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GENERIC DIFFERENTIABILITY OF CONVEX FUNCTIONS ON THE DUAL OF A BANACH SPACE

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We study a class of Banach spaces which have the property that every continuous convex function on an open convex subset of the dual possessing a weak * continuous subgradient at points of a dense G_δ subset of its domain, is Fréchet differentiable on a dense G_δ subset of its domain. A smaller more amenable class consists of Banach spaces where every minimal weak * cusco from a complete metric space into subsets of the second dual which intersect the embedding from a residual subset of the domain is single-valued and norm upper semi-continuous at the points of a residual subset of the domain. It is known that all Banach spaces with the Radon-Nikodym property belong to these classes as do all with equivalent locally uniformly rotund norm. We show that all with an equivalent weakly locally uniformly rotund norm belong to these classes. The condition closest to a characterisation is that the Banach space have its weak topology fragmentable by a metric whose topology on bounded sets is stronger than the weak topology. We show that the space $\ell_\infty(\Gamma)$, where Γ is uncountable, does not belong to our special classes.

We say that a Banach space is a *dual differentiability space* (*DD* space) if every continuous convex function on an open convex subset of the dual possessing a weak * continuous subgradient at points of a dense G_δ subset of its domain, is Fréchet differentiable on a dense G_δ subset of its domain. Spaces of this class include those with the Radon-Nikodym property, and all those which can be equivalently renormed to be locally uniformly rotund. In the paper [**K-G**, p. 472] it was shown that spaces which can be equivalently renormed to have every point of the unit sphere a denting point of the closed unit ball are spaces of this class, and in the paper [**G-M1**, p. 264] it was shown that spaces which can be equivalently renormed to have every point of the unit sphere an α denting point of the closed unit ball, (α is Kuratowski's index of non-compactness), are spaces of this class; Troyanski [**T1**, p. 306] and [**T2**, p. 179] has shown that spaces with either of these properties can be equivalently renormed to be locally uniformly rotund. In paper [**G-M2**, p. 111], the denting point property was weakened using an index of non-WCG.

Information about the class of DD spaces is more easily obtained through the study of a subclass defined by certain set-valued mappings having special continuity properties. A set-valued mapping Φ from a topological space A into subsets of a topological space X is *upper semi-continuous* at $t \in A$ if given an open subset W where $\Phi(t) \subseteq W$ there exists an open neighbourhood U of t such that $\Phi(U) \subseteq W$. If X is a linear topological space and $\Phi(t)$ is non-empty compact and convex for each $t \in A$ and Φ is upper semi-continuous on A we call Φ a *cusco* on A. A cusco Φ on A is said to be a *minimal cusco* if its graph does not contain the graph of any other cusco on A.

We say that a Banach space X is a *generic continuity space* (GC space) if every minimal weak * cusco Φ from a complete metric space A into subsets of the second dual X^{**} for which the set $\left\{t \in A : \Phi(t) \cap \widehat{X} \neq \emptyset\right\}$ is residual in A, is single-valued and norm upper semi-continuous at the points of a residual subset of A.

An open subset of a complete metric space is itself completely metrisable and a continuous convex function ϕ on an open convex subset of a Banach space generates a subdifferential mapping $x \mapsto \partial\phi(x)$ which is a minimal weak * cusco. The subdifferential mapping being single-valued and norm upper semi-continuous at a point is equivalent to the convex function being Fréchet differentiable at the point . So the class of GC spaces is contained in the class of DD spaces.

In Section 1 we show that for any Banach space X, minimal weak * cuscos from a complete metric space A into subsets of the second dual X^{**} which satisfy a certain generic property are always single-valued and norm upper semi-continuous at the points of a residual subset of A. We use this general result to show that Banach spaces which satisfy certain geometrical properties are GC spaces. In particular, we show that those Banach spaces which have an equivalent weakly locally uniformly rotund norm are GC spaces. In Section 2 we show that a Banach space is a GC space if its weak topology is fragmentable by a metric whose topology on bounded sets is stronger than the weak topology. We conclude in Section 3 by showing that the Banach space $\ell_\infty(\Gamma)$, where Γ is an uncountable set, is not a GC space.

1. A general property implying geometrical conditions for membership of the class of GC spaces.

For our general result we need the following characterisations of a minimal cusco.

Lemma 1.1. [**G-M1**, Lemma 2.5]. *Consider a cusco Φ from a topological space A into subsets of a separated locally convex space X. The following are* *equivalent*

(i) Φ *is a minimal cusco on* A,

(ii) *given any open set* U *in* A *and closed convex set* K *in* X *where* $\Phi(U) \not\subseteq K$ *there exists a non-empty open set* $V \subseteq U$ *such that* $\Phi(V) \cap K = \emptyset$,

(iii) *given any open set* U *in* A *and open half-space* W *in* X *where* $\Phi(U) \cap W \neq \emptyset$ *there exists a non-empty open set* $V \subseteq U$ *such that* $\Phi(V) \subseteq W$.

We also use a continuity condition defined in terms of *Kuratowski's index of non-compactness*. Given a bounded set E in a metric space X such an index is

$$\alpha(E) \equiv \inf\{r : E \text{ is covered by a finite family of sets of diameter less than } r\}.$$

Given a set-valued mapping Φ from a topological space A into subsets of a metric space X we say that Φ is α *upper semi-continuous* at $t \in A$ if given $\epsilon > 0$ there exits an open neighbourhood U of t such that $\alpha(\Phi(U)) < \epsilon$. Such α upper semi-continuous mappings have single-valued properties.

Lemma 1.2. [**G-M1**, p. 253]. *Consider a minimal weak * cusco* Φ *from a Baire space* A *into subsets of the second dual* X^{**} *of a Banach space* X. *If* Φ *is* α *upper semi-continuous on a dense subset of* A *then* Φ *is single-valued and norm upper semi-continuous at the points of a residual subset of* A.

The proof of our general theorem follows a similar method of proof as was used to prove Lemma 1.2 which is similar to a theorem of Christensen, [**Chr**, p. 651].

Theorem 1.3. *A minimal weak * cusco* Φ *from a complete metric space* A *into subsets of the second dual* X^{**} *of a Banach space* X *where the set*

$$E \equiv \left\{ t \in A : \Phi(t) \subseteq \overline{\Phi(t) \cap \widehat{X}}^{w^*} \right\}$$

is residual in A, *is single-valued and norm upper semi-continuous at the points of a residual subset of* A.

Proof. Given $\epsilon > 0$ consider the open set $O_\epsilon \equiv \bigcup\{\text{open sets } U \text{ in } A : \alpha(\Phi(U)) \leq 2\epsilon\}$. Suppose that O_ϵ is not dense in A. Then there exists a non-empty open set V_0 in A such that $V_0 \cap O_\epsilon = \emptyset$. Consider a dense G_δ subset D of A contained in E. Now D is completely metrisable and we consider it with such a metric d.

We proceed by induction. Consider $t_1 \in V_0 \cap D$ and $\hat{x}_1 \in \Phi(t_1) \cap \widehat{X}$. Now $\Phi(V_0) \not\subseteq \hat{x}_1 + \epsilon B(X^{**})$ for otherwise $V_0 \cap O_\epsilon \neq \emptyset$. Since Φ is a minimal weak * cusco, by Lemma 1.1, there exists a non-empty open set V_1 such that $\overline{V}_1 \subseteq V_0$ and $\Phi(V_1) \cap (\widehat{x}_1 + \epsilon B(X^{**})) = \emptyset$. We may assume that the d-diam$(V_1 \cap D) < 1$.

Suppose that the first n iterations of this procedure have been completed. Then we have a non-empty open set V_n such that $\overline{V}_n \subseteq V_{n-1}$ and $\Phi(V_n) \cap (\text{co}\{\widehat{x}_1, \widehat{x}_2, \dots, \widehat{x}_n\} + \epsilon B(X^{**})) = \emptyset$ where $\widehat{x}_i \in \Phi(t_i) \cap \widehat{X}$ and $t_i \in V_{i-1} \cap D$ for $i \in \{1, 2, \dots, n\}$. Now consider $t_{n+1} \in V_n \cap D$ and $\widehat{x}_{n+1} \in \Phi(t_{n+1}) \cap \widehat{X}$. Again $\Phi(V_n) \not\subseteq \text{co}\{\widehat{x}_1, \widehat{x}_2, \dots, \widehat{x}_{n+1}\} + \epsilon B(X^{**})$ for otherwise $V_0 \cap O_\epsilon \supseteq V_n \neq \emptyset$. Since Φ is a minimal weak * cusco, by Lemma 1.1 there exists a non-empty open set V_{n+1} with d-diam$(V_{n+1} \cap D) < \frac{1}{2^n}$ such that $\overline{V}_{n+1} \subseteq V_n$ and $\Phi(V_{n+1}) \cap (\text{co}\{\widehat{x}_1, \widehat{x}_2, \dots, \widehat{x}_{n+1}\} + \epsilon B(X^{**})) = \emptyset$. Continuing in this way we form a Cauchy sequence $\{t_n\}$ in D which converges to some $t_\infty \in \bigcap_{n\in\mathbb{N}} \overline{V}_n = \bigcap_{n\in\mathbb{N}} V_n \subseteq D$.

Then for each $n \in \mathbb{N}$, $\Phi(t_\infty) \cap (\text{co}\{\widehat{x}_1, \widehat{x}_2, \dots, \widehat{x}_n\} + \epsilon B(X^{**})) = \emptyset$ and so

$$\Phi(t_\infty) \cap \left(\text{co} \bigcup_{n\in\mathbb{N}} \{\widehat{x}_n\} + \epsilon B(X^{**})\right) = \Phi(t_\infty) \cap \left(\bigcup_{n\in\mathbb{N}} \text{co}\{\widehat{x}_1, \widehat{x}_2, \dots, \widehat{x}_n\} + \epsilon B(X^{**})\right) = \emptyset.$$

So there exists an $f \in X^*$, which strongly separates $\Phi(t_\infty) \cap \widehat{X}$ and $\overline{\text{co}} \bigcup_{n\in\mathbb{N}} \{\widehat{x}_n\}$ and so there is a weak * open half space W generated by f containing $\overline{\Phi(t_\infty) \cap \widehat{X}}^{w^*}$ and disjoint from $\overline{\text{co}} \bigcup_{n\in\mathbb{N}} \{\widehat{x}_n\}$. Since $t_\infty \in E$, we have $\Phi(t_\infty) \subseteq W$. Since Φ is weak * upper semi-continuous at t_∞ there exists an open neighbourhood U of t_∞ such that $\Phi(U) \subseteq W$. However, for $n \in \mathbb{N}$ sufficiently large, $t_n \in U$ and then $\widehat{x}_n \in \Phi(t_n) \cap \widehat{X} \subseteq W$ contradicting the separation by f. We conclude that O_ϵ is dense in A and that Φ is α upper semi-continuous at the points of $\bigcap_{n\in\mathbb{N}} O_{\frac{1}{n}}$ a dense G_δ subset of A. Our result now follows from Lemma 1.2. □

We can now make the following deductions from Theorem 1.3.

Corollary 1.4. *A minimal weak * cusco Φ from a complete metric space A into subsets of the second dual X^{**} of a Banach space X where the set $\left\{t \in A : \Phi(t) \subseteq \widehat{X}\right\}$ is residual in A, is single-valued and norm upper semi-continuous at the points of a residual subset of A.*

A special case of a theorem of Namioka [**N**, p. 525] can be deduced from Theorem 1.3.

Corollary 1.5. *A weakly continuous single-valued mapping from a complete metric space A into a Banach space X is norm continuous at the points of a residual subset of A.*

A Banach space X is *weak Asplund* if every continuous convex function on an open convex subset A of X is Gâteaux differentiable on a residual subset of A. A Banach space X belongs to Stegall's class $\mathcal{S}$ if and only if every minimal weak * cusco Φ from a Baire space A into subsets of X^* is single-valued on a residual subset of A. It has been shown [**K-O**, Corol. 4.5] that a Banach space X belongs to Stegall's class $\mathcal{S}$ if and only if every minimal weak * cusco Φ from a complete metric space A into subsets of X^* is single-valued on a residual subset of A.

Corollary 1.6. *A Banach space X is*

(i) *a DD space if its dual X^* is weak Asplund,*

(ii) *a GC space if its dual X^* belongs to Stegall's class $\mathcal{S}$.*

Proof. We consider only the proof of (ii). A minimal weak * cusco Φ from a complete metric space A into subsets of X^{**} has the set $\{t \in A : \Phi(t) \text{ is singleton}\}$ residual in A. So if the set $\left\{t \in A : \Phi(t) \cap \widehat{X} \neq \emptyset\right\}$ is residual in A then the set $\left\{t \in A : \Phi(t) \subseteq \widehat{X}\right\}$ is residual in A and we deduce from Corollary 1.4 that X is a GC space. □

We should note the Banach space ℓ_1 has dual ℓ_∞ which is not weak Asplund, [**P**, p. 13]. However ℓ_1 has the Radon-Nikodym property and so the property given in Corollary 1.6 is a sufficient but not necessary condition for a Banach space to be a DD space or a GC space.

It has recently been proved, that a Banach space belongs to Stegall's class $\mathcal{S}$ if it has an equivalent norm Gâteaux differentiable away from the origin, [**P-P-N**].

Corollary 1.7. *A Banach space X is a GC space if the dual X^* has an equivalent norm Gâteaux differentiable away from the origin.*

We note that the equivalent norm on X^* need not be a dual norm.

Corollary 1.8. *A Banach space X is a GC space if it can be mapped into a GC space Y, by a continuous linear mapping T whose conjugate T' has a dense range.*

Proof. Consider a minimal weak * cusco Φ from a complete metric space A into subsets of X^{**} where the set $\left\{t \in A : \Phi(t) \cap \widehat{X} \neq \emptyset\right\}$ is residual in A. As a conjugate, T'' is continuous when X^{**} and Y^{**} have their weak * topologies so $T'' \circ \Phi$ is a minimal weak * cusco from A into subsets of Y^{**}. Since Y is a GC space and the set $\left\{t \in A : T'' \circ \Phi(t) \cap \widehat{Y} \neq \emptyset\right\}$ is residual in A, so $T'' \circ \Phi$ is single-valued on a residual subset A. Since T' has dense range then T'' is one-to-one, so Φ is single-valued on a residual subset of A and we have by Theorem 1.3 that Φ is single-valued and norm upper semi-continuous at the points of a residual subset of A. □

It is well known that a closed linear subspace of a Banach space with the Radon-Nikodym property has the Radon-Nikodym property. The following is an extension of this result.

Theorem 1.9. *If a Banach space X is a GC space then every closed linear subspace Y of X is a GC space.*

Proof. The conjugate of the inclusion mapping maps X^* onto Y^* and so the result follows from Corollary 1.8. □

This subspace property holds for the larger class of DD spaces, but the proof uses a different technique.

Theorem 1.10. *If a Banach space X is a DD space then every closed linear subspace Y of X is a DD space.*

Proof. Consider ϕ a continuous convex function on an open convex subset B of Y^* where the set $\left\{g \in B : \partial\phi(g) \cap \widehat{Y} \neq \emptyset\right\} \supseteq E$ a dense G_δ subset of B. Consider T the inclusion mapping of Y into X. The conjugate T' maps X^* onto Y^*. Further, $\phi \circ T'$ is a continuous convex function on the open convex set $A \equiv (T')^{-1}(B)$ in X^*. Since T' is onto it is an open mapping and therefore $D \equiv (T')^{-1}(E)$ is a dense G_δ subset of A. But further, if $f_0 \in D$ then exists a $y_0 \in Y$ such that $\widehat{y}_0 \in \partial\phi(T'f_0)$. Then

$$\widehat{y}_0(T'f) - \widehat{y}_0(T'f_0) \leq \phi(T'f) - \phi(T'f_0) \text{ for all } f \in A$$

so

$$\widehat{y}_0(f) - \widehat{y}_0(f_0) \leq (\phi \circ T')(f) - (\phi \circ T')(f_0) \text{ for all } f \in A;$$

that is, $\widehat{y}_0 \in \partial(\phi \circ T')(f_0)$.

Then $\left\{f \in A : \partial(\phi \circ T')(f) \cap \widehat{X} \neq \emptyset\right\} \supseteq D$ a dense G_δ subset of A. Since X is a DD space there exists a dense G_δ subset G of A where $\phi \circ T'$ is Fréchet differentiable. That is, for $f \in G$,

$$\lim_{\lambda \to 0} \frac{(\phi \circ T')(f + \lambda g) - (\phi \circ T')(f)}{\lambda}$$

exists and is approached uniformly for all $g \in X^*$, $||g|| = 1$. Using the fact the T' is the restriction of each element of X^* to Y and that each restriction has a norm preserving extension on X then

$$\lim_{\lambda \to 0} \frac{\phi(T'(f+\lambda g)) - \phi(T'f)}{\lambda}$$

exists and is approached uniformly for all $T'g \in Y^*$, $||T'g|| = 1$. So ϕ is Fréchet differentiable on $T'(G)$ which is a dense subset of B. Since the set of points where a continuous convex function is Fréchet differentiable is always a G_δ subset, [**P**, p. 15], ϕ is Fréchet differentiable on a dense G_δ subset of B. We conclude that Y is a DD space. □

A Banach space X is said to be *weakly locally uniformly rotund* if for each $x_0 \in X$, $||x_0|| = 1$, given $\epsilon > 0$ and $f \in X^*$, $||f|| = 1$ there exists a $\delta(\epsilon, x_0, f) > 0$ such that $|f(x - x_0)| < \epsilon$ for all $x \in X$, $||x|| \leq 1$ when $||x + x_0|| > 2 - \delta$. A weakly locally uniformly rotund space is rotund but not necessarily locally uniformly rotund. However, such a geometrical property on a Banach space does have rotundity implications for the second dual space.

Lemma 1.11. *Consider a weakly locally uniformly rotund Banach space X. Given $x_0 \in X$, $||x_0|| = 1$, for every $F \in X^{**}$, $||F|| = 1$, $F \neq \widehat{x}_0$, we have $||F + \widehat{x}_0|| < 2$.*

Proof. Suppose that there exists an $F \in X^{**}$, $||F|| = 1$, $F \neq \widehat{x}_0$, such that $||F + \widehat{x}_0|| = 2$. Since $F \neq \widehat{x}_0$ there exists an $f_0 \in X^*$, $||f_0|| = 1$ and an $r > 0$ such that $|(F - \widehat{x}_0)(f_0)| > r$. Since X is weakly locally uniformly rotund, given $0 < \epsilon < \frac{r}{2}$ there exists a $\delta(\epsilon, x_0, f_0) > 0$ such that $|f_0(x - x_0)| < \epsilon$ for all $x \in X$, $||x|| \leq 1$ when $||x + x_0|| > 2 - \delta$. Since the norm on X^{**} is weak * lower semi-continuous the set $\{G \in X^{**} : ||G + \widehat{x}_0|| > 2 - \delta\}$ is weak *open in X^{**} and contains F. By Goldstine's Theorem $B\left(\widehat{X}\right)$ is weak * dense in $B(X^{**})$ so there exists some $\widehat{x} \in B\left(\widehat{X}\right)$ such that $||\widehat{x} + \widehat{x}_0|| > 2 - \delta$ and $|(F - \widehat{x})(f_0)| < \epsilon$. Then for such an $\widehat{x} \in B\left(\widehat{X}\right)$ we have $|f_0(x - x_0)| < \epsilon$ and therefore

$$|(F - \widehat{x}_0)(f_0)| \leq |(F - \widehat{x})(f_0)| + |f_0(x - x_0)| < 2\epsilon < r$$

which contradicts the initial separation property. □

We need the following property of minimimal weak * cuscos.

Lemma 1.12. [**K-G**, p. 471]. *Given a minimal weak * cusco Φ from a Baire space A into subsets of the dual X^* of a Banach space X, there exists* *a residual subset of A at each point t of which, $\Phi(t)$ lies in the face of a sphere of X^*.*

Theorem 1.13. *A Banach space X is a GC space if it can be equivalently renormed to be weakly locally uniformly rotund.*

Proof. Consider X so renormed. Then since Φ is a minimal weak * cusco on A we have by Lemma 1.12 that there exists a residual subset D of A at each point t of which, $\Phi(t)$ lies in the face of a sphere of X^{**}. So if the set $G \equiv \left\{t \in A : \Phi(t) \cap \widehat{X} \neq \emptyset\right\}$ is residual in A then $G \cap D$ is residual in A. But by Lemma 1.11, Φ is single-valued on $G \cap D$ and so $\Phi(G \cap D) \subseteq \widehat{X}$ and we deduce from Theorem 1.3 that X is a GC space. □

We do not need so strong a geometrical condition as weak local uniform rotundity. To be a GC space it would be sufficient for the space X to have an equivalent norm such that given $x_0 \in X$, $||x_0|| = 1$, for every $F \in X^{**} \setminus \widehat{X}$, $||F|| = 1$ we have $||F + \widehat{x}_0|| < 2$. Such an equivalent norm is not necessarily rotund. However, it is difficult to find a characterisation of this property on X.

2. Fragmentability conditions for membership of the class of *GC* spaces.

We aim to find fragmentability conditions which imply that a Banach space is a GC space.

Consider a bounded subset E in a Banach space X. Given $f \in X^*$, $||f|| = 1$ and $\delta > 0$, a *slice* of E defined by f and δ is the subset

$$S(E, f, \delta) \equiv \{x \in E : f(x) > \sup f(E) - \delta\}.$$

A slice of a bounded set E in the dual X^* defined by a weak * continuous linear functional on X^* is called a *weak * slice* of E.

We need the following local boundedness property of minimal weak * cuscos.

Lemma 2.1. *A minimal weak * cusco Φ from a Baire space A into subsets of the dual X^* of a Banach space X is locally bounded on a dense open subset of A.*

Proof. It is sufficient to show that there exists an open subset of A on which Φ is bounded. For each $n \in \mathbb{N}$, consider the set

$$E_n \equiv \{t \in A : \Phi(t) \subseteq nB(X^*)\}.$$

Clearly, $\bigcup_{n\in\mathbb{N}} E_n = A$. Since A is Baire there exists an $n_0 \in \mathbb{N}$ such that $\operatorname{int}\overline{E_{n_0}} \neq \emptyset$. Consider an open set $U \subseteq \overline{E}_{n_0}$. Suppose for some $t_0 \in U \setminus E_{n_0}$ there exists an $f_0 \in \Phi(t_0)\setminus n_0B(X^*)$. Then f_0 can be strongly separated from $n_0B(X^*)$ by a weak * continuous linear functional on X^* which generates a weak * open half space W containing f_0 and $n_0B(X^*) \subseteq C(W)$. Then since Φ is a minimal weak * cusco, by Lemma 1.1 there exists a non-empty open set $V \subseteq U$ such that $\Phi(V) \subseteq W$. But this contradicts the fact that there are points of E_{n_0} in V which map into $n_0B(X^*)$. □

The following characterisation of the class of GC spaces simplifies our computation.

Theorem 2.2. *A Banach space X is a GC space if and only if every minimal weak * cusco Φ from a complete metric space A into subsets of X^{**} where $\Phi(t) \cap \widehat{X} \neq \emptyset$ for all $t \in A$ is single-valued and norm upper semi-continuous at the points of a residual subset of A.*

Proof. Consider a minimal weak * cusco Φ from a complete metric space A into subsets of X^{**} where $\left\{t \in A : \Phi(t) \cap \widehat{X} \neq \emptyset\right\} \supseteq A_1$ a dense G_δ subset of A. Then A_1 is completely metrisable, [**K-N**, p. 96]. Consider the set-valued mapping Φ_1 the restriction of Φ to A_1. Now Φ_1 is also a minimal weak * cusco on A_1 and $\Phi_1(t)\cap\widehat{X} \neq \emptyset$ for all $t \in A_1$. So Φ_1 is single-valued and norm upper semi-continuous at the points of a dense G_δ subset D of A_1 which is also a dense G_δ subset of A.

Consider $t_0 \in D$. Since Φ_1 is norm upper semi-continuous at t_0 there exists an open neighbourhood U of t_0 such that $\Phi_1(U \cap A_1) \subseteq B[\Phi(t_0); \epsilon]$. We will show that $\Phi(U) \subseteq B[\Phi(t_0); \epsilon]$. Suppose not, then since Φ is a minimal weak * cusco, by Lemma 1.1 there exists a non-empty open set $V \subseteq U$ such that $\Phi(V) \cap B[\Phi(t_0); \epsilon] = \emptyset$. But this contradicts the fact that A_1 is dense in A and Φ_1 is norm upper semi-continuous at t_0.

The converse is obvious. □

The following norm fragmenting theorem generalises a characterisation of Banach spaces with the Radon-Nikodym property.

Theorem 2.3. *A Banach space X is a GC space if there exists a weak * lower semi-continuous norm $|||\cdot|||$ on X^{**} and every non-empty bounded subset of X has slices of arbitrarily small $|||\cdot|||$-diameter.*

Proof. Consider a minimal weak * cusco Φ from a complete metric space A into subsets of X^{**} where $\Phi(t) \cap \widehat{X} \neq \emptyset$ for all $t \in A$. Consider the mapping $\widetilde{\Phi}$ from A into subsets of $\widehat{X}$ defined by

$$\widetilde{\Phi}(t) = \Phi(t) \cap \widehat{X}.$$

Given $\epsilon > 0$, consider the set

$$O_\epsilon \equiv \bigcup \left\{\text{open sets } V \text{ such that } |||\cdot||| - \text{diam } \widetilde{\Phi}(V) < \epsilon\right\}.$$

Now O_ϵ is open; we show that it is dense in A. By Lemma 2.1 we may assume that $\widetilde{\Phi}$ is locally bounded. Consider any non-empty open set U in A where $\widetilde{\Phi}(U)$ is bounded. Then there is a weak * slice of $\widetilde{\Phi}(U)$ with $|||\cdot|||$-diameter less than ϵ. Since Φ is a minimal weak * cusco, by Lemma 1.1 there exists a non-empty open set $V \subseteq U$ such that $\widetilde{\Phi}(V)$ lies inside this slice and so $|||\cdot|||$-diam $\widetilde{\Phi}(V) < \epsilon$. So O_ϵ is dense in A. Then $D \equiv \bigcap_{n\in\mathbb{N}} O_{\frac{1}{n}}$ is a dense G_δ of A and $\widetilde{\Phi}$ is single-valued and $|||\cdot|||$-upper semi-continuous at the points of D.

Consider $t_0 \in D$. Suppose that there exists an $F_0 \in \Phi(t_0) \setminus \widehat{X}$. For $r \equiv \frac{1}{2}|||F_0 - \widehat{x}_0|||$, consider $B_{|||\cdot|||}[\widehat{x}_0; r]$. Since $|||\cdot||$ is weak * lower semi-continuous, $B_{|||\cdot|||}[\widehat{x}_0; r]$ is weak * closed. So F_0 and $B_{|||\cdot|||}[\widehat{x}_0; r]$ can be strongly separated by a weak * continuous linear functional which generates a weak * open half-space W containing F_0 and $B_{|||\cdot|||}[\widehat{x}_0; r] \subseteq C(W)$. Since $\widetilde{\Phi}$ is $|||\cdot|||$-upper semi-continuous at t_0, there exists an open neighbourhood U of t_0, such that $\widetilde{\Phi}(U \cap D) \subseteq B_{|||\cdot|||}[\widehat{x}_0; r]$. Now $\Phi(U) \cap W \neq \emptyset$ and since Φ is a minimal weak * cusco, by Lemma 1.1 there exists a non-empty open set $V \subseteq U$ such that $\Phi(V) \subseteq W$. But this contradicts the fact that $\Phi(t) \cap C(W) \neq \emptyset$ for each $t \in V \cap D$. So we conclude that Φ is single-valued on D and maps into $\widehat{X}$. It follows from Theorem 1.3 that Φ is single-valued and norm upper semi-continuous at the points of a residual subset of A. □

We note that the weak * lower semi-continuous norm $||\cdot|||$ on X^{**} need not be an equivalent norm for X^{**}.

A Banach space has the Radon-Nikodym property if and only if every non-empty bounded subset has slices of arbitrarily small diameter, [**P**, p. 72]. So we could deduce the following known result from Theorem 2.3.

Corollary 2.4. *A Banach space with the Radon-Nikodym property is a GC space.*

It is possible to give a characterisation for *GC* spaces in terms of the behavior of set-valued mappings from a complete metric space into subsets of the original space. To do this we generalise the idea of minimality for set-valued mappings from the characterisation of minimal cuscos given in Lemma 1.1.

We say that a set-valued mapping Φ from a topological space A into subsets of a separated locally convex space X is *minimal* if for any open half-space W in X and open subset U in A where $\Phi(U) \cap W \neq \emptyset$ there exists a non-empty open set $V \subseteq U$ such that $\Phi(V) \subseteq W$.

We use the following selection property of minimal set-valued mappings.

Lemma 2.5. *Consider a Banach space X with a separated locally convex topology τ where the norm closed balls are also τ-closed and a τ-minimal set-valued mapping Φ from a topological space A into subsets of X. If there exists a selection $\widetilde{\Phi}$ on a dense set D in A which is norm continuous on D then Φ is single-valued and norm upper semi-continuous at the points of D.*

Proof. Suppose that at $t_0 \in D$, Φ is not single-valued and norm upper semi-continuous. Then there exists an $r > 0$ and in every neighbourhood U of t_0 there exists a $t_1 \in U$ such that $\Phi(t_1) \not\subseteq B\left(\widetilde{\Phi}(t_0); r\right)$. Now $x_1 \in \Phi(t_1) \setminus B\left(\widetilde{\Phi}(t_0); r\right)$ can be strongly separated from $B\left[\widetilde{\Phi}(t_0); \frac{r}{2}\right]$ by a τ-continuous linear functional which generates a τ-open half-space W containing x_1 and $B\left[\widetilde{\Phi}(t_0); \frac{r}{2}\right] \subseteq C(W)$. Since $\widetilde{\Phi}$ is norm continuous at t_0 there exists an open neighbourhood U of t_0, such that $\widetilde{\Phi}(U \cap D) \subseteq B\left(\widetilde{\Phi}(t_0); \frac{r}{2}\right)$. But Φ is τ-minimal and $\Phi(U) \cap W \neq \emptyset$. So there exists a non-empty open set $V \subseteq U$ such that $\Phi(V) \subseteq W$. But this contradicts $\widetilde{\Phi}(V \cap D) \subseteq C(W)$. So we conclude that Φ is single-valued and norm upper semi-continuous at the points of D. □

The following theorem characterises a *GC* space X by the behavior of weakly minimal mappings into X.

Theorem 2.6. *For a Banach space X the following are equivalent*

(i) *X is a GC space,*

(ii) *every weakly minimal locally bounded set-valued mapping Φ from a complete metric space A into subsets of X is single-valued and norm upper semi-continuous at the points of a residual subset of A,*

(iii) *every weakly minimal locally bounded single-valued mapping ϕ from a complete metric space A into X is norm continuous at the points of a residual subset of A.*

Proof. (i) ⇒ (ii). Consider a weakly minimal locally bounded set-valued mapping Φ from A into subsets of X, and weak * cusco $\overline{\Phi}$ from A into subsets of X^{**} generated by Φ where

$$\overline{\Phi}(t) = \bigcap \left\{ \overline{co}^{w^*} \widehat{\Phi(U)} \text{ where } U \text{ is a neighbourhood of } t \right\}, \text{ [B-F-K, p. 472].}$$

Since Φ is weakly minimal then from Lemma 1.1 we see that $\overline{\Phi}$ is minimal weak * cusco. But also $\overline{\Phi}(t) \cap \widehat{X} \neq \emptyset$ for all $t \in A$. Since X is a *GC* space we deduce that $\overline{\Phi}$ is single-valued and norm upper semi-continuous at the points of a residual subset of A, and then so is Φ also.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (i) Consider a minimal weak * cusco Φ from a complete metric space A into subsets of X^{**} where $\Phi(t) \cap \widehat{X} \neq \emptyset$ for all $t \in A$. By Lemma 2.1, we may suppose that Φ is locally bounded on A. Consider a selection $\widetilde{\Phi}$ from A into X. Now $\widetilde{\Phi}$ is a weakly minimal, locally bounded single-valued mapping from A into X so is norm-continuous at the points of a residual subset D of A. It follows from Lemma 2.5 that Φ is single-valued and norm upper semi-continuous at the points of D. □

Although this characterisation enables our computation, it is somewhat unsatisfactory in that it does not give us significant information about the specific properties which identity GC spaces. When looking for a characterisation of GC spaces, it is logical to look for a condition which includes the sufficiency conditions which we have already given. A unifying condition can be found in the concept of fragmentability and its generalisation, [**R1**, p. 247].

Given a topological space X we say that a function $\lambda : X \times X \to \mathbb{R}$ is a *premetric* on X if

(i) $\lambda(x,y) \geq 0$ for all $x, y \in X$ and

(ii) $\lambda(x,y) = 0$ if and only if $x = y$, [**Sc**, p. 225].

We define what we will call the λ-*topology* on X as follows. A subset U of X is said to be λ-open if for every $x_0 \in U$ there exists an $r > 0$ such that $\{x \in X : \lambda(x, x_0) < r\} \subseteq U$. Given $x_0 \in X$ and $\epsilon > 0$, a subset of the form $\{x \in X : \lambda(x, x_0) < \epsilon\}$ is fundamental in defining the λ-topology but it is not necessarily λ-open. We say that λ *fragments* X if, given $\epsilon > 0$, for every non-empty subset E of X there exists a relatively open subset U of E such that

$$\lambda - \operatorname{diam}(U) \equiv \sup\{\lambda(x,y) : x, y \in U\} < \epsilon.$$

We note that the λ-topology on a subset E of X is *stronger* than the relative topology on E if for every $x_0 \in E$ and open set W containing x_0 there exists a $\delta > 0$ such that $\{x \in E : \lambda(x, x_0) < \delta\} \subseteq W$.

If a topological space X has a fragmenting premetric then there exists a fragmenting metric on X, [**R1**, p. 246]. A Banach space which has an equivalent rotund norm has a fragmenting metric for its weak topology, [**R2**]. We recall that $\ell_\infty(\mathbb{N})$ can be equivalently renormed to be rotund but $\ell_\infty(\Gamma)$, where Γ is uncountable, cannot, [**D**, p. 120; 123].

Theorem 2.7. *A Banach space X is a GC space if it possesses a premetric λ where every non-empty bounded set has slices of arbitrarily small λ-diameter, and where the λ-topology on bounded sets is stronger than the* *weak topology.*

Proof. Consider a weakly minimal locally bounded set-valued mapping Φ from a complete metric space A into subsets of X. Given $\epsilon > 0$, consider the set $O_\epsilon \equiv \cup\{\text{open sets } V \text{ in } A \text{ such that } \lambda\text{-diam } \Phi(V) < \epsilon\}$. Now O_ϵ is open in A; we show that it is dense in A. Consider any non-empty open set U in A where $\Phi(U)$ is bounded. Then there is a slice of $\Phi(U)$ with λ-diameter less than ϵ. Since Φ is weakly minimal, there exists a non-empty open set $V \subseteq U$ such that $\Phi(V)$ lies inside this slice and so λ-diam $\Phi(V) < \epsilon$. So O_ϵ is dense in A. Then $D \equiv \bigcap_{n\in\mathbb{N}} O_{\frac{1}{n}}$ is a dense G_δ subset of A where Φ is single-valued. Since the λ-topology is stronger than the weak topology on bounded set, Φ is single-valued and weakly continuous at the points of D. Now D is a dense G_δ subset of the complete metric space A so D is completely metrisable, [**K-N**, p. 96]. Then by Corollary 1.5 there exists a dense G_δ subset E of D and so of A where $\Phi|_D$ is norm continuous . We conclude from Lemma 2.5 that Φ is single-valued and norm upper semi-continuous at the points of E. Our result now follows from Theorem 2.6. □

We show that Theorem 2.7 includes Theorem 1.13. We do this using the following premetric. Given a rotund normed linear space X and using the notation $[x, y] \equiv \{\alpha x + (1 - \alpha)y : 0 \leq \alpha \leq 1\}$, we define the function $\lambda : X \times X \to \mathbb{R}$ by

$$\lambda(x, y) = \max\{||[x, y]||\} - \min\{||[x, y]||\}, [\mathbf{Sc}, \text{ p. } 226].$$

Clearly, $\lambda(x, x) = 0$. If $x \neq y$ then by rotundity $\lambda(x, y) \geq \max\{||[, y]||\} - \frac{1}{2}||x + y|| > 0$. So λ is a premetric on X.

We need the following properties of this premetric. Given $x_0 \in X$ and $r > 0$ we use the notation

$$B_\lambda(x_0; r) \equiv \{x \in X : \lambda(x, x_0) < r\}.$$

Lemma 2.8. *Given a rotund normed linear space X,*

(i) $\lambda(Kx, y) \leq \lambda(x, y) + 2|1 - K|\,||x||$ *for all* $K \neq 0$ *and* $x, y \in X$,

(ii) $B_\lambda(x; r) \subseteq (||x|| + r)B(X)$ *for all* $x \in X$,

(iii) *given* $x \in X$, *for* $K > 1$ *and* $0 < r < (K - 1)||x||$,

$$B_\lambda(x; r) \subseteq B_\lambda(Kx; r + 2|1 - K|\,||x||) \cap K||x||B(X).$$

Proof. (i) For $0 \leq \alpha \leq 1$, $||\alpha Kx + (1-\alpha)y|| \leq ||\alpha x + (1-\alpha)y|| + \alpha|1-K|\,||x||$, so $\max\{||[Kx, y]||\} \leq \max\{||[x, y]||\} + |1-K|\,||x||$. But also, $||\alpha x + (1-\alpha)y|| \leq ||\alpha Kx + (1-\alpha)y|| + \alpha|1-K|\,||x||$, so $\min\{||[x, y]||\} \leq \min\{||[Kx, y]||\} + |1-$

$K|\,||x||$. Therefore, $\max\{|||[Kx,y]|||\} - \min\{|||[Kx,y]|||\} \le \max\{|||[x,y]|||\} - \min\{|||[x,y]|||\} + 2|1-K|\,||x||$.

(ii) and (iii) come directly from the definition of λ and (i). □

We notice that if X is a weakly locally uniformly rotund normed linear space then given $x_0 \in X$, $x_0 \neq 0$ and $\epsilon > 0$ and $f \in X^*$, $||f|| = 1$, there exists $\delta(\epsilon, x_0, f) > 0$ such that $|f(x_0 - x)| < ||x_0||\epsilon$ when $x \in ||x_0||B(X)$ and $||x + x_0|| > ||x_0||(2-\delta)$. So if $\lambda(x, x_0) < ||x_0||\frac{\delta}{2}$ and $x \in ||x_0||B(X)$ then

$$\begin{aligned}\frac{1}{2}||x+x_0|| > \min\{|||[x,x_0]|||\} &> \max\{|||[x,x_0]|||\} - ||x_0||\frac{\delta}{2}\\ &> ||x_0||\left(1-\frac{\delta}{2}\right)\end{aligned}$$

so $||x+x_0|| > ||x_0||(2-\delta)$ and it follows that $|f(x_0 - x)| < ||x_0||\epsilon$.

Proposition 2.9. *A Banach space X which has an equivalent weakly locally uniformly rotund norm has a premetric λ where every non-empty bounded subset of X has slices of arbitrarily small λ-diameter and where the λ-topology is stronger than the weak topology.*

Proof. Consider X so renormed and the premetric λ defined above. Consider a non-empty bounded subset A of X and write $s \equiv \sup\{||x|| : x \in A\}$. If $s = 0$ then it is trivially true. If $s \neq 0$ then given $\epsilon > 0$ there exists an $f \in X^*$, $||f|| = 1$ such that the set $E \equiv A \cap S(sB(X), f, \epsilon) \neq \emptyset$. For $x, y \in E$ and writing $r \equiv \max\{||x||, ||y||\} \le s$ we note that $x, y \in S(rB(X), f, \epsilon + r - s)$ and so $\lambda\text{-diam}\, E < \epsilon$.

To show that the λ-topology is stronger than the weak topology it is sufficient to show that each subbasic weak open set is λ-open. At 0 the norm and λ-topologies agree so we consider neighbourhoods of $x_0 \in X$, $x_0 \neq 0$. Given $\epsilon > 0$ consider the weak open subbasic set

$$W \equiv \{x \in X : |f(x) - f(x_0)| < 3\epsilon||x_0||\} \text{ where } f \in X^*,\ ||f|| = 1.$$

Now we have that there exists a $\delta(\epsilon, x_0, f) > 0$ such that $|f(x_0 - x)| < ||x_0||\epsilon$ when $\lambda(x, x_0) < ||x_0||\frac{\delta}{2}$ and $x \in ||x_0||B(X)$. Choose $1 < K < 2$ such that $K - 1 < \min\left\{\frac{\delta}{8}, \frac{\epsilon||x_0||}{|f(x_0)|+1}\right\}$ and then choose $0 < r < \min\left\{||x_0||\frac{\delta}{4}, (K-1)||x_0||\right\}$. From Lemma 2.8(iii) we have that

$$\begin{aligned}B_\lambda(x_0; r) &\subseteq B_\lambda(Kx_0; r + 2(K-1)||x_0||) \cap K||x_0||B(X)\\ &\subseteq B_\lambda\left(Kx_0; ||x_0||\frac{\delta}{2}\right) \cap K||x_0||B(X)\end{aligned}$$

by the choice of K and r.

So $B_\lambda(x_0;r) \subseteq B_\lambda(Kx_0; K||x_0||\frac{\delta}{2}) \cap K||x_0||B(X)$. Therefore $|f(Kx_0 - x)| < K||x_0||\epsilon$ when $x \in B_\lambda(x_0;r)$. But then

$$\begin{aligned} |f(x_0) - f(x)| &\le |f(x_0) - Kf(x_0)| + |f(Kx_0) - f(x)| \\ &< (K-1)|f(x_0)| + K||x_0||\epsilon \\ &< \frac{\epsilon||x_0||\,|f(x_0)|}{|f(x_0)|+1} + K||x_0||\epsilon \\ &< 3\epsilon||x_0||. \end{aligned}$$

So $B_\lambda(x_0;r) \subseteq W$ and we conclude that the λ-topology is stronger than the weak topology on X. □

It is straight forward to show that Theorem 2.7 includes Theorem 2.3. This follows directly from the following lemma.

Lemma 2.10. *A Banach space X where there exists a weak * lower semi-continuous norm $|||\cdot|||$ on X^{**} has the $|||\cdot|||$-topology stronger than the weak topology on bounded subsets of X.*

Proof. Consider a bounded subset A of X, $x_0 \in A$ and a subbasic weak open neighbourhood of x_0 in A, $W \equiv \{x \in A : |f(x) - f(x_0)| < \epsilon\}$ for $\epsilon > 0$ and $f \in X^*$, $||f|| = 1$. Given $r > 0$ the closed ball $B^{**}_{|||\cdot|||}[\widehat{x}_0;r]$ is weak * closed so $B^{**}_{|||\cdot|||}[\widehat{x}_0;r] \cap (A \setminus W)$ is weak * compact. If $B^{**}_{|||\cdot|||}[\widehat{x}_0;\frac{1}{n}] \cap (A \setminus W) \neq \emptyset$ for all $n \in \mathbb{N}$ then there exists an $F \in \bigcap_{n\in\mathbb{N}} B^{**}_{|||\cdot|||}[\widehat{x}_0;\frac{1}{n}] \cap (A \setminus W)$. But this would contradict the fact that $F \neq \widehat{x}_0$. So there exists an $r > 0$ such that $B^{**}_{|||\cdot|||}(x_0;r) \subseteq W$ and we conclude that the $|||\cdot|||$-topology is stronger than the weak topology on A. □

3. A Banach space which is not a *GC* space.

The Banach space $\ell_\infty(\Gamma)$, where Γ is uncountable, is not a *GC* space. To show this we exhibit a complete metric space P and a weakly minimal, locally bounded set-valued mapping Φ from P into subsets of $\ell_\infty(\Gamma)$ where for each $p \in P$, $\Phi(p)$ is not singleton. Our argument is completed by an appeal to the characterisation given in Theorem 2.6. The construction is based on an example of Talagrand [**Ta**].

We denote by X the set of characteristic functions of countable subsets of Γ with the topology of uniform convergence on countable subsets of Γ. A base of neighbourhoods for $x_0 \in X$ is given by sets of the form $U(x_0, J) \equiv \{x \in X : x|_J = x_0|_J\}$ where J is a countable subset of Γ.

We use the technique of the Banach-Mazur game played on the topological space X, [**C**, p. 115]. This is a game between two players α and β where each player chooses alternately a non-empty open set contained in the other's previously chosen set. Player β begins by choosing U_1. When β chooses U_n then α chooses V_n where $U_n \supseteq V_n$; when α chooses V_n then β chooses U_{n+1} where $V_n \supseteq U_{n+1}$. The sequence of open sets

$$U_1 \supseteq V_1 \supseteq U_2 \supseteq V_2 \supseteq \cdots \supseteq U_n \supseteq V_n \supseteq \cdots$$

is called a *play*. The player α *wins* this play if $\bigcap_{n \in \mathbb{N}} V_n \neq \emptyset$. The game is said to be *α-favourable* if there exists a *winning tactic* by which α chooses V_n dependent only on how β chooses U_n so that α always wins.

Although the following lemma was proved in [**Ta**, p. 160], we will subsequently need to refer to the α-winning tactic used in our proof.

Lemma 3.1. *The topological space X is α-favourable.*

Proof. We define an α-tactic as follows:

For each open set U in X choose a point $x \in U$ and a basic neighbourhood

$$V \equiv U(x, J) \subseteq U.$$

Each play, $U_1 \supseteq V_1 \supseteq U_2 \supseteq V_2 \supseteq \cdots \supseteq U_n \supseteq V_n \supseteq \cdots$ generates a decreasing sequence of basic neighbourhoods

$$V_1 \equiv U(x_1, J_1) \supseteq V_2 \equiv U(x_2, J_2) \supseteq \cdots V_n \equiv U(x_n, J_n) \supseteq \cdots.$$

Clearly, $J_n \subseteq J_{n+1}$ for each $n \in \mathbb{N}$ and each x_{n+1} is an extension of $x_n|_{J_n}$ to J_{n+1}. So we can define a function x_* on Γ as an extension of $x_n|_{J_n}$ for each $n \in \mathbb{N}$ on $J \equiv \bigcup_{n \in \mathbb{N}} J_n$ and zero on $\Gamma \setminus J$. Since J is countable, $x_* \in X$. But also $x_* \in \bigcap_{n \in \mathbb{N}} U(x_n, J_n)$ so we have an α-winning tactic. $\square$

We note that $U(x_*, J) \subseteq \bigcap_{n \in \mathbb{N}} U(x_n, J_n)$ and $U(x_*, J)$ has infinitely many elements.

In Lemma 3.1 we produced an α-winning tactic. We now consider the set $\mathcal{P}$ of all plays

$$p \equiv (U_n, V_n) \equiv U_1 \supseteq V_1 \supseteq U_2 \supseteq V_2 \supseteq \cdots \supseteq U_n \supseteq V_n \supseteq \cdots$$

which follow such an α-winning tactic, with metric ρ defined by

$$\rho(p, p) = 0 \text{ for each } p \in \mathcal{P} \text{ and}$$
$$\text{if } p' \equiv (U'_n, V'_n) \neq (U''_n, V''_n) \equiv p'' \text{ then}$$
$$\rho(p', p'') = \frac{1}{n} \text{ where } n \text{ is the first integer where } U'_n \neq U''_n.$$

If for some $n \in \mathbb{N}$, $U'_n = U''_n$ then from the definition of the play for such an α-winning tactic, $V'_n = V''_n$.

Lemma 3.2. *The metric space $\mathcal{P}$ is complete.*

Proof. Consider a Cauchy sequence $\{p^k \equiv (U^k_n, V^k_n)\}$ in $\mathcal{P}$. Then for every $n \in \mathbb{N}$ there exists some $k_n \geq n$ such that $U^{k_n}_i = U^k_i, V^{k_n}_i = V^k_i$ whenever $1 \leq i \leq n$ and $k \geq k_n$. So we can define a new play $p^* \in \mathcal{P}$ by

$$p^* \equiv (U^{k_n}_n, V^{k_n}_n) \text{ and } \rho(p^k, p^*) \to 0 \text{ as } k \to \infty.$$

□

A similar metric space was studied in [**K-O**, Prop. 2.1].

We now consider the natural embedding π of the topological space X into the Banach space $\ell_\infty(\Gamma)$. For $x_1, x_2 \in X$, $x_1 \neq x_2$ we have that $||\pi(x_1) - \pi(x_2)||_\infty = 1$ and so it is clear that this embedding is nowhere norm continuous on X. However, the natural embedding π of X into $\ell_\infty(\Gamma)$ with its weak topology is continuous at every point of X. We will establish this through two preliminary lemmas.

Given $x \in X$, we denote by $s(x)$ the *support* of x; that is, $s(x) = \{t \in \Gamma : x(t) = 1\}$. Our first result follows from Zorn's lemma.

Lemma 3.3. *Given $f \in \ell^*_\infty(\Gamma)$ which is not identically zero on $\pi(X)$ there exists a non-empty subset A of X which is maximal with respect to the properties*

(i) *$\{s(x) : x \in A\}$ is disjoint family in Γ; that is, for $x_1, x_2 \in A$, $x_1 \neq x_2$ we have $s(x_1) \cap s(x_2) = \emptyset$, and*

(ii) *$f(\pi(x)) \neq 0$ for each $x \in A$.*

Lemma 3.4. *The set A is countable.*

Proof. Given $\epsilon > 0$, consider the set $A_\epsilon \equiv \{x \in A : |f(\pi(x))| \geq \epsilon\}$. Now $A = \bigcup_{n \in \mathbb{N}} A_{\frac{1}{n}}$ so it is sufficient to prove that for every $\epsilon > 0$, A_ϵ is finite. Suppose that for some $r > 0$, A_r is infinite. Then one of the sets $A^+_r \equiv \{x \in A : f(\pi(x)) > r\}$ or $A^-_r \equiv \{x \in A : f(\pi(x)) < -r\}$ will be infinite. We may suppose that A^+_r is infinite. For any finite subset A' of A^+_r we have from property (i) of Lemma 3.3 that $\sum_{x \in A'} \pi(x)$ belongs to the closed unit ball $B(\ell_\infty(\Gamma))$. But $f\left(\sum_{x \in A'} \pi(x)\right) = \sum_{x \in A'} f(\pi(x)) > |A'|r$ where $|A'|$ denotes the number of elements in the finite set A'. But this implies that f is not bounded on $B(\ell_\infty(\Gamma))$ which contradicts the continuity of f. □

We are now in a position to establish our continuity property.

Lemma 3.5. *The natural embedding π of the topological space X into $\ell_\infty(\Gamma)$ with its weak topology is continuous at every point of X.*

Proof. Consider $f \in \ell^*_\infty(\Gamma)$. If f is identically zero on $\pi(X)$ then the result is obvious. Suppose f is not identically zero on $\pi(X)$. Then from Lemma 3.4,

$$J^* \equiv \bigcup\{s(x) : x \in A\} \text{ is a countable subset of } \Gamma.$$

Denote by x^* the characteristic function of J^* on Γ. For every $x \in X$ we have $x = x.x^* + x.(1 - x^*)$, so $f(\pi(x)) = f(\pi(x.x^*)) + f(\pi(x.(1 - x^*)))$. But $s(x.(1 - x^*)) \subseteq s(x) \cap (\Gamma \setminus J^*)$ so $x.(1 - x^*) \in X \setminus A$. Since A is maximal with respect to properties (i) and (ii) of Lemma 3.3, we deduce that $f(\pi(x.(1 - x^*))) = 0$. Therefore, $f(\pi(x)) = f(\pi(x.x^*))$ for all $x \in X$. Now consider $x_0 \in X$ and a basic neighbourhood $U(x_0, J^*)$. For any $x \in U(x_0, J^*)$ we have $x|_{J^*} = x_0|_{J^*}$ and so $x.x^* = x_0.x^*$. Then $f(\pi(x)) = f(\pi(x.x^*)) = f(\pi(x_0.x^*)) = f(\pi(x_0))$. This implies the required continuity of the natural embedding π. □

We now consider the set-valued mapping Φ from $\mathcal{P}$ into subsets of $\ell_\infty(\Gamma)$ defined for the play $p \equiv (U_n, V_n) \in \mathcal{P}$ by

$$\Phi(p) = \bigcap_{n \in \mathbb{N}} \pi(U_n) = \bigcap_{n \in \mathbb{N}} \pi(V_n).$$

It is this set-valued mapping which establishes that $\ell_\infty(\Gamma)$ is not a GC space.

Theorem 3.6. *The set-valued mapping Φ from $\mathcal{P}$ into subsets of $\ell_\infty(\Gamma)$ is weakly minimal, locally bounded and for each $p \in \mathcal{P}$, $\Phi(p)$ is not singleton.*

Proof. Clearly, for each $p \in \mathcal{P}$, $\Phi(p) \subseteq B(\ell_\infty(\Gamma))$. For each play $p \equiv (U_n, V_n)$ we note from Lemma 3.1 that the set $E_p \equiv \bigcap_{n \in \mathbb{N}} U_n = \bigcap_{n \in \mathbb{N}} V_n$ is a subset of X which contains more than one point. So for each $p \in \mathcal{P}$, $\Phi(p) = \bigcap_{n \in \mathbb{N}} \pi(U_n)$ is not singleton.

Consider $f \in \ell^*_\infty(\Gamma)$ generating a weak open half-space W in $\ell_\infty(\Gamma)$ and play $p^o \equiv (U^o_n, V^o_n) \in \mathcal{P}$ such that $x^o \in \Phi(p^o) \cap W$. Now by Lemma 3.5, the natural embedding π of X into $\ell_\infty(\Gamma)$ is weakly continuous so $\pi^{-1}(W)$ is a non-empty open subset of X. Given $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $n_0 > \frac{1}{\delta}$ consider any play $p' \equiv (U'_n, V'_n) \in \mathcal{P}$ such that $U'_i = U^o_n$, $V'_i = V^o_i$ for all $1 \le i \le n_0$ and $U'_{n_0+1} = U^o_{n_0+1} \cap \pi^{-1}(W)$. Now $\rho(p', p^o) < \frac{1}{n_0} < \delta$. But since $\pi(U'_{n_0+1}) \subseteq W$ we have $\Phi(p') \subseteq W$. So Φ is weakly minimal. □

Note added in proof

Professor Isaac Namioka has recently given an example to show that $\ell_\infty(\mathbb{N})$ is not a GC space.

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