phi\|_{cb}$ as in the linear case. In this last subsection, we will indicate how to generalize all these results to multilinear Schur products on $B(\ell_p^n)$.

In the sequel, we will simply denote by R_n and C_n the p-matrix normed spaces $R_n^{\mathbb{C}}$ and $C_n^{\mathbb{C}}$ defined by (4.1) and (4.2). Similarly, the notation SQ_p will stand for $SQ_p(\mathbb{C})$ and $\mathcal{C}$ will stand for $\mathcal{C}_{\mathbb{C},\mathbb{C}}$. Let $(\varepsilon_i)_{1\leq i\leq n}$ and $(\varepsilon'_j)_{1\leq j\leq n}$ be the canonical bases of R_n and C_n respectively. We set:

$$G_n = \text{Span }\left\{\varepsilon_i \otimes \varepsilon'_j \;/\; i \neq j\right\} \subset R_n \otimes_h C_n.$$

Recall that $(R_n \otimes_h C_n)^* = B(\ell_p^n)$ (see Lemma 2.7 for example). In this duality, $G_n^\perp$ is clearly identified with the space of diagonal operators on $B(\ell_p^n)$. Thus $G_n^\perp = \ell_\infty^n$ and therefore we have isometrically:

$$\ell_1^n = \frac{R_n \otimes_h C_n}{G_n}. \tag{6.5}$$

Now the quotient formula (6.5) defines a p-matrix structure on ℓ_1^n (see the Subsection 6.1). In the sequel we will always consider ℓ_1^n as the p-matrix normed space defined above. Note that from Lemma 5.2, we have $R_n \otimes_h C_n \in \mathcal{C}$. Thus by (6.1) we obtain that $\ell_1^n \in \mathcal{C}$. Note also that when $p = 2$, this space is nothing but Max (ℓ_1^n). Thus the following is not really surprising.

Lemma 6.4. *Let E, F be Banach spaces and let $A : \ell_1^n \to B(E,F)$ be a linear map. Assume that $E \in SQ_p$ and $F \in SQ_p$. Then we have $\|A\|_{cb} = \|A\|$.*

Proof. Let $(\eta_i)_{1\leq i\leq n}$ be the canonical basis of ℓ_1^n. For any $1 \leq i \leq n$, let $T_i = A(\eta_i) \in B(E,F)$. We define $\widetilde{A} : F_r^* \otimes_h R_n \otimes_h C_n \otimes_h E_c \to \mathbb{C}$ by setting:

$$\forall\ 1 \leq i,j \leq n,\ \ \widetilde{A}(f^*, \varepsilon_i, \varepsilon'_j, e) = \delta_{ij}\langle T_i(e), f^*\rangle.$$

By Lemma 2.7 and Proposition 6.1, we have $\left\|\widetilde{A}\right\| = \|A\|_{cb}$. Since $E, F \in SQ_p$, Proposition 6.3 implies that $C_n \otimes_h E_c = (\ell_p^n(E))_c$ and $F_r^* \otimes_h R_n = (\ell_p^n(F))_r^*$ completely isometrically. Thus by Lemma 2.7 again:

$$(F_r^* \otimes_h R_n \otimes_h C_n \otimes_h E_c)^* = M_n(B(E, F^{**})).$$

Under this identification, $\widetilde{A}$ becomes the diagonal matrix $\begin{pmatrix} T_1 & & \\ & \ddots & \\ & & T_n \end{pmatrix}$. Therefore $\left\|\widetilde{A}\right\| = \mathrm{Sup}_{i\leq n} \|T_i\|$. Since $\|A\| = \mathrm{Sup}_{i\leq n} \|T_i\|$, the result follows. □

We now turn to multilinear Schur products. Let $N \geq 1$ and let $n_0, \ldots, n_N$ be some fixed positive integers. We give ourselves a finite family of complex numbers $a = (a_{i_N,\ldots,i_0})_{\substack{0\leq j\leq N \\ 1\leq i_j\leq n_j}}$. Note that any $m(j) \in B\left(\ell_p^{n_{j-1}}, \ell_p^{n_j}\right)$ has a canonical matrix representation $m(j) = [m(j)_{i_j,i_{j-1}}]_{i_j,i_{j-1}}$ with respect to the canonical bases of $\ell_p^{n_{j-1}}$ and $\ell_p^{n_j}$. We define the N-linear Schur product

$$\Phi_a : B\left(\ell_p^{n_{N-1}}, \ell_p^{n_N}\right) \otimes_h \cdots \otimes_h B\left(\ell_p^{n_1}, \ell_p^{n_2}\right) \otimes_h B\left(\ell_p^{n_0}, \ell_p^{n_1}\right) \to B\left(\ell_p^{n_0}, \ell_p^{n_N}\right)$$

associated to a as follows. For any $1 \leq j \leq N$, let $m(j) = [m(j)_{i_j,i_{j-1}}]_{i_j,i_{j-1}} \in B\left(\ell_p^{n_{j-1}}, \ell_p^{n_j}\right)$. Then we set

$$\Phi_a(m(N), \ldots, m(1)) =$$
$$\left[\sum\nolimits_{\substack{1\leq j\leq N-1 \\ 1\leq i_j\leq n_j}} a_{i_N,\ldots,i_0}\ m(N)_{i_N,i_{N-1}} \ldots m(2)_{i_2,i_1}\ m(1)_{i_1,i_0}\right]_{i_N,i_0} \in B\left(\ell_p^{n_0},\ \ell_p^{n_N}\right).$$

We now introduce another map naturally asociated to a. For any $0 \leq j \leq N$, let us denote by $(\eta_{i_j})_{1 \leq i_j \leq n_j}$ the canonical basis of $\ell_1^{n_j}$. Then we define $\varphi_a : \ell_1^{n_N} \otimes_h \ldots \otimes_h \ell_1^{n_1} \otimes_h \ell_1^{n_0} \to \mathbb{C}$ by setting $\varphi_a (\eta_{i_N}, \ldots, \eta_{i_0}) = a_{i_N, \ldots, i_0}$. We are now ready to state our last result. We keep the notation above.

Theorem 6.5. *The following are equivalent.*
(i) $\|\Phi_a\| \leq 1$
(ii) $\|\Phi_a\|_{cb} \leq 1$
(iii) $\|\varphi_a\| \leq 1$
(iv) *There are Banach spaces $K_1, \ldots, K_N$ which are all in SQ_p and there are linear contractions $T_{i_0} : \mathbb{C} \to K_1$ $(1 \leq i_0 \leq n_0)$, $T_{i_j} : K_j \to K_{j+1} (1 \leq j \leq N-1, 1 \leq i_j \leq n_j)$, $T_{i_N} : K_N \to \mathbb{C}(1 \leq i_N \leq n_N)$ such that for all $i_0, \ldots, i_N$:*

$$a_{i_N, \ldots, i_0} = T_{i_N} \circ \cdots \circ T_{i_1} \circ T_{i_0}.$$

Proof. Recall that for any $0 \leq j \leq N$, the p-matrix normed space $\ell_1^{n_j}$ belongs to $\mathcal{C}$. Thus the equivalence (iii) $\iff$ (iv) follows from Theorem 5.1, Remarks 5.4, 5.5 and Lemma 6.4. Let us now check that (ii) $\iff$ (iii).

Let $S = B\left(\ell_p^{n_{N-1}}, \ell_p^{n_N}\right) \otimes_h \ldots \otimes_h B\left(\ell_p^{n_1}, \ell_p^{n_2}\right) \otimes_h B\left(\ell_p^{n_0}, \ell_p^{n_1}\right)$.

By Proposition 6.3, each $B\left(\ell_p^{n_{j-1}}, \ell_p^{n_j}\right)$ may be completely isometrically identified with $C_{n_j} \otimes_h R_{n_{j-1}}$. Thus by Lemma 2.7, this yields:

$$CB\left(S, B\left(\ell_p^{n_0}, \ell_p^{n_N}\right)\right) = \left(R_{n_N} \otimes_h C_{n_N} \otimes_h \cdots \otimes_h C_{n_1} \otimes_h R_{n_0} \otimes_h C_{n_0}\right)^*.$$

Now since $\otimes_h$ is projective (see Proposition 6.1), $(\ell_1^{n_N} \otimes_h \cdots \otimes_h \ell_1^{n_1} \otimes_h \ell_1^{n_0})^*$ may be viewed as a subspace of $\left(R_{n_N} \otimes_h C_{n_N} \otimes_h \cdots \otimes_h C_{n_0}\right)^*$. As a consequence, we obtain an isometric embedding $\rho : (\ell_1^{n_N} \otimes_h \cdots \otimes_h \ell_1^{n_0})^* \to CB\left(S, B\left(\ell_p^{n_0}, \ell_p^{n_N}\right)\right)$. Now it is not hard to see that the range of ρ is exactly the set of N-linear Schur products from S into $B\left(\ell_p^{n_0}, \ell_p^{n_N}\right)$ and that $\rho(\varphi_a) = \Phi_a$. This achieves the proof of (ii) $\iff$ (iii).

Since (ii) $\implies$ (i) is obvious, it remains to show that (i) $\implies$ (ii). We follow the approach of [**S**, Theorem 2.1]. First note that given $\beta \in B\left(\ell_p^{n_N}\right)$, $\alpha \in B\left(\ell_p^{n_0}\right)$ and $m(j) \in B\left(\ell_p^{n_{j-1}}, \ell_p^{n_j}\right)$ $(1 \leq j \leq N)$, we may set $\beta(m(N) \otimes \cdots \otimes m(1))\alpha = \beta m(N) \otimes \cdots \otimes m(1)\alpha$. By linearity this allows us to consider the product $\beta s \alpha$ for all $s \in S$. It is easy to check that for any $\alpha_1, \ldots, \alpha_m \in$

$B\left(\ell_p^{n_0}\right), \beta_1, \dots \beta_m \in B\left(\ell_p^{n_N}\right)$, $s = [s_{\ell k}] \in M_m(S)$:

$$\left\| \sum_{1 \leq \ell, k \leq m} \beta_\ell s_{\ell k} \alpha_k \right\| \leq \|s\| \left\| \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} \right\| \|(\beta_1, \dots, \beta_m)\| . \tag{6.6}$$

We now define $D_{n_0} \subset B\left(\ell_p^{n_0}\right)$ (resp. $D_{n_N} \subset B(\ell_p^{n_N})$) as the space of all the diagonal operators on $B\left(\ell_p^{n_0}\right)$ (resp. $B\left(\ell_p^{n_N}\right)$). A main feature of Schur products is that:

$$\forall\ (\beta, s, \alpha) \in D_{n_N} \times S \times D_{n_0}, \quad \Phi_a(\beta s \alpha) = \beta \Phi_a(s) \alpha. \tag{6.7}$$

We are now ready to show that $\|\Phi_a\|_{cb} \leq 1$. In order to achieve this, take $s = [s_{\ell k}] \in M_m(S)$ and $x_1, \dots, x_m \in \ell_p^{n_0}$, $y_1^*, \dots, y_m^* \in \left(\ell_p^{n_N}\right)^* = \ell_q^{n_N}$ such that $\|s\| \leq 1$ and

$$\sum_{k=1}^{m} \|x_k\|^p \leq 1 \text{ and } \sum_{\ell=1}^{m} \|y_\ell^*\|^q \leq 1. \tag{6.8}$$

We thus have to show that:

$$\left| \sum_{1 \leq \ell, k \leq m} \langle \Phi_a(s_{\ell k}) x_k, y_\ell^* \rangle \right| \leq 1. \tag{6.9}$$

For any $1 \leq \ell, k \leq m$, write $x_k = (x_k(i_0))_{1 \leq i_0 \leq n_0}$ and $y_\ell^* = (y_\ell^*(i_N))_{1 \leq i_N \leq n_N}$. We define $\widehat{x} \in \ell_p^{n_0}$ and $\widehat{y}^* \in \ell_q^{n_N}$ by letting $\widehat{x}(i_0) = \left(\sum_{k=1}^{m} |x_k(i_0)|^p\right)^{1/p}$ and $\widehat{y}^*(i_N) = \left(\sum_{\ell=1}^{m} |y_\ell^*(i_N)|^q\right)^{1/q}$. Thus (6.8) imply:

$$\|\widehat{x}\| \leq 1 \quad \text{and} \quad \|\widehat{y}^*\| \leq 1. \tag{6.10}$$

Now we define $\alpha_k \in D_{n_0}$ as follows. We set $\alpha_k(i_0) = \dfrac{x_k(i_0)}{\widehat{x}(i_0)}$ for any $1 \leq i_0 \leq n_0$ $\left(\text{with the usual convention } \frac{0}{0} = 0\right)$ and we let $\alpha_k = \begin{pmatrix} \alpha_k(1) & & \\ & \ddots & \\ & & \alpha_k(n_0) \end{pmatrix}$.

Similarly we define $\beta_\ell = \begin{pmatrix} \beta_\ell(1) & & \\ & \ddots & \\ & & \beta_\ell(n_N) \end{pmatrix} \in D_{n_N}$ by $\beta_\ell(i_N) = \dfrac{y_\ell^*(i_N)}{\widehat{y}^*(i_N)}$.

Obviously, we have for all $1 \le k, \ell \le m$: $x_k = \alpha_k(\widehat{x})$ and $y_\ell^* = \beta_\ell^*(\widehat{y}^*)$.
Hence we have:

$$\begin{aligned}\left|\textstyle\sum_{1\le\ell,k\le m}\langle\Phi_a(s_{\ell k})x_k, y_\ell^*\rangle\right| &= \left|\textstyle\sum_{\ell,k}\langle\Phi_a(s_{\ell k})\alpha_k(\widehat{x}), \beta_\ell^*(\widehat{y}^*)\rangle\right| \\ &= \left|\left\langle\Phi_a\left(\textstyle\sum_{\ell,k}\beta_\ell s_{\ell k}\alpha_k\right)\widehat{x}, \widehat{y}^*\right\rangle\right| \quad \text{by } (6.7) \\ &\le \|(\beta_1, \dots, \beta_m\| \left\|\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix}\right\| \quad \text{by } (6.6) \text{ and } (6.10).\end{aligned}$$

Clearly we have

$$\left\|\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_m\end{pmatrix}\right\| = \sup_{1\le i_0\le m}\left(\sum_{k=1}^m |\alpha_k(i_0)|^p\right)^{1/p}.$$

Hence we have

$$\left\|\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix}\right\| \le 1.$$

Similarly, $\|(\beta_1, \dots, \beta_m)\| \le 1$ and therefore, (6.9) follows. □

Remark 6.6. In the particular case $N = 1$, the previous factorization theorem can be refined as follows. We give ourselves a family $a = (a_{i,j})_{\substack{1\le i\le n\\ 1\le j\le m}}$ to which we associate a Schur product map $\Phi_a : B(\ell_p^m, \ell_p^n) \to B(\ell_p^m, \ell_p^n)$ as above as well as the linear map $u_a : \ell_1^m \to \ell_\infty^n$ of canonical matrix a. Then the following are equivalent:

(i) $\|\Phi_a\| \le 1$

(ii) The map u_a factors contractively through L_p-spaces, i.e. there exist a measure space (Ω, μ) and linear contractions $T_1 : \ell_1^m \to L_p(\Omega, \mu)$, $T_2 : L_p(\Omega, \mu) \to \ell_\infty^n$ such that $u_a = T_2 T_1$.

Indeed by Theorem 6.5, $\|\Phi_a\| \le 1$ if and only if u_a factors contractively through SQ_p-spaces. From the lifting property of ℓ_1 and the extension property of ℓ_∞, this is equivalent to (ii).

The (linear) result mentioned in this remark was learned to me by G. Pisier. It is stated in [**Pi2**, Chapter 5] where explanations on its origine are given.

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