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A TYPE OF UNIQUENESS FOR THE DIRICHLET PROBLEM ON A HALF-SPACE WITH CONTINUOUS DATA

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Dedicated to Professor F.-Y. Maeda on his 60th birthday

In this paper, we shall prove a property of the harmonic function H defined on a half-space T which is represented by the generalized Poisson integral with a slowly growing continuous function f on the boundary ∂T of T. Then we shall investigate the difference between H and more general harmonic functions having the same boundary value f on ∂T. These give a kind of positive answer to a question asked by Siegel.

1. Introduction.

Let $\mathbb{R}$ and $\mathbb{R}_+$ be the sets of all real numbers and of all positive real numbers, respectively. We introduce the spherical coordinate (r, Θ), $\Theta = (\theta_1, \theta_2, \dots, \theta_{n-1})$, in the n-dimensional Euclidean space $\mathbb{R}^n$ $(n \geq 2)$ which are related to the cartesian coordinates (X, y), $X = (x_1, x_2, \dots, x_{n-1}, y)$ by the formulas

$$x_1 = r\left(\prod_{j=1}^{n-1} \sin\theta_j\right), \qquad y = r\cos\theta_1,$$

and if $n \geq 3$,

$$x_{n+1-k} = r\left(\prod_{j=1}^{k-1} \sin\theta_j\right)\cos\theta_k \qquad (2 \leq k \leq n-1),$$

where

$$0 \leq r < +\infty,\ -2^{-1}\pi \leq \theta_{n-1} < 2^{-1}3\pi$$

and if

$$n \geq 3,\ 0 \leq \theta_j \leq \pi\ (1 \leq j \leq n-2).$$

The unit sphere (the unit circle, if $n = 2$) and the upper half unit sphere $\{(1, \theta_1, \theta_2, \dots, \theta_{n-1}) \in \mathbb{R}^n;\ 0 \leq \theta_1 < \frac{\pi}{2}\}$ (the upper half circle $\{(1, \theta_1) \in \mathbb{R}^2;$

$-2^{-1}\pi < \theta_1 < 2^{-1}\pi\}$ if $n=2$) in $\mathbb{R}^n$ $(n \geq 2)$ are denoted by $\mathbb{S}^{n-1}$ and $\mathbb{S}_+^{n-1}$, respectively. The half-space

$$\{(X,y) \in \mathbb{R}^n;\ X \in \mathbb{R}^{n-1}, y>0\} = \{(r,\Theta) \in \mathbb{R}^n;\ \Theta \in \mathbb{S}_+^{n-1},\ 0<r<+\infty\}$$

is denoted by $\mathbb{T}_n$. Then the boundary $\partial\mathbb{T}_n$ of $\mathbb{T}_n$ in $\mathbb{R}^n$ $(n \geq 2)$ is identified with $\mathbb{R}^{n-1}$, which is represented as

$$\{Q=(t,\xi) \in \mathbb{R}^{n-1};\quad |Q| = t \geq 0,\ \xi \in \partial\mathbb{S}_+^{n-1}\}$$

by the spherical coordinates, where $\partial\mathbb{S}_+^{n-1}$ is the boundary of $\mathbb{S}_+^{n-1}$ in $\mathbb{S}^{n-1}$ (if $n \geq 3$, then $\partial\mathbb{S}_+^{n-1} = \mathbb{S}^{n-2}$ and if $n=2$, then $\partial\mathbb{S}_+^1 = \{-\frac{\pi}{2}, \frac{\pi}{2}\}$, $(t, \frac{\pi}{2}) = t \in \mathbb{R}$ and $(t, -\frac{\pi}{2}) = -t \in \mathbb{R}$ $(t \geq 0)$).

Given a continuous function f on $\partial\mathbb{T}_n$, we say that h is a solution of the (classical) Dirichlet problem on $\mathbb{T}_n$ with f, if h is harmonic in $\mathbb{T}_n$ and

$$\lim_{P \in \mathbb{T}_n, P \to Q} h(P) = f(Q)$$

for every $Q \in \partial\mathbb{T}_n$.

Helms [**4**, p. 42 and p. 158] states that even if f is a bounded continuous function on $\partial\mathbb{T}_n$, the solution of the Dirichlet problem on $\mathbb{T}_n$ with f is not unique and to obtain the unique solution $H(P)$ $(P=(X,y) \in \mathbb{T}_n)$ we must specify the behavior of $H(P)$ as $y \to +\infty$. With respect to this fact, Siegel [**6**, Theorems 1] proved the following result. *Let F_ℓ $(\ell \geq 0)$ be the set of continuous functions $f(x)$ on $\mathbb{R}$ such that*

$$\int_{-\infty}^{+\infty} \frac{|f(x)|}{1+|x|^{2+\ell}}\, dx < +\infty.$$

If $f \in F_\ell$, then there exists a solution $H_{\ell,2}(f)(P)$ of the Dirichlet problem on $\mathbb{T}_2$ with f satisfying

$$H_{\ell,2}(f)(P) = o(r^{\ell+1}/\cos\theta_1) \qquad (r \to +\infty)$$
$$(P = (r\sin\theta_1,\ r\cos\theta_1) \in \mathbb{T}_2).$$

If $h(P)$ is a solution of the Dirichlet problem on $\mathbb{T}_2$ with this f such that

$$h(P) = o(r^{\ell+1}/\cos\theta_1) \qquad (r \to +\infty)\,(P = (r\sin\theta_1,\ r\cos\theta_1) \in \mathbb{T}_2),$$

then

$$h(P) = H_{\ell,2}(f)(P) + U(h)(P)$$

for every $P \in \mathbb{T}_2$, where $U(h)(P)$ is a harmonic polynomial (of $P=(x,y) \in \mathbb{R}^2$) of degree at most ℓ vanishing on $\partial\mathbb{T}_2 = \{(x,0) \in \mathbb{R}^2;\ x \in \mathbb{R}\}$. Further he stated the following result without proof (Siegel [**6**, Theorem 3]). *Let ℓ be a non-negative integer. If f is a continuous function on $\partial\mathbb{T}_n$ $(n \geq 2)$ such that*

$$(1.1)\qquad |f(Q)| \leq F(x) \qquad (Q \in \partial\mathbb{T}_n = \mathbb{R}^{n-1},\quad |Q| = x)$$

for some $F(x) \in F_\ell$, $F(x) = F(-x)$ $(x \in \mathbb{R})$, then there exists a solution $H_{\ell,n}(f)(P)$ of the Dirichlet problem on $\mathbb{T}_n$ with f satisfying

$$(1.2)\quad H_{\ell,n}(f)(P) = o(r^{\ell+1}/\cos\theta_1) \qquad (r \to +\infty)$$
$$(P = (r,\Theta) \in \mathbb{T}_n,\ \Theta = (\theta_1, \theta_2, \dots, \theta_{n-1}))\,.$$

If $h(P)$ is a solution of the Dirichlet problem on $\mathbb{T}_n$ with this f satisfying

$$(1.3)\quad h(P) = o(r^{\ell+1}/\cos\theta_1) \qquad (r \to +\infty)$$
$$(P = (r,\Theta) \in \mathbb{T}_n,\ \Theta = (\theta_1, \theta_2, \dots, \theta_{n-1}))\,,$$

then

$$h(P) = H_{\ell,n}(f)(P) + U(h)(P) \qquad (P \in \mathbb{T}_n),$$

where $U(h)(P)$ is a harmonic polynomial of $P = (x_1, x_2, \dots, x_{n-1}, y) \in \mathbb{R}^n$ of degree at most ℓ vanishing on $\partial\mathbb{T}_n = \{(X,0) \in \mathbb{R}^n;\ X \in \mathbb{R}^{n-1}\}$.

In connection with these results, Siegel [**6**, p. 8] asked whether the condition (1.1) of $f(Q)$ can be replaced by more natural condition

$$(1.4)\qquad \int_{\mathbb{R}^{n-1}} \frac{|f(Q)|}{1 + |Q|^{n+\ell}}\, dX < +\infty \qquad (\ell \geq 0),$$

under which $H_{\ell,n}(f)(P)$ exists.

A special case of the following result of Yoshida shows that this question is solved affirmatively in the case where $\ell = 0$. To state it, we need the following notations. Let $\Phi(r,\Theta)$ be a function on $\mathbb{T}_n$. We put

$$N(\Phi)(r) = \int_{\mathbb{S}_+^{n-1}} \Phi(r,\Theta)\cos\theta_1\, d\sigma_\Theta \quad (\Theta = (\theta_1, \theta_2, \dots, \theta_{n-1}))$$

and

$$\mu_0(\Phi) = \lim_{r\to\infty} r^{-1} N(\Phi)(r),$$

if they exist, where $d\sigma_\Theta$ is the surface element on $\mathbb{S}^{n-1}$. Let $G_n(P_1, P_2)(P_1, P_2 \in \mathbb{T}_n)$ be the Green function of $\mathbb{T}_n$. By $K_{0,n}(P,Q)$ $(P \in \mathbb{T}_n,\ Q \in \partial\mathbb{T}_n)$, we denote the ordinary Poisson kernel of $\mathbb{T}_n$

$$c_n^{-1}\frac{\partial}{\partial\nu} G_n(P,Q) = \frac{2y}{s_n}|P-Q|^{-n} \qquad c_n = \begin{cases} 2\pi, & (n=2) \\ (n-2)s_n, & (n \geq 3) \end{cases},$$

where $\dfrac{\partial}{\partial\nu}$ denotes the differentiation at Q along the inward normal into $\mathbb{T}_n$ and s_n is the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\mathbb{S}^{n-1}$.

Theorem A. (Yoshida [**8**, Theorem 3 and Lemma 3]). *Let $f(Q)$ be a continuous function on $\mathbb{T}_n$ $(n \geq 2)$ satisfying*

$$\int^{+\infty} t^{-2}\left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\, d\sigma_\xi\right) dt < +\infty, \tag{1.5}$$

where $d\sigma_\xi$ is the surface element of $\partial\mathbb{S}_+^{n-1} = \mathbb{S}^{n-2}$ $(n \geq 3)$ and

$$\int_{\partial\mathbb{S}_+^1} |f(t,\xi)|\, d\sigma_\xi = \left|f\left(t, \frac{\pi}{2}\right)\right| + \left|f\left(t, -\frac{\pi}{2}\right)\right| \quad (n = 2).$$

Then the Poisson integral

$$H_{0,n}(f)(P) = \int_{\partial\mathbb{T}_n} f(Q) K_{0,n}(P,Q)\, d\sigma_Q$$

is a solution of the classical Dirichlet problem on $\mathbb{T}_n$ with f such that

$$\mu_0\left(H_{0,n}(|f|)\right) = 0.$$

If $h(P)$ is a solution of the classical Dirichlet problem on $\mathbb{T}_n$ with this f, then two limits $\mu_0(h)$ $(-\infty < \mu_0(h) \leq +\infty)$ and $\mu_0(|h|)$ $(0 \leq \mu_0(|h|) \leq +\infty)$ exist, and if

$$\mu_0(|h|) < +\infty, \tag{1.6}$$

then

$$h(P) = H_{0,n}(f)(P) + 2n s_n^{-1} \mu_0(h) y \tag{1.7}$$

for any $P = (X, y) \in \mathbb{T}_n$.

We remark that (1.5) is equivalent to

$$\int_{\mathbb{R}^{n-1}} \frac{|f(Q)|}{1 + |Q|^n}\, dQ < +\infty.$$

If h is a solution of the Dirichlet problem on $\mathbb{T}_n$ with this f such that $h = o(r/\cos\theta_1)$ $(r \to \infty)$, then $\mu_0(|h|) = 0$, $\mu_0(h) = 0$ and hence $h(P) = H_{0,n}(f)(P)$. This shows that Theorem A gives a positive answer to Siegel's question in the case where $\ell = 0$. However Theorem A gives a form of h not only in the case where $\mu_0(|h|) = 0$ but also in the case where $0 < \mu_0(|h|) < +\infty$.

In this paper we shall show that a solution of the Dirichlet problem on $\mathbb{T}_n$ with f satisfying (1.4) satisfies a natural condition weaker than (1.2) (Theorem 1) and other solutions with this f satisfying some growth condition different from (1.3) are specified in a certain sense (Theorem 2), which contains a positive answer to Siegel's question in every case (Corollary 1) and gives a generalized form of Theorem A (Corollary 2). We shall also state Theorem 2 in more general form (Theorem 3).

I would like to thank the referee for suggesting a much simpler proof of Lemma 3.

2. Statement of results.

We denote the origin of $\mathbb{R}^n$ by O. Let k $(k \geq 0)$ and n $(n \geq 2)$ be two integers and let $L_{k,n+2}$ be the $(n+2)$-dimensional Legendre polynomial of degree k, where $L_{0,n+2} = 1$. We also put

$$c_{k,n+2} = \binom{k+n-1}{k}.$$

We note that $c_{k,n+2}L_{k,n+2}(t)$ is equal to the ultraspherical (or Gegenbauer) polynomial $P_k^{n/2}$ of degree k associated with $\frac{n}{2}$ (see Stein and Weiss [**7**, p. 148]).

The following theorem gives the Fourier expansion of $K_{0,n}(P,Q)$.

Theorem B. (Armitage [**1**, Theorem E] and Gardiner [**3**, Theorem B]). *Let $Q = (Z) = (t, \xi) \in \mathbb{R}^{n-1} - \{O\}$, $|Q| = t$, $\xi \in \mathbb{S}^{n-2}$ $(n \geq 2)$. The function $J_{k,n,Q}$ of $P = (X,y) = (r,\Theta) \in \mathbb{R}^n$, $\Theta = (\theta_1, \theta_2, \dots, \theta_{n-1})$, given by*

$$J_{k,n,Q}(P) = r^{k+1}\cos\theta_1 L_{k,n+2}(\sin\theta_1\cos\gamma) \tag{2.1}$$

$$(\gamma \quad \textit{is the angle between} \quad (X,0) \quad \textit{and} \quad (Z,0))$$

is a homogeneous harmonic polynomial of degree $k+1$. Further the function independent of t and r

$$I_{k,n,\xi}(\Theta) = r^{-k-1}J_{k,n,Q}(P)$$

(which is the restriction to the surface $\mathbb{S}^{n-1}$ of $J_{k,n,Q}(P)$ and hence a spherical harmonic of degree $k+1$) satisfies

$$|I_{k,n,\xi}(\Theta)| \leq \cos\theta_1 \tag{2.2}$$

for each $P = (r,\Theta) \in \mathbb{R}^n$. If $r < t$ and $\Theta \in \mathbb{S}_+^{n-1}$ then $K_{0,n}(P,Q)$ is given by

$$K_{0,n}(P,Q) = \frac{2}{s_n}\sum_{k=0}^{\infty} c_{k,n+2}t^{-k-n}r^{k+1}I_{k,n,\xi}(\Theta).$$

For an integer $\ell \geq 1$ and two points $P = (r, \Theta) \in \mathbb{T}_n$, $Q = (t, \xi) \in \partial\mathbb{T}_n$, we put

$$V_{\ell,n}(P,Q) = \frac{2}{s_n} \sum_{k=0}^{\ell-1} c_{k,n+2} t^{-k-n} r^{k+1} I_{k,n,\xi}(\Theta).$$

We see from Theorem B that for any fixed $Q \in \partial\mathbb{T}_n$ the function $V_{\ell,n}(P,Q)$ of $P \in \mathbb{T}_n$ is harmonic on $\mathbb{T}_n$ and vanishes on $\partial\mathbb{T}_n$. We define another function

$$W_{\ell,n}(P,Q) = \begin{cases} V_{\ell,n}(P,Q) & (P \in \mathbb{T}_n, Q = (t,\xi) \in \partial\mathbb{T}_n,\ 1 \leq t < +\infty) \\ 0 & (P \in \mathbb{T}_n, Q = (t,\xi) \in \partial\mathbb{T}_n,\ 0 \leq t < 1). \end{cases}$$

In addition to $K_{0,n}(P,Q)$, the *Poisson kernel* $K_{\ell,n}(P,Q)$ $(P \in \mathbb{T}_n,\ Q \in \partial\mathbb{T}_n)$ of order $\ell\,(\ell \geq 1)$ is defined by

$$K_{\ell,n}(P,Q) = K_{0,n}(P,Q) - W_{\ell,n}(P,Q)$$

(see Siegel [**6**, p. 7] and also see Armitage [**1**, p. 56]).

Let ℓ be a non-negative integer. Given a function $\Phi(r,\Theta)$ on $\mathbb{T}_n$, we set

$$\mu_\ell(\Phi) = \lim_{r\to\infty} r^{-\ell-1} N(\Phi)(r),$$

if it exists. By $F_{\ell,n}$ we denote the set of continuous functions $f(Q)$ on $\partial\mathbb{T}_n = \mathbb{R}^{n-1}$ $(n \geq 2)$ such that

$$\int_{\mathbb{R}^{n-1}} \frac{|f(Q)|}{1+|Q|^{n+\ell}}\, dQ < +\infty, \tag{2.3}$$

which is equivalent to

$$\int^{+\infty} t^{-\ell-2} \left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\, d\sigma_\xi \right) dt < +\infty.$$

Hence $F_{\ell,2}$ is equal to F_ℓ.

Theorem 1. *Let $\ell\,(\ell \geq 0)$, $n\,(n \geq 2)$ be two integers and $f \in F_{\ell,n}$. Then*

$$H_{\ell,n}(f)(P) = \int_{\partial\mathbb{T}_n} f(Q) K_{\ell,n}(P,Q)\, d\sigma_Q$$

is a solution of the classical Dirichlet problem on $\mathbb{T}_n$ with f satisfying

$$\mu_\ell\left(|H_{\ell,n}(f)|\right) = 0. \tag{2.4}$$

Remark 1. Further, suppose in Theorem 1 that $f \in F_{\ell',n}$ for some ℓ' less than ℓ. Then

$$H_{\ell',n}(f)(P) - H_{\ell,n}(f)(P) = \frac{2}{s_n} \sum_{k=\ell'}^{\ell-1} c_{k,n+2} J^*_{k,n}(f)(P),$$

where

$$J^*_{k,n}(f)(P) = r^{k+1} \int_1^{+\infty} t^{-k-2} \left(\int_{\partial\mathbb{S}^{n-1}_+} I_{k,n,\xi}(\Theta) f(t,\xi)\, d\sigma_\xi \right) dt \quad P = (r,\Theta).$$

We note from (2.2) that

$$\left| J^*_{k,n}(f)(P) \right| \le r^{k+1} \cos\theta_1 \int_1^{+\infty} t^{-k-2} \left(\int_{\partial\mathbb{S}^{n-1}_+} |f(t,\xi)|\, d\sigma_\xi \right) dt < +\infty.$$

Put $J_{k,n,Q}(P) = y\Upsilon_{k,n,Q}(P)$, and observe from (2.1) that $\Upsilon_{k,n,Q}(P)$ is a polynomial of $P = (x_1, x_2, \dots, x_{n-1}, y) \in \mathbb{R}^n$ of degree at most k and even with respect to the variable y. Hence, if we set $J^*_{k,n}(f)(P) = y\Upsilon^*_{k,n}(f)(P)$, then $\Upsilon^*_{k,n}(f)(P)$ is a polynomial of $P = (x_1, x_2, \dots, x_{n-1}, y)$ of degree at most k and even with respect to y ($k = \ell', \ell'+1, \ell'+2, \dots, \ell-1$). Thus

$$H_{\ell,n}(f)(P) = H_{\ell',n}(f)(P) + yL(f)(P),$$

where $L(f)(P)$ is a polynomial of $P = (x_1, x_2, \dots, x_{n-1}, y) \in \mathbb{R}^n$ of degree at most $\ell - 1$ and even with respect to y.

Remark 2. If (1.2) is satisfied, then (2.4) also holds. Since Siegel assumed (1.1) which is stronger than (2.3), he could obtain (1.2). It is interesting to ask whether (1.2) follows under (2.3) or not.

The following result is just a generalization of Picard's theorem stating that a positive harmonic function in the Euclidean space is a constant. *Let $H(r,\Theta)$ be harmonic on $\mathbb{R}^m$ ($m \ge 2$). If, for some positive $t > 1$,*

$$r^{-t-1}\mathcal{M}(H^+)(r) \to 0 \quad (r \to +\infty), \quad \mathcal{M}(H^+)(r) = \int_{\mathbb{S}^{m-1}_+} H^+(r,\Theta)\, d\sigma_\Theta,$$

then for some positive integer ℓ less than t

$$H(r,\Theta) = C + \sum_{k=1}^{\ell} \Xi_k(r,\Theta) \qquad ((r,\Theta) \in \mathbb{R}^m),$$

where C is a constant and $\Xi_k(r,\Theta) = r^k Y_k(\Theta)$ is a homogeneous harmonic polynomial of order k ($Y_k(\Theta)$ is a spherical harmonic function) (see e.g. Brelot [**2**, Appendix; §26]).

It is well known that many results on harmonic functions in $\mathbb{R}^n$ can easily obtained by a passage to $\mathbb{R}^{n+2}$. By using this fact and the result with $m = n+2$ stated above, Kuran proved the following Theorem C. To state it, for a function $\Phi(r,\Theta)$ on $\mathbb{T}_n$ we define

$$\mathcal{D}(y\Phi, r) = (\sigma_r^+)^{-1} \int_{\mathbb{S}_r^+} y\Phi(r,\Theta)\, d\mathbb{S}_r^+,$$

if it exists, where $\mathbb{S}_r^+ = \{(r, \Theta) \in \mathbb{T}_n;\ \Theta \in \mathbb{S}_+^{n-1}\}$, σ_r^+ is the surface area of the spherical part of $\mathbb{S}_r^+$ and $d\mathbb{S}_r^+$ is the surface element of $\mathbb{S}_r^+$.

Theorem C. (Kuran [**5**, Theorem 10]). *Let $h(X,y)$ $(= h(r,\Theta))$ be a harmonic function on $\mathbb{T}_n$ such that h vanishes continuously on $\partial\mathbb{T}_n$.*

If, for some positive t,

$$\lim_{r\to\infty} r^{-t-2}\mathcal{D}(yh^+, r) = 0, \tag{2.5}$$

then

$$h = y\Pi(h)$$

in $\mathbb{T}_n$, where $\Pi(h)$ is a polynomial of $(x_1, x_2, \dots, x_{n-1}, y) \in \mathbb{R}^n$ of degree less than t and even with respect to the variable y.

Remark 3. Let $\Phi(r,\Theta)$ be a function on $\mathbb{T}_n$. Then

$$\mathcal{D}(y\Phi, r) = 2s_n^{-1} r N(\Phi)(r), \tag{2.6}$$

if they exist. Hence (2.5) is equivalent to

$$\lim_{r\to\infty} r^{-(t+1)} N(h^+)(r) = 0.$$

The following theorem answers affirmatively Siegel's question in the case where ℓ is a positive integer.

Theorem 2. *Let ℓ $(\ell \geq 1)$, n $(n \geq 2)$ be two integers and*

$$f \in F_{\ell,n}. \tag{2.7}$$

If $h(r,\Theta)$ is a solution of the Dirichlet problem on $\mathbb{T}_n$ with f satisfying

$$\mu_\ell(h^+) = 0, \tag{2.8}$$

then

$$h(P) = H_{\ell,n}(f)(P) + y\Pi(h)(P) \tag{2.9}$$

for every $P = (X,y) \in \mathbb{T}_n$, where $\Pi(h)(P)$ is a polynomial of $P = (x_1, x_2, \dots, x_{n-1}, y) \in \mathbb{R}^n$ of degree at most $\ell - 1$ and even with respect to the variable y.

The result obtained by Siegel immediately follows from the remark following Theorem A (the case $\ell = 0$) and Theorem 2 (the case $\ell \geq 1$).

Corollary 1. *Let ℓ be a non-negative integer and $f(Q)$ be a continuous function on $\partial\mathbb{T}_n = \mathbb{R}^{n-1}$ $(n \geq 2)$ satisfying*

$$|f(Q)| \leq F(x) \qquad (Q \in \mathbb{R}^{n-1},\ |Q| = x > 0)$$

for some $F(x) \in F_\ell$ $(\ell \geq 0)$, $F(x) = F(-x)$ $(x \in \mathbb{R})$. If $h(P)$ is a solution of the Dirichlet problem on $\mathbb{T}_n$ with f such that

$$h(P) = o(r^{\ell+1}/\cos\theta_1) \quad (r \to \infty) \quad (P = (r, \Theta) \in \mathbb{T}_n),$$

then

$$h(P) = H_{\ell,n}(f)(P) + U(h)(P) \qquad (P = (r, \Theta) \in \mathbb{T}_n),$$

where $U(h)(P)$ is a harmonic polynomial of $P = (x_1, x_2, \ldots, x_{n-1}, y) \in \mathbb{R}^n$ of degree at most ℓ vanishing on $\partial\mathbb{T}_n$.

Theorems 1, 2 and Remark 1 also give a generalized form of Theorem A.

Corollary 2. *Let ℓ be a positive integer and $f(Q)$ be a continuous function on $\partial\mathbb{T}_n$ $(n \geq 2)$ satisfying $f \in F_{\ell-1,n}$. Then the Poisson integral*

$$H_{\ell-1,n}(f)(P) = \int_{\partial\mathbb{T}_n} f(Q) K_{\ell-1,n}(P, Q)\, d\sigma_Q$$

is a solution of the classical Dirichlet problem on $\mathbb{T}_n$ with f satisfying

$$\mu_{\ell-1}\left(|H_{\ell-1,n}(f)|\right) = 0. \tag{2.10}$$

If $h(P)$ is any solution of the classical Dirichlet problem on $\mathbb{T}_n$ with this f satisfying

$$\mu_\ell(h^+) = 0,$$

then

$$h(P) = H_{\ell-1,n}(f)(P) = y\Pi^*(h)(P) \tag{2.11}$$

for every $P = (X, y) \in \mathbb{T}_n$, where $\Pi^*(h)(P)$ is a polynomial of P with degree at most $\ell - 1$ and even with respect to the variable y.

Remark 4. Since

$$\mu_{\ell-1}(h) = \mu_{\ell-1}(y\Pi^*(h))$$

from (2.10) and (2.11) and

$$y\Pi^*(h)(P) = r^\ell \varphi(h)(\Theta)\{1 + o(1)\} \qquad (r \to +\infty) \quad (P = (r, \Theta) \in \mathbb{T}_n)$$

for some $\varphi(h)(\Theta)$ on $\mathbb{S}^{+}_{n-1}$, it follows that

$$\mu_{\ell-1}(h) = \int_{\mathbb{S}^{n-1}_{+}} \varphi(h)(\Theta) \cos\theta_1 \, d\sigma_\Theta$$

exists. Put $\ell = 1$ in Corollary 2. Then $\Pi^*(h)(P)$ is a constant C and $\mu_0(h) = C\mu_0(y) = \frac{C}{2n} s_n$. Thus we obtain (1.7) under the weaker condition $\mu_1(h^+) = 0$ than (1.6).

It may be more desirable to restate Theorem 2 in the following form.

Theorem 3. *If $h(r,\Theta)$ is a solution of the Dirichlet problem on $\mathbb{T}_n$ $(n \geq 2)$ with some $f \in F_{\ell,n}$ $(\ell \geq 0)$ satisfying*

$$\overline{\lim}_{r\to\infty} \frac{\log N(h^+)(r)}{\log r} < +\infty,$$

then

$$h(P) = H_{\ell,n}(f)(P) + y\Lambda(h)(P)$$

for every $P = (X,y) \in \mathbb{T}_n$, where $\Lambda(h)(P)$ is a polynomial of $P = (x_1, x_2, \dots, x_{n-1}, y) \in \mathbb{R}^n$ and even with respect to the variable y.

3. Proofs of the Theorems 1, 2, 3 and Corollary 2.

For a set E, $E \subset \mathbb{R}_+ \cup \{0\}$, we denote $\{(r,\Theta) \in \mathbb{T}_n;\ r \in E\}$ and $\{(r,\Theta) \in \partial\mathbb{T}_n;\ r \in E\}$ by $\mathbb{T}_n E$ and $\partial\mathbb{T}_n E$, respectively.

Lemma 1. *For a positive integer ℓ we have*

$$|K_{0,n}(P,Q) - V_{\ell,n}(P,Q)| \leq C_1 r^{\ell+1} t^{-n-\ell} \cos\theta_1$$

for any $P = (r,\Theta) \in \mathbb{T}_n$, $\Theta = (\theta_1, \theta_2, \dots, \theta_{n-1})$ and any $Q = (t,\xi) \in \partial\mathbb{T}_n - \{O\}$ $(n \geq 2)$ satisfying $0 < \frac{2r}{t} \leq 1$, where C_1 is a constant depending only on ℓ and n.

Proof. Take any $P = (r,\Theta) \in \mathbb{T}_n$ and any $Q = (t,\xi) \in \partial\mathbb{T}_n - \{O\}$. Put $R_1 = \frac{2r}{t}$, $a = \frac{t}{2}$ and $\Theta_1 = \Theta$ in

$$a^{n-2} G_n\left((aR_1, \Theta_1), (aR_2, \Theta_2)\right) = G_n\left((R_1,\Theta_1), (R_2,\Theta_2)\right)$$
$$(a \in \mathbb{R}_+,\ (R_1,\Theta_1),\ (R_2,\Theta_2) \in \mathbb{T}_n).$$

When (R_2, Θ_2) approach to $(2,\xi) \in \partial\mathbb{T}_n$ along the inward normal, we obtain

$$\left(\frac{1}{2}t\right)^{n-1} K_{0,n}((r,\Theta),\ (t,\xi)) = K_{0,n}\left(\left(\frac{2r}{t}, \Theta\right),\ (2,\xi)\right). \tag{3.1}$$

Suppose that $0 < \frac{2r}{t} \leq 1$. From Theorem B and (2.2) we have that

(3.2)

$$\begin{aligned}&\left|K_{0,n}\left(\left(\frac{2r}{t},\Theta\right),\ (2,\xi)\right) - 2s_n^{-1}\sum_{k=0}^{\ell-1} c_{k,n+2}2^{-k-n}\left(\frac{2r}{t}\right)^{k+1} I_{k,n,\xi}(\Theta)\right|\\ &\leq s_n^{-1}2^{-n+1}\sum_{k=\ell}^{\infty} c_{k,n+2}2^{-k}\left(\frac{2r}{t}\right)^{k+1}|I_{k,n,\xi}(\Theta)|\\ &\leq s_n^{-1}2^{-n+1}\left(\frac{2r}{t}\right)^{\ell+1}\cos\theta_1\sum_{k=\ell}^{\infty} c_{k,n+2}2^{-k}.\end{aligned}$$

Since

$$\sum_{k=\ell}^{\infty} c_{k,n+2}2^{-k} = \frac{(n+\ell-1)!}{(n-1)!(\ell-1)!}\int_0^{1/2}\left(\frac{1}{2}-u\right)^{\ell-1}(1-u)^{-n-\ell}\,du = C_1'$$

is finite, we immediately have

$$\begin{aligned}&\left|K_{0,n}((r,\Theta),\ (t,\xi)) - 2s_n^{-1}\sum_{k=0}^{\ell-1} c_{k,n+2}t^{-n-k}r^{k+1}I_{k,n,\xi}(\Theta)\right|\\ &\leq C_1 t^{-n-\ell}r^{\ell+1}\cos\theta_1 \qquad (C_1 = 2^{\ell+1}s_n^{-1}C_1')\end{aligned}$$

from (3.1) and (3.2), which is the conclusion. □

Lemma 2. *Let ℓ be any positive integer. Let $f(Q)$ be a locally integrable function on $\partial\mathbb{T}_n$ ($n \geq 2$) satisfying (2.3). Then $H_{\ell,n}(f)(P)$ is a harmonic function on $\mathbb{T}_n$.*

Proof. For any fixed $P = (r,\Theta) \in \mathbb{T}_n$, take a number R satisfying $R \geq \max(1, 2r)$. Then from Lemma 1 we have

(3.3)

$$\begin{aligned}&\int_{\partial\mathbb{T}_n[R,+\infty)} |f(Q)|\,|K_{\ell,n}(P,Q)|\,d\sigma_Q\\ &\qquad = \int_{\partial\mathbb{T}_n[R,+\infty)} |f(Q)|\,|K_{0,n}(P,Q) - V_{\ell,n}(P,Q)|\,d\sigma_Q\\ &\qquad \leq C_1 r^{\ell+1}\cos\theta_1\int_R^{+\infty} t^{-\ell-2}\left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\,d\sigma_\xi\right)dt < +\infty.\end{aligned}$$

Thus $H_{\ell,n}(f)(P)$ is finite for any $P \in \mathbb{T}_n$. Since $K_{\ell,n}(P,Q)$ is a harmonic function of $P \in \mathbb{T}_n$ for any fixed $Q \in \partial\mathbb{T}_n$, $H_{\ell,n}(f)(P)$ is also a harmonic function of $P \in \mathbb{T}_n$. □

Lemma 3. *Let ℓ be any positive integer. Let $f(Q)$ be a locally integrable and finite-valued upper semicontinuous function on $\partial\mathbb{T}_n$ $(n \geq 2)$ satisfying* (2.3). *Then*

$$\overline{\lim}_{P\to Q^*,\ P\in\mathbb{T}_n} H_{\ell,n}(f)(P) \leq f(Q^*)$$

for any $Q^* \in \partial\mathbb{T}_n$.

Proof. Let $Q^* = (t^*, \xi^*)$ be any fixed point of $\partial\mathbb{T}_n$ and ε be any positive number. Take a positive number δ, $\delta < 1$, such that

$$f(Q) < f(Q^*) + \varepsilon \tag{3.4}$$

for any $Q \in \partial\mathbb{T}_n \cap U_\delta(Q^*)$, where $U_\delta(Q^*) = \{P \in \mathbb{R}^n;\ |P - Q^*| < \delta\}$. From (3.3), we can choose a number R^*, $R^* > 2(t^* + 1)$, such that

$$\int_{\partial\mathbb{T}_n[R^*,+\infty)} |f(Q)|\ |K_{\ell,n}(P,Q)|\ d\sigma_Q < \varepsilon, \tag{3.5}$$

for any $P \in \mathbb{T}_n \cap U_\delta(Q^*)$. Now we write

$$\begin{aligned}
H_{\ell,n}(f)(P) &= \int_{\partial\mathbb{T}_n\cap U_\delta(Q^*)} f(Q)K_{\ell,n}(P,Q)\,d\sigma_Q \\
&\quad + \int_{\partial\mathbb{T}_n[0,R^*)-U_\delta(Q^*)} f(Q)K_{\ell,n}(P,Q)\,d\sigma_Q \\
&\quad + \int_{\partial\mathbb{T}_n[R^*,+\infty)} f(Q)K_{\ell,n}(P,Q)\,d\sigma_Q \\
&= I_1(P) + I_2(P) + I_3(P), \\
I_1(P) &= \int_{\partial\mathbb{T}_n\cap U_\delta(Q^*)} f(Q)K_{0,n}(P,Q)\,d\sigma_Q \\
&\quad - \int_{\partial\mathbb{T}_n\cap U_\delta(Q^*)} f(Q)W_{\ell,n}(P,Q)\,d\sigma_Q \\
&= I_{1,1}(P) + I_{1,2}(P)
\end{aligned}$$

and

$$\begin{aligned}
I_2(P) &= \int_{\partial\mathbb{T}_n[0,R^*)-U_\delta(Q^*)} f(Q)K_{0,n}(P,Q)\,d\sigma_Q \\
&\quad - \int_{\partial\mathbb{T}_n[0,R^*)-U_\delta(Q^*)} f(Q)W_{\ell,n}(P,Q)\,d\sigma_Q \\
&= I_{2,1}(P) + I_{2,2}(P).
\end{aligned}$$

First we see from (3.5) that

$$|I_3(P)| < \varepsilon \tag{3.6}$$

for any $P \in \mathbb{T}_n \cap U_\delta(Q^*)$. Since

$$\begin{aligned}1 - \int_{\partial\mathbb{T}_n \cap U_\delta(Q^*)} K_{0,n}(P,Q)\, d\sigma_Q &= \int_{\partial\mathbb{T}_n - U_\delta(Q^*)} K_{0,n}(P,Q)\, d\sigma_Q \\ &= \frac{2y}{s_n} \int_{\partial\mathbb{T}_n - U_\delta(Q^*)} |P-Q|^{-n}\, d\sigma_Q\end{aligned}$$

for any $P = (X,y) \in \mathbb{T}_n$, we have

$$\lim_{P\in\mathbb{T}_n,\, P\to Q^*} \int_{\partial\mathbb{T}_n \cap U_\delta(Q^*)} K_{0,n}(P,Q)\, d\sigma_Q = 1$$

and hence from (3.4)

$$\overline{\lim}_{P\in\mathbb{T}_n,\, P\to Q^*} I_{1,1}(P) \le f(Q^*) + \varepsilon. \tag{3.7}$$

Also observe that

$$|I_{2,1}(P)| \le \frac{2y}{s_n}\left(\frac{\delta}{2}\right)^{-n} \int_0^{R^*} t^{n-2}\left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\, d\sigma_\xi\right) dt \tag{3.8}$$

for any $P = (X,y) \in \mathbb{T}_n \cap U_{\delta/2}(Q^*)$. Since

$$\int_{\partial\mathbb{T}_n[0,R^*)} |f(Q)|\; |W_{\ell,n}(P,Q)|\; d\sigma_Q \le C_2 \cos\theta_1$$

for any $P = (r,\Theta) \in \mathbb{T}_n \cap U_\delta(Q^*)$, $\Theta = (\theta_1, \theta_2, \dots, \theta_{n-1})$, where

$$C_2 = 2s_n^{-1} \sum_{k=0}^{\ell-1} c_{k,n+2}(t^*+1)^{k+1} \int_1^{R^*} t^{-k-2}\left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\, d\sigma_\xi\right) dt,$$

we obtain that

$$\begin{aligned}|I_{1,2}(P)| &\le \int_{\partial\mathbb{T}_n \cap U_\delta(Q^*)} |f(Q)|\, |W_{\ell,n}(P,Q)|\; d\sigma_Q \\ &\le C_2 \cos\theta_1 \to 0\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}|I_{2,2}(P)| &\le \int_{\partial\mathbb{T}_n[0,R^*) - U_\delta(Q^*)} |f(Q)|\, |W_{\ell,n}(P,Q)|\; d\sigma_Q \\ &\le C_2 \cos\theta_1 \to 0,\end{aligned} \tag{3.10}$$

as $P=(r,\Theta)\to Q^*$. All (3.6), (3.7), (3.8), (3.9) and (3.10) give

$$\overline{\lim}_{P\in\mathbb{T}_n,\ P\to Q^*} H_{\ell,n}(f)(P)\le f(Q^*)+2\varepsilon,$$

from which the conclusion immediately follows. □

Proof of Theorem 1. If $\ell=0$, then Theorem 1 is included in Theorem A. Hence we can assume that $\ell\ge 1$. It immediately follows from Lemma 2 and Lemma 3 that $H_{\ell,n}(f)(P)$ is a harmonic function on $\mathbb{T}_n$ and

$$\lim_{p\in\mathbb{T}_n,\ P\to Q} H_{\ell,n}(f)(P)=f(Q^*)$$

for any $Q^*\in\partial\mathbb{T}_n$.

To prove (2.4), we see first that

$$\begin{aligned}(3.11)\qquad & N\left(|H_{\ell,n}(f)|\right)(r)\\ &\le\int_{\mathbb{S}_+^{n-1}}\left(\int_{\partial\mathbb{T}_n}|f(Q)|\,|K_{\ell,n}(P,Q)|\,d\sigma_Q\right)\cos\theta_1\,d\sigma_\Theta\\ &=I_1(r)+I_2(r)\end{aligned}$$

for any $P=(r,\Theta)\in\mathbb{T}_n$, $\Theta=(\theta_1,\theta_2,\dots,\theta_{n-1})$, where

$$I_1(r)=\int_{\mathbb{S}_+^{n-1}}\left(\int_{\partial\mathbb{T}_n[2r,+\infty)}|f(Q)|\,|K_{\ell,n}(P,Q)|\,d\sigma_Q\right)\cos\theta_1\,d\sigma_\Theta$$

and

$$I_2(r)=\int_{\mathbb{S}_+^{n-1}}\left(\int_{\partial\mathbb{T}_n[0,2r)}|f(Q)|\,|K_{\ell,n}(P,Q)|\,d\sigma_Q\right)\cos\theta_1\,d\sigma_\Theta.$$

Let ε be any positive number. Take a sufficiently large number r_0 such that

$$\int_{2r_0}^{+\infty}t^{-\ell-2}\left(\int_{\partial\mathbb{S}_+^{n-1}}|f(t,\xi)|\,d\sigma_\xi\right)dt<n\left(C_1s_n\right)^{-1}\varepsilon,$$

where C_1 is the constant in Lemma 1. Since

$$(3.12)\qquad \int_{\mathbb{S}_+^{n-1}}\cos^2\theta_1\,d\sigma_\Theta=(2n)^{-1}s_n,$$

we have from (3.3)

$$(3.13)\qquad I_1(r)\le\frac{\varepsilon}{2}r^{\ell+1}$$

for any $P = (r, \Theta) \in \mathbb{T}_n,\ r \geq r_0$.

Suppose $P = (r, \Theta) \in \mathbb{T}_n[\frac{1}{2}, +\infty)$. For any $Q = (t, \xi) \in \partial\mathbb{T}_n\,(0 < t \leq 2r)$ we obtain

$$\begin{aligned}|V_{\ell,n}(P,Q)| &\leq 2s_n^{-1}t^{-n}r\cos\theta_1\sum_{k=0}^{\ell-1}2^{-k}c_{k,n+2}(2r/t)^k\\ &\leq C_3t^{-n-\ell+1}r^\ell\cos\theta_1 \qquad \Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})\end{aligned}$$

from (2.2) and hence

$$|K_{\ell,n}(P,Q)| \leq \begin{cases} K_{0,n}(P,Q) + C_3r^\ell t^{-n-\ell+1}\cos\theta_1, & (t \geq 1)\\ K_{0,n}(P,Q), & (0 < t < 1),\end{cases}$$

where

$$C_3 = \ell 2^\ell s_n^{-1}\max_{0\leq k\leq \ell-1}2^{-k}c_{k,n+2}.$$

Hence we have

$$I_2(r) \leq I_{2,1}(r) + I_{2,2}(r) \tag{3.14}$$

from (3.12), where

$$I_{2,1}(r) = \int_{\partial\mathbb{T}_n[0,2r)}|f(Q)|\left(\int_{\mathbb{S}_+^{n-1}}K_{0,n}(P,Q)\cos\theta_1\,d\sigma_\Theta\right)d\sigma_Q$$

and

$$I_{2,2}(r) = C_3(2n)^{-1}s_nr^\ell\int_1^{2r}t^{-\ell-1}\left(\int_{\partial\mathbb{S}_+^{n-1}}|f(t,\xi)|\,d\sigma_\xi\right)dt.$$

Here, consider the function $K_{0,n}(P,Q)$ of $P = (r,\Theta) \in \mathbb{T}_n$ for any fixed $Q = (t,\xi) \in \partial\mathbb{T}_n$. Then we see from (2.5) that

$$N\left(K_{0,n}\right)(r) = \frac{s_n}{2r}\mathcal{D}\left(yK_{0,n}, r\right)$$

and from Kuran [**5**, Lemma 2] and Helms [**4**, p. 109; Example 2] that

$$n\mathcal{D}\left(yK_{0,n}, r\right) = \begin{cases}2r^2s_n^{-1}r^{-n}, & (t \leq r)\\ 2r^2s_n^{-1}t^{-n}, & (r \leq t)\end{cases},$$

which gives

$$\int_{\mathbb{S}_+^{n-1}}K_{0,n}(P,Q)\cos\theta_1\,d\sigma_\Theta = \begin{cases}n^{-1}r^{1-n}, & (t \leq r)\\ n^{-1}rt^{-n}, & (r \leq t)\end{cases} \leq n^{-1}r^{1-n}$$
$$(\Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})).$$

Hence we obtain

(3.15)

$$\begin{aligned} I_{2,1}(r) &\le n^{-1}r^{1-n}\int_0^{2r} t^{n-2}\left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\,d\sigma_\xi\right) dt \\ &= n^{-1}r^{1-n}\int_0^{1} t^{n-2}\left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\,d\sigma_\xi\right) dt \\ &\quad + n^{-1}r^{1-n}\int_1^{2r} t^{n-2}\left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\,d\sigma_\xi\right) dt \\ &\le C_4 n^{-1}r^{1-n} + n^{-1}r^{1-n}\int_1^{2r} t^{-\ell-1}(2r)^{n+\ell-1}\left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\,d\sigma_\xi\right) dt \\ &= C_4 n^{-1}r^{1-n} + n^{-1}2^{n+\ell-1}r^{\ell}\psi(r), \end{aligned}$$

where

$$C_4 = \int_0^1 t^{n-2}\left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\,d\sigma_\xi\right) dt$$

and

$$\psi(r) = \int_1^{2r} t^{-\ell-1}\left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\,d\sigma_\xi\right) dt.$$

Then

$$I_{2,2}(r) = C_3(2n)^{-1}s_n r^{\ell}\psi(r). \tag{3.16}$$

Thus if we can show

$$\psi(r) = o(r) \qquad (r \to \infty), \tag{3.17}$$

then we have

$$I_{2,1}(r) = o(r^{\ell+1}) \qquad (r \to \infty)$$

from (3.15),

$$I_{2,2}(r) = o(r^{\ell+1}) \qquad (r \to \infty)$$

from (3.16) and hence from (3.14) we can find a number r_1 such that

$$I_2(r) < \frac{\varepsilon}{2}r^{\ell+1} \tag{3.18}$$

for any $r \ge r_1$.

To see (3.17), we note that $\psi(r)$ is increasing,

$$\int_1^{+\infty} \frac{\psi'(r)}{r}\, dr = 2\int_2^{+\infty} t^{-\ell-2}\left(\int_{\partial \mathbb{S}_+^{n-1}} |f(t,\xi)|\, d\sigma_\xi\right) dt \le 2C_5$$

and

$$\frac{\psi(r)}{r} \le 2\int_1^{2r} t^{-\ell-2}\left(\int_{\partial \mathbb{S}_+^{n-1}} |f(t,\xi)|\, d\sigma_\xi\right) dt < 2C_5,$$

where

$$C_5 = \int_1^{\infty} t^{-\ell-2}\left(\int_{\partial \mathbb{S}_+^{n-1}} |f(t,\xi)|\, d\sigma_\xi\right) dt.$$

From these we see

$$\int_1^{+\infty} r^{-2}\psi(r)\, dr < +\infty$$

by the integration by parts. Since

$$\frac{\psi(r)}{r} = \psi(r)\int_r^{+\infty} x^{-2}\, dx \le \int_r^{+\infty} x^{-2}\psi(x)\, dx,$$

this gives (3.17).

If we put $r_2 = \max(r_0, r_1)$, then we finally have from (3.11), (3.13) and (3.18)

$$r^{-\ell-1}N\left(|H_{\ell,n}(f)|\right)(r) < \varepsilon$$

for any r, $r \ge r_2$, which gives (2.14). $\square$

Proof of Theorem 2. Consider the function $h - H_{\ell,n}(f)$. Then it follows from Theorem 1 that this is harmonic in $\mathbb{T}_n$ and vanishes continuously on $\partial\mathbb{T}_n$. Since

$$0 \le \{h - H_{\ell,n}(f)\}^+(P) \le h^+(P) + \{H_{\ell,n}(f)\}^-(P) \tag{3.19}$$

for any $P \in \mathbb{T}_n$ and

$$\mu_\ell\left(\{H_{\ell,n}(f)\}^-\right) = 0$$

from (2.4) of Theorem 1, (2.8) gives that

$$\mu_\ell\left(\{h - H_{\ell,n}(f)\}^+\right) = 0.$$

From Remark 3 and Theorem C we see that

$$h(P) - H_{\ell,n}(f)(P) = y\Pi(h)(P)$$

for every $P = (X, y) \in \mathbb{T}_n$, where $\Pi(h)$ is a polynomial in $\mathbb{R}^n$ of degree at most $\ell - 1$ and even with respect to the variable y, which gives the conclusion of Theorem 2. $\square$

Proof of Corollary 2. The first part follows from Theorem 1. Since $f \in F_{\ell,n}$, Theorem 2 gives

$$h(P) = H_{\ell,n}(f)(P) + y\Pi(h)(P)$$

for every $P = (X, y) \in \mathbb{T}_n$, where $\Pi(h)(P)$ is a polynomial of $P \in \mathbb{R}^n$ with degree at most $\ell - 1$ and even with respect to the variable y. Remark 1 also gives

$$H_{\ell,n}(f)(P) = H_{\ell-1,n}(f)(P) + yL(f)(P)$$

for every $P = (X, y) \in \mathbb{T}_n$, where $L(f)(P)$ is a polynomial of $P \in \mathbb{R}^n$ with degree at most $\ell - 1$ and even with respect to the variable y. From these, we evidently obtain (2.11). $\square$

Proof of Theorem 3. Put

$$\overline{\lim}_{r\to\infty} \frac{\log N(h^+)(r)}{\log r} = \alpha.$$

It immediately follows that $\mu_{[\alpha]+1}(h^+) = 0$. Take an integer ℓ^* satisfying $\ell^* \geq \max(\ell, [\alpha] + 1)$. Since $f \in F_{\ell^*,n}$ and $\mu_{\ell^*}(h^+) = 0$, Theorem 2 gives that

$$h(P) = H_{\ell^*,n}(f)(P) + y\Pi(h)(P), \tag{3.20}$$

where $\Pi(h)(P)$ is a polynomial of P and even with respect to y. If $\ell = \ell^*$, then (3.20) gives the conclusion. Suppose that $\ell^* > \ell$. From Remark 1 we also see

$$H_{\ell^*,n}(f)(P) = H_{\ell,n}(f)(P) + yL(f)(P), \tag{3.21}$$

where $L(f)(P)$ is a polynomial of P and even with respect to y. From (3.20) and (3.21) we have

$$h(P) = H_{\ell,n}(f)(P) + y\Lambda(h)(P), \quad \Lambda(h)(P) = \Pi(h)(P) + L(h)(P),$$

which is also the conclusion of Theorem 3. $\square$

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