PACIFIC JOURNAL OF MATHEMATICS
Vol. 173, No. 1, 1996

THE WEYL QUANTIZATION OF POISSON $SU(2)$

ALBERT JEU-LIANG SHEU

In this paper, we consider the problem of quantizing the canonical multiplicative Poisson structure on $SU(2)$ by C*-algebraic deformation, a notion introduced by Rieffel, and show that there is such a deformation which is also a coalgebra homomorphism. Parallel to the algebraic development of quantum group theory, Woronowicz successfully quantized the group structure of $SU(2)$ (and other groups) through deformation in the context of Hopf C*-algebras. It is known that there exists a C*-algebraic deformation quantization of the multiplicative Poisson structure on $SU(2)$ which is 'compatible' with Woronowicz's deformation (of the group structure) on the C*-algebra level. Although that deformation preserves the important symplectic leaf structure on $SU(2)$ in a natural way, it does not preserve the group struture in the sense that it is not a coalgebra homomorphism. We show that the Weyl transformation introduced by Dubois-Violette gives a different C*-algebraic deformation quantization which is compatible with Woronowicz's deformation and does preserve the group structure.

1. Introduction.

In recent years, there has been a fast growing interest in the theory of deformation quantization (initiated in [**Ge**], [**BFFLS**]), which fits nicely with the concept of non-commutative geometry [**C1**]. There are many papers in this area using different approaches, either analytic, algebraic, geometric, or physical, and some of them have an extensive reference list [**R5**], [**Wo2**], [**K**], [**Gi**], [**Dr**], [**We-X**]. We refer readers to these sources for a broad overview.

It is known [**Sh1**] that there exists a C*-algebraic deformation quantization, a concept introduced by Rieffel [**R1**], of the multiplicative Poisson structure on $SU(2)$ [**Lu-We**] which is compatible with Woronowicz's C*-algebraic deformation quantization of the group structure on $SU(2)$ [**Wo1**], in the sense that the C*-algebras obtained in these two processes are isomorphic. This result shows that on the algebra level, the multiplicative Poisson structure can be deformed in a way compatible with the deformed group structure of $SU(2)$ during quantization.

However, Rieffel's [**R1**], [**R2**], [**R3**] and Woronowicz's [**Wo1**], [**Wo2**] formulations of (deformation) quantization are different in one aspect, i.e. in Rieffel's formulation most of the smooth functions (on the Poisson manifold) must be explicitly deformed to specific operators (or elements in a C*-algebra) for each Plank constant h, but Woronowicz's formulation only requires explicit deformation of (finitely many) generators of the commutative function algebra (of the manifold) so that the deformed generators satisfy some specific commutation relations. It is highly desirable to have a Rieffel's deformation of the Poisson $SU(2)$ which when restricted to those generators satisfies Woronowicz's commutation relations. But the deformation found in [**Sh1**] does not seem to have this property. Furthermore, since the coalgebra structure gives the group (action) structure, it is also desired to have a deformation which respects the coalgebra structure. It is now known that the above deformation is not a coalgebra homomorphism [**Sh4**].

In [**Du**], an interesting correspondence called the Weyl transformation from regular functions on $SU(2)$ to elements of the algebra $C(S_\mu U(2))$ of "functions" on the twisted $SU(2)$ [**Wo1**] is proposed, and is shown to be a coalgebra isomorphism. In this paper, we show that this Weyl transformation gives indeed a C*-algebraic deformation quantization of the Poisson $SU(2)$. One obvious advantage of this deformation is that its restriction to the generators of $C(SU(2))$ gives exactly the deformed generators in Woronowicz's theory and hence establishes a direct link between Rieffel's and Woronowicz's deformation theories on the Poisson Lie group $SU(2)$. This strengthens our claim in [**Sh1**] that the group structure and the Poisson structure on $SU(2)$ are not only compatible on the classical level, but also compatible on the quantum level, since now they are not only compatible globally (i.e. as algebras) but also locally (i.e. for individual functions on $SU(2)$).

Comparing the deformation quantizations obtained here and in [**Sh1**], we have the following remark. In [**Sh1**], the deformation quantization is obtained by a (symplectic) "leaf-wise" quantization through pseudo-differential operators and hence is "leaf-preserving" in the sense to be explained in Section 5. This property allows us to view the quantization as giving some kind of singular foliation C*-algebra [**Sh3**] resembling the important example of foliation operator algebras in non-commutative geometry [**C2**]. Moreover the approach used there, now, fits nicely in the far more general framework developed in [**R5**] which is involved with external $\mathbb{R}^d$-actions. All these show some advantage in that approach. However, that approach does not work well with the internal $SU(2)$-action or the comultiplication structure [**Sh4**]. On the other hand, the C*-algebraic deformation quantization obtained in this paper is compatible with Woronowicz's quantization of group structure, and with the internal $SU(2)$-action and hence the comultiplication on $C(S_\mu U(2))$ [**Du**]. But it applies to an agebra (consisting of regular functions) smaller than the algebra of smooth functions dealt with by the earlier approach, and it is not "leaf-preserving" with respect to the foliation of $SU(2)$ by symplectic leaves.

The author would like to dedicate this paper to the memory of Professor John W. Bunce, an honorable colleague and mathematician whose contribution in the theory of C*-algebras, for example, the Bunce-Deddens algebra, is already a fundamental part of the theory. After this paper was written, we received Bauval's preprint [**B**] with similar results. However the concepts of universal Poisson algebra and leaf-wise deformation quantization introduced in this paper are not discussed there.

2. Quantum $SU(2)$.

In this section, we shall realize the C*-algebra family $C(S_\mu U(2))$ of twisted $SU(2)$, introduced and studied by Woronowicz in [**Wo1**], as a continuous field of C*-algebras.

Let $\boldsymbol{A}$ be the universal C*-algebra generated by $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, and ν, subject to the relations

$$
\begin{array}{ll}
\boldsymbol{\alpha}^*\boldsymbol{\alpha} + \boldsymbol{\gamma}\boldsymbol{\gamma}^* = 1, & \boldsymbol{\alpha}\boldsymbol{\alpha}^* + \nu^2\boldsymbol{\gamma}\boldsymbol{\gamma}^* = 1, \\
\boldsymbol{\alpha}\boldsymbol{\gamma} = \nu\boldsymbol{\gamma}\boldsymbol{\alpha}, & \boldsymbol{\alpha}\boldsymbol{\gamma}^* = \nu\boldsymbol{\gamma}^*\boldsymbol{\alpha}, \\
\boldsymbol{\gamma}\boldsymbol{\gamma}^* = \boldsymbol{\gamma}^*\boldsymbol{\gamma}, & \nu\boldsymbol{\alpha} = \boldsymbol{\alpha}\nu, \quad \nu\boldsymbol{\gamma} = \boldsymbol{\gamma}\nu, \\
\nu^* = \nu, & \nu\nu^* \leq 1.
\end{array}
\tag{1}
$$

Clearly the C*-algebra $C^*(\nu)$ generated by ν is a central subalgebra of $\boldsymbol{A}$. It is well known that $C([-1,1])$ is the universal C*-algebra generated by a self-adjoint element of norm bounded by 1, and hence there is a homomorphism form $C([-1,1])$ to $\boldsymbol{A}$ identifying the identity function $id : \mu \longmapsto \mu$ with ν. On the other hand, the universality of $\boldsymbol{A}$ implies the existence of a homomorphism sending $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, and ν to 1, 0, and id in $C([-1,1])$, respectively. From this observation, it is clear that $C^*(\nu) \cong C([-1,1])$.

Let $\mathcal{I}_\mu = \{f \in C([-1,1]) \mid f(\mu) = 0\}$ for $\mu \in [-1,1]$. Then the surjective homomorphism $\pi_\mu : \boldsymbol{A} \longrightarrow C(S_\mu U(2))$ sending $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, and ν to α, γ, and μ, respectively, factors through $\boldsymbol{A}/(\mathcal{I}_\mu \boldsymbol{A})$ and hence induces a homomorphism $\pi'_\mu : \boldsymbol{A}/(\mathcal{I}_\mu \boldsymbol{A}) \longrightarrow C(S_\mu U(2))$. We use the same symbols α, γ to denote the generators of $C(S_\mu U(2))$for all μ. Hopefully the context will clarify any possible ambiguity. When $\mu = 1$, $C(S_\mu U(2))$ is identified with $C(SU(2))$ so that α, γ are identified with the canonical coordinate functions on $SU(2)$ sending $\begin{pmatrix} \alpha & -\bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix}$ in $SU(2)$ to α, γ respectively, and in this case, $\alpha^* = \bar{\alpha}$,

$\gamma^* = \bar{\gamma}$. But by the universality of $C(S_\mu U(2))$, there is a homomorphism sending α, γ in $C(S_\mu U(2))$ to the calsses of $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$ in $\boldsymbol{A}/(\mathcal{I}_\mu \boldsymbol{A})$, respectively, which can be easily seen to be an (the) inverse of π'_μ. So we have $\boldsymbol{A}/(\mathcal{I}_\mu \boldsymbol{A}) \cong C(S_\mu U(2))$.

Thus by Proposition 1.2 of [**R4**], the family $\{C(S_\mu U(2))\}_{\mu\in[-1,1]}$ is an upper semi-continuous field of C*-algebras over $[-1,1]$ with $\mu \longmapsto \|\pi_\mu(a)\|$ upper semi-continuous for any $a \in \boldsymbol{A}$. Furthermore, since $C([-1,1])$ is unital, $\boldsymbol{A} = C([-1,1])\boldsymbol{A}$ is identified with a maximal algebra of cross sections of $\{C(S_\mu U(2))\}_{\mu\in[-1,1]}$ by identifying $\boldsymbol{a} \in \boldsymbol{A}$ with the map $\mu \longmapsto \{\pi_\mu(\boldsymbol{a})\}_{\mu\in[-1,1]}$.

Next we would like to show the lower semi-continuity of the field $\{C(S_\mu U(2))\}_{\mu\in[-1,1]}$. In order to do so, it is helpful, as suggested in [**R4**], to have a concrete faithful representation of $\boldsymbol{A}$ disintegrated into a field of representations of $C(S_\mu U(2))$.

First there is a well-known faithful representation σ_μ of $C(S_\mu U(2))$ with $|\mu| < 1$ which realizes α and γ in $C(S_\mu U(2))$ as the operators

$$\sigma_\mu(\alpha) = 1 \otimes \left(\sum_{k\in\mathrm{N}} \sqrt{1-\mu^{2k}}\, e_{k,k+1}\right)$$

and

$$\sigma_\mu(\gamma) = \tau \otimes \left(\sum_{k\in\mathrm{N}} \mu^{k-1} e_{k,k}\right)$$

on the Hilbert space $L^2(\mathbb{T}) \otimes \ell^2(\mathbb{N})$, where τ is the multiplication operator on $L^2(\mathbb{T})$ by the identity map $e^{i\theta}$ on the unit circle $\mathbb{T}$ and $e_{i,j}$ is the matrix element on $\ell^2(\mathbb{N})$ sending the canonical basis δ_k to δ_i if $k = j$ and to 0 if $k \neq j$. Since $\sigma_\mu(\alpha)$ and $\sigma_\mu(\gamma)$ are norm continuous in μ, it is easy to see that $\sigma_\mu(a)$ is norm continuous and hence $\|\pi_\mu(\boldsymbol{a})\|$ is continuous in $\mu \in (-1,1)$ for any $\boldsymbol{a} \in A$. So we only need to show the lower semi-continuity at $\mu = 1$.

In [**Wo1**], Woronowicz showed that $\pi_\mu(\boldsymbol{\alpha}^k\boldsymbol{\gamma}^n\boldsymbol{\gamma}^{*m})$ and $\pi_\mu(\boldsymbol{\alpha}^{*k}\boldsymbol{\gamma}^n\boldsymbol{\gamma}^{*m})$ form a linear basis of the smooth algebra $C(S_\mu U(2))^\infty$ of regular functions on the twisted pseudogroup $S_\mu U(2)$ [**V-So**], i.e. the *-algebra generated by α, γ (algebraically) in $C(S_\mu U(2))$, for $\mu \in (0,1]$, and there exists a faithful state h_μ, the Haar measure, on $C(S_\mu U(2))^\infty$ satisfying

$$h_\mu((\gamma^*\gamma)^m) = \frac{1-\mu^2}{1-\mu^{2m+2}} \tag{2}$$

and

$$h_\mu(\alpha^k\gamma^n\gamma^{*m}) = h_\mu(\alpha^{*k}\gamma^n\gamma^{*m}) = 0 \tag{3}$$

for $k \neq 0$, or $n \neq m$, for $0 < \mu < 1$. Since h_μ is faithful, $C(S_\mu U(2))^\infty$ is embedded in a Hilbert space $\mathcal{H}_\mu$ and the simple C*-algebra $C(S_\mu U(2))$ is represented faithfully by extensions of multiplication operators on $\mathcal{H}_\mu$, through the GNS construction [**P**]. Let σ_μ be these representations. Clearly $\alpha^k \gamma^n \gamma^{*m}$ and $\alpha^{*k}\gamma^n\gamma^{*m}$ with k, m, n nonnegative integers form a complete linearly independent set in each $\mathcal{H}_\mu$.

The identification of a point (α, γ) in the unit sphere $\mathbb{S}^3 \subset \mathbb{C}^2$ with $\begin{pmatrix} \alpha & -\bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix} \in SU(2)$ gives an identification of the Lebesgue measure on the sphere with the Haar measure on $SU(2)$ (up to the constant factor $1/(8\pi^2)$). In fact, using the parametrization of $\mathbb{S}^3$ by $\Psi : \mathbb{D} \times [0, 2\pi] \longrightarrow \mathbb{S}^3$ sending (α, θ) to $(\alpha, \gamma) = (\alpha, \sqrt{1 - |\alpha|^2} e^{i\theta})$, we can verify that

$$\int_{\mathbb{S}^3} \alpha^k \gamma^n \bar{\gamma}^m = \begin{cases} 0 & \text{if } k \neq 0, \text{ or } n \neq m \\ \frac{2\pi^2}{n+1} & \text{if } k = 0 \text{ and } n = m. \end{cases}$$

So the normalized Haar measure h_1 on $SU(2)$ satisfies that

$$h_1(\alpha^k \gamma^n \bar{\gamma}^m) = \begin{cases} 0 & \text{if } k \neq 0, \text{ or } n \neq m \\ \frac{1}{n+1} & \text{if } k = 0 \text{ and } n = m. \end{cases} \tag{4}$$

Similarly the above equation holds when α is replaced by $\bar{\alpha}$. Note that the faithful Haar measure h_1 on $C(SU(2))$ gives rise to the canonical faithful representation σ_1 of $C(SU(2))$ on $\mathcal{H}_1 = L^2(SU(2), h_1)$ by multiplication operators.

From Equations 2, 3, and 4, we have

$$\lim_{\mu \to 1} h_\mu(\pi_\mu(\boldsymbol{a})) = h_1(\pi_1(\boldsymbol{a}))$$

and hence

$$\lim_{\mu \to 1} \langle \pi_\mu(\boldsymbol{a}), \pi_\mu(\boldsymbol{b}) \rangle_\mu = \lim_{\mu \to 1} h_\mu(\pi_\mu(\boldsymbol{b})^* \pi_\mu(\boldsymbol{a})) = \lim_{\mu \to 1} h_\mu(\pi_\mu(\boldsymbol{b}^* \boldsymbol{a}))$$

$$= h_1(\pi_\mu(\boldsymbol{b}^* \boldsymbol{a})) = h_1(\pi_\mu(\boldsymbol{b})^* \pi_\mu(\boldsymbol{a})) = \langle \pi_\mu(\boldsymbol{a}), \pi_\mu(\boldsymbol{b}) \rangle_1 \tag{5}$$

for any linear combinations $\boldsymbol{a}, \boldsymbol{b}$ of $\boldsymbol{\alpha}^k \boldsymbol{\gamma}^n \boldsymbol{\gamma}^{*m}$'s and $\boldsymbol{\alpha}^{*k} \boldsymbol{\gamma}^n \boldsymbol{\gamma}^{*m}$'s, where $\langle \cdot, \cdot \rangle_\mu$ is the inner product on $\mathcal{H}_\mu$.

Observe that for any $\boldsymbol{a}$ in the *-algebra $\boldsymbol{\mathcal{A}}$ generated by ν, $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ in $\boldsymbol{A}$, i.e. $\boldsymbol{a}$ is a linear combination of monomials in ν, $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, $\boldsymbol{\alpha}^*$, and $\boldsymbol{\gamma}^*$, there are (finitely many) nonnegative integers k_i, k'_j, n_i, n'_j, m_i, m'_j (independent of

μ) such that

$$\pi_\mu(\boldsymbol{a})\pi_\mu(\boldsymbol{\alpha}^k\boldsymbol{\gamma}^n\boldsymbol{\gamma}^{*m}) = \sum_i f_i(\mu,\mu^{-1})\pi_\mu(\boldsymbol{\alpha}^{k_i}\boldsymbol{\gamma}^{n_i}\boldsymbol{\gamma}^{*m_i}) + \sum_j g_j(\mu,\mu^{-1})\pi_\mu\left(\boldsymbol{\alpha}^{*k'_j}\boldsymbol{\gamma}^{n'_j}\boldsymbol{\gamma}^{*m'_j}\right)$$

for all $\mu \in (0,1]$ with f_i, g_j polynomials in two variables, because of the commutation relations in Equation 1. Similar formula holds for

$$\pi_\mu(\boldsymbol{a})\pi_\mu(\boldsymbol{\alpha}^{*k}\boldsymbol{\gamma}^n\boldsymbol{\gamma}^{*m}),$$

too. Let $\mathcal{K}$ be a linear space with basis B consisting of finitely many monomials of the form $\boldsymbol{\alpha}^k\boldsymbol{\gamma}^n\boldsymbol{\gamma}^{*m}$ or $\boldsymbol{\alpha}^{*k}\boldsymbol{\gamma}^n\boldsymbol{\gamma}^{*m}$, and let $\mathcal{K}_\mu$ be $\pi_\mu(\mathcal{K})$ embedded in $\mathcal{H}_\mu$. By the above observation, it is easy to see that there is a suitably large $\mathcal{K}'$ with basis B' consisting of monomials of the form $\boldsymbol{\alpha}^k\boldsymbol{\gamma}^n\boldsymbol{\gamma}^{*m}$ or $\boldsymbol{\alpha}^{*k}\boldsymbol{\gamma}^n\boldsymbol{\gamma}^{*m}$ such that $\pi_\mu(\boldsymbol{a})\pi_\mu(\mathcal{K}) \subset \pi_\mu(\mathcal{K}')$ and hence $\sigma_\mu(\pi_\mu(\boldsymbol{a}))\mathcal{K}_\mu \subset \mathcal{K}'_\mu$. With respect to the bases $\pi_\mu(B)$ and $\pi_\mu(B')$ embedded in $\mathcal{H}_\mu$, the entries in the matrix representation of $\sigma_\mu(\pi_\mu(\boldsymbol{a}))$ are polynomials $f_{i,j}(\mu,\mu^{-1})$ which are clearly continuous in $\mu \in (0,1]$. Since $\mathcal{K}_\mu, \mathcal{K}'_\mu$ are finite dimensional with inner products satisfying Equation 5, we have

$$\lim_{\mu\to 1} \|\sigma_\mu(\pi_\mu(\boldsymbol{a}))\|_{\mathcal{B}(\mathcal{K}_\mu,\mathcal{K}'_\mu)} = \|\sigma_1(\pi_\mu(\boldsymbol{a}))\|_{\mathcal{B}(\mathcal{K}_1,\mathcal{K}'_1)}.$$

Since $\|\pi_1(\boldsymbol{a})\| = \|\sigma_1(\pi_1(\boldsymbol{a}))\|_{\mathcal{B}(\mathcal{H}_1)}$ can be approximated by

$$\|\sigma_1(\pi_1(\boldsymbol{a}))\|_{\mathcal{B}(\mathcal{K}_1,\mathcal{K}'_1)}$$

with sufficiently large $\mathcal{K}$ and

$$\|\sigma_\mu(\pi_\mu(\boldsymbol{a}))\|_{\mathcal{B}(\mathcal{K}_\mu,\mathcal{K}'_\mu)} \le \|\sigma_\mu(\pi_\mu(\boldsymbol{a}))\|_{\mathcal{B}(\mathcal{H}_\mu)} = \|\pi_\mu(\boldsymbol{a})\|,$$

we get $\|\pi_\mu(\boldsymbol{a})\|$ lower semi-continuous at $\mu = 1$. Since linear combinations of monomials in $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$ are dense in $\boldsymbol{A}$, the conclusion holds for any $\boldsymbol{a} \in \boldsymbol{A}$.

Summarizing, we have

Theorem 1. *The universal C^*-algebra A is identified with a maximal algebra of cross sections of $\{C(S_\mu U(2))\}_{\mu\in[-1,1]}$ by the map $\boldsymbol{a} \longmapsto \{\pi_\mu(a)\}_{\mu\in[-1,1]}$. The family $\{C(S_\mu U(2))\}_{\mu\in[-1,1]}$ is an upper semi-continuous field of C^*-algebras over $[-1,1]$ with $\mu \longmapsto \|\pi_\mu(\boldsymbol{a})\|$ continuous in $\mu \in (-1,1]$ for any $\boldsymbol{a} \in \boldsymbol{A}$.*

After this paper was written up, the author learned that G. Nagy has obtained results which generalize the above theorem (c.f. [**N**]).

Since our goal is to find a deformation quantization of $SU(2)$, we are only interested in the behavior of $C(S_\mu U(2))$ for μ close to 1, and so in the rest of this paper we shall fix an $\epsilon > 0$ and add

$$\nu \geq \epsilon \tag{6}$$

to the identities in Equation 1. We use the same notion $\boldsymbol{A}$ to denote the corresponding universal C*-algebra. Then it is easy to see that the above argument still works and $\boldsymbol{A}$ is identified with a maximal algebra of cross sections of $\{C(S_\mu U(2))\}_{\mu\in[\epsilon,1]}$ by the map $\boldsymbol{a} \longmapsto \{\pi_\mu(\boldsymbol{a})\}_{\mu\in[\epsilon,1]}$. The family $\{C(S_\mu U(2))\}_{\mu\in[\epsilon,1]}$ is a continuous field of C*-algebras over $[\epsilon, 1]$ with $\mu \longmapsto \|\pi_\mu(\boldsymbol{a})\|$ continuous on $[\epsilon, 1]$ for any $\boldsymbol{a} \in \boldsymbol{A}$. Note that the generator ν is now invertible in A. So we now use $\boldsymbol{\mathcal{A}}$ to denote the *-algebra generated by ν, ν^{-1}, $\boldsymbol{\alpha}$, and $\boldsymbol{\gamma}$ in $\boldsymbol{A}$.

3. Weyl transformation.

Let $\mathcal{I} = C([\epsilon, 1])$ be the C*-subalgebra generated by ν in $\boldsymbol{A}$. Then we denote by $\boldsymbol{A} \otimes_{\mathcal{I}} \boldsymbol{A}$ the tensor algebra $\boldsymbol{A} \otimes \boldsymbol{A}$ modulo the ideal generated by $\nu \otimes 1 - 1 \otimes \nu$. It is easy to see that $f\boldsymbol{a} \otimes \boldsymbol{b} = \boldsymbol{a} \otimes f\boldsymbol{b}$ in $\boldsymbol{A} \otimes_{\mathcal{I}} \boldsymbol{A}$ for any $\boldsymbol{a}$, $\boldsymbol{b} \in \boldsymbol{A}$ and $f \in \mathcal{I}$. As in [**Wo1**], by the universality of $\boldsymbol{A}$ we can define the comultiplication $\Phi : \boldsymbol{A} \longrightarrow \boldsymbol{A} \otimes_{\mathcal{I}} \boldsymbol{A}$ on the algebra $\boldsymbol{A}$ as the C*-algebra homomorphism determined by

$$\Phi(\boldsymbol{\alpha}) = \boldsymbol{\alpha} \otimes \boldsymbol{\alpha} - \nu\boldsymbol{\gamma}^* \otimes \boldsymbol{\gamma}$$

and

$$\Phi(\boldsymbol{\gamma}) = \boldsymbol{\gamma} \otimes \boldsymbol{\alpha} + \boldsymbol{\alpha}^* \otimes \boldsymbol{\gamma}.$$

Applying the commutation relation $\boldsymbol{\alpha\gamma}^* = \nu\boldsymbol{\gamma}^*\boldsymbol{\alpha}$ to the first factor in the tensor product, we can easily see that, for $n \geq 0$ and $0 \leq k \leq n$,

$$\Phi(\boldsymbol{\alpha}^{n-k}\boldsymbol{\gamma}^{*k}) = (\boldsymbol{\alpha} \otimes \boldsymbol{\alpha} - \nu\boldsymbol{\gamma}^* \otimes \boldsymbol{\gamma})^{n-k}(\boldsymbol{\gamma}^* \otimes \boldsymbol{\alpha}^* + \boldsymbol{\alpha} \otimes \boldsymbol{\gamma}^*)^k$$

is of the form

$$\sum_{0\leq i\leq n} \boldsymbol{\alpha}^{n-i}\boldsymbol{\gamma}^{*i} \otimes \boldsymbol{\omega}^n_{ik}$$

where $\boldsymbol{\omega}^n_{ik}$ is a finite $\mathbb{Z}$-linear sum of monomials in the elements ν, ν^{-1}, $\boldsymbol{\alpha}$, $\boldsymbol{\alpha}^*$, $\boldsymbol{\gamma}$, and $\boldsymbol{\gamma}^*$ in $\boldsymbol{A}$. (Although there may be several such representations of $\boldsymbol{\omega}^n_{ik}$, we fix one for each $\boldsymbol{\omega}^n_{ik}$ throughout the discussion.) For example, we have

$$\begin{array}{ll} \boldsymbol{\omega}^0_{00} = 1, & \boldsymbol{\omega}^1_{00} = \boldsymbol{\alpha}, \\ \boldsymbol{\omega}^1_{10} = -\nu\boldsymbol{\gamma}, & \boldsymbol{\omega}^2_{11} = \boldsymbol{\alpha\alpha}^* - \boldsymbol{\gamma\gamma}^*, \\ \boldsymbol{\omega}^2_{02} = \boldsymbol{\gamma}^{*2}, & \boldsymbol{\omega}^2_{12} = \nu\boldsymbol{\alpha}^*\boldsymbol{\gamma}^* + \nu^{-1}\boldsymbol{\alpha}^*\boldsymbol{\gamma}^*. \end{array} \tag{7}$$

It is easy to see that $\{\pi_\mu(\boldsymbol{\omega}^n_{ik}) \mid n \geq 0,\ 0 \leq i,k \leq n\}$ is the set of matrix elements of all the canonical irreducible quantum group representations of $C(S_\mu U(2))$ found in [**Wo1**].

In [**Du**], a notion of Weyl transformation W_h is proposed. In the next section, we shall show that this transformation actually defines a C*-algebraic deformation quantization of the Poisson $SU(2)$ in the sense of Rieffel [**R1**]. Here we recall its definition. It is well known that a regular function f on $SU(2)$ can be written in a unique way as a linear combination $\sum_{n,i,k} \hat{f}^n_{ik}\pi_1(\boldsymbol{\omega}^n_{ik})$ of the orthogonal matrix elements $\pi_1(\boldsymbol{\omega}^n_{ik}) \in L^2(SU(2))$ of the irreducible representations of $SU(2)$, where

$$\hat{f}^n_{ik} = \|\pi_1(\boldsymbol{\omega}^n_{ik})\|_2^{-2} \int_{SU(2)} f\pi_1(\boldsymbol{\omega}^n_{ik})$$

are the Fourier coefficients of f. The Weyl transformation defined in [**Du**] (see the remark following Theorem 2) when restricted to the case of $SU(2)$ is the linear map $W_h : C(SU(2))^\infty \longrightarrow C(S_\mu U(2))^\infty$ with $\mu = e^{-h/2}$ and

$$W_h(f) = \sum_{n,i,k} \hat{f}^n_{ik}\pi_\mu(\boldsymbol{\omega}^n_{ik}). \tag{8}$$

By Theorem 3.1 of [**Du**], W_h is a coalgebra isomorphism. Note that we may define a linear map $W : C(SU(2))^\infty \longrightarrow \boldsymbol{A}$ by

$$W(f) = \sum_{n,i,k} \hat{f}^n_{ik}\boldsymbol{\omega}^n_{ik}$$

and then we have $W_h = \pi_\mu \circ W$.

Summarizing, we have

Theorem 2 (Dubois-Violette and Woronowicz). *The Weyl transformation $W_h : C(SU(2))^\infty \longrightarrow C(S_\mu U(2))^\infty$ definde by $W_h = \pi_\mu \circ W$ where $W : C(SU(2))^\infty \longrightarrow \boldsymbol{A}$ is defined by*

$$W\left(\sum_{n,i,k} c^n_{ik}\pi_1(\boldsymbol{\omega}^n_{ik})\right) = \sum_{n,i,k} c^n_{ik}\boldsymbol{\omega}^n_{ik}$$

is a coalgebra isomorphism for each $h = -2\ln(\mu)$. In particular, $W_1 = \pi_1 \circ W$ is the identity map on $C(SU(2))^\infty$.

We remark that in [**Du**], the term, Weyl transformation, is reserved for more restrictive transforms defined in the same way, but using only irreducible unitary representations to get the matrix elements $\boldsymbol{\omega}^n_{ik}$. It can be shown that the irreducible representations $\boldsymbol{\omega}^n$ which we use here can be unitarized by conjugating the matrices $(\omega_{ik}^n)_{ik}$ by some invertible diagonal matrices with entries in $C([\epsilon, 1])$ (actually polynomials of ν and ν^{-1}). We denote the matrix elements of the irreducible unitary representations constructed in this way by u_{ik}^n.

The proof of our result on deformation quantization of Poisson $SU(2)$, namely, Theorem 3, does not depend on the choice of the representatives of irreducible representations in defining Weyl transformations. That is, we can define W_h using u_{ik}^n's instead of ω_{ik}^n's. One reason for using only unitary irreducible representations to define Weyl transformations in [**Du**] is probably the claim made there saying that there are suitable "consistent" choices of the representatives of an irreducible unitary representation and its (presumably non-equivalent) conjugate irreducible unitary representation, the Weyl transformations will be *-preserving. This claim may actually be wrong since each of the unitary irreducible representations $\pi_\mu(u^n)$ mentioned above and its conjugate representation are in the same equivalent class of representations, and there is no way to make "consistent" choices of representatives to make the Weyl transform *-preserving. So the claim in [**Du**] that the Weyl transformation is a *-coalgebra isomorphism is not true here.

4. Universal Poisson algebras.

In this section, we consider the idea of noncommutative Poisson algebras for the convenience of later discussion. Usually, a Poisson algebra refers to a commutative algebra P with a Lie bracket $\{\cdot,\cdot\} : P \times P \longrightarrow P$ such that $\{a, \cdot\}$ and $\{\cdot, a\}$ are derivations of P, i.e.

$$\begin{aligned} \{a, bc\} &= \{a, b\}c + b\{a, c\}, \\ \{ab, c\} &= \{a, c\}b + a\{b, c\}, \end{aligned} \tag{9}$$

for all $a,\ b,\ c \in P$. But this definition does not need the commutativity of P in order to make sense. In fact, a (noncommutative) algebra A endowed with the commutator bracket $[a, b] = ab - ba$ for $a,\ b \in A$ is an example. So in this paper, Poisson algebra need not be commutative.

We consider the concept of universal Poisson algebra $\mathcal{P}(S)$ generated by elements in a set S (with no additional relations imposed). More precisely, $\mathcal{P}(S)$ is a Poisson algebra endowed with an embedding (i.e. an injection) $\iota : S \longrightarrow \mathcal{P}(S)$ such that any function from S to a Poisson algebra A extends uniquely to a Poisson algebra homomorphism from $\mathcal{P}(S)$ to A. Such a universal Possion algebra is clearly unique up to isomorphism. Actually it exists and can be constructed in the following way. We take one representative from each congruence class of morphisms $f : S \longrightarrow A_f$, where f is a function, A_f is a Poisson algebra which has no proper Poisson subalgebra containing $f(S)$, and two such morphisms f, g are congruent if there is a Poisson algebra isomorphism $h : A_f \longrightarrow A_g$ such that $g = h \circ f$. Let P be the Poisson algebra direct product $\prod_{[f]} A_f$ with exactly one A_f from each congruence class $[f]$. Then we define $\mathcal{P}(S)$ to be the smallest Poisson subalgebra of P containing all $\prod_{[f]} f(s)$, $s \in S$. Because the free (noncommutative) algebra generated by elements of S is such an A_f containing a faithful copy of S, the Poisson algebra $\mathcal{P}(S)$ is well-defined and clearly satisfies the universality condition. It seems to be plausible but yet is not clear how to construct such universal Poisson algebra using the constructions of free Lie algebras and the enveloping algebras of Lie algebras.

Let a_i, b_j be some generators in $S \subset \mathcal{P}(S)$. Then by repeated applications of Equation 9, we can check that $\{a_1a_2...a_m, b_1b_2...b_n\}$ is a finite sum of monomials of the form $s_1s_2...s_k\{s_0, s_0\}t_1t_2..t_l$ with s_i, t_j genarators in S. In fact,

$$\{a_1a_2...a_m, b_1b_2...b_n\} = \sum a_1a_2...a_{i-1}b_1b_2...b_{j-1}\{a_i, b_j\}b_{j+1}...b_na_{i+1}..a_m.$$

Note that, for each pair of monomials $a = a_1a_2...a_m$ and $b = b_1b_2...b_n$, this finite sum expression of their Poisson bracket is not unique. But we shall fix one summation formula

$$p(a, b) = \sum s_1s_2...s_k\{s_0, t_0\}t_1t_2..t_l \tag{10}$$

for each pair of monomials a, b in the generators of $\mathcal{P}(S)$ where s, t are finite sequences of generators in S of various lengths. Note that by the universality of $\mathcal{P}(S)$, given any surjection $F : S \longrightarrow X$ with X a set of algebra generators of a Poisson algebra A, then for any pair of monomials x, y in elements of X, we can find monomials a, b in elements of S such that

$$\{x, y\} = \phi(p(a, b)),$$

where ϕ is the unique Poisson algebra homomorphism from $\mathcal{P}(S)$ to A extending the function F. In fact, any monomials a and b with $\phi(a) = x$ and $\phi(b) = y$ satisfy $\phi(p(a, b)) = \phi(\{a, b\}) = \{\phi(a), \phi(b)\} = \{x, y\}$.

5. Weyl quantization of Poisson $SU(2)$.

It was shown in [**Lu-We**] that there is a Poisson structure on $SU(2)$ compatible with the group structure, in the sense that the multiplication $m : SU(2) \times SU(2) \to SU(2)$ is a Poisson map where $SU(2) \times SU(2)$ is endowed with the product Poisson structure. We refer readers to [**We**] for the theory of Poisson manifolds.

We first give a definition of a C*-algebraic deformation quantization in Rieffel's sense [**R1**].

Definition 1. Given a Poisson manifold $(M, \{\cdot,\cdot\})$ and a smooth Poisson algebra $\mathcal{A}$ (of functions on M), i.e. a (sup-norm) dense Poisson *-subalgebra of $C_b^\infty(M) \subset C_b(M)$, a family of linear surjections $\rho_h : \mathcal{A} \longrightarrow \mathcal{A}_h$ with $0 \le h < \delta$ for some $\delta > 0$ is called a C*-algebraic deformation quantization of $\mathcal{A}$ if

(1) each $\mathcal{A}_h$ is a *-subalgebra of a C*-algebra A_h with $\mathcal{A}_0 = \mathcal{A}$ and ρ_0 the identity map,

(2) $\|\rho_h(f)\|$ is continuous in $h \in [0, \delta)$ for any fixed $f \in \mathcal{A}$,

(3) $\lim_{h\to 0} \|\rho_h(f^*) - \rho_h(f)^*\| = 0$ for any $f \in \mathcal{A}$,

(4) $\lim_{h\to 0} \|\rho_h(f)\rho_h(g) - \rho_h(fg)\| = 0$ for any $f, g \in \mathcal{A}$,

(5) $\lim_{h\to 0} \|(ih)^{-1}[\rho_h(f), \rho_h(g)] - \rho_h(\{f,g\})\| = 0$ for any $f, g \in \mathcal{A}$.

In the following, with $\mu = e^{-h/2}$, we take $A_h = C(S_\mu U(2))$ and $\mathcal{A}_h = C(S_\mu U(2))^\infty$, the *-algebra generated by α, γ in $C(S_\mu U(2))$. In particular, $\mathcal{A} = \mathcal{A}_0$ is the algebra of regular functions on $SU(2)$. We claim that

$$\rho_h = W_h : \mathcal{A}_0 \longrightarrow \mathcal{A}_h \subset C(S_\mu U(2))$$

defines a C*-algebraic deformation quantization of $\mathcal{A}_0$.

Clearly (1) in Definition 1 is satisfied, and Theorem 1 implies that $\|W_h(f)\| = \|\pi_\mu(W(f))\|$ is continuous in $\mu \in [\epsilon, 1]$ and hence in $h \in [0, -\ln(\epsilon)]$, which is condition (2) in Definition 1.

Since $W(f)W(g) - W(fg)$ is in $\boldsymbol{A}$, we have

$$\begin{aligned}
&\lim_{h\to 0} \|W_h(f)W_h(g) - W_h(fg)\| = \lim_{\mu\to 1} \|\pi_\mu(W(f)W(g) - W(fg))\| \\
&= \|\pi_1(W(f)W(g) - W(fg))\| = \|\pi_1(W(f))\pi_1(W(g)) - \pi_1(W(fg))\| \\
&= \|fg - fg\| = 0
\end{aligned}$$

by Theorem 1 and Theorem 2. So condition (3) is satisfied.

We also have

$$\begin{aligned}
&\lim_{h\to 0} \|W_h(\bar{f}) - W_h(f)^*\| = \lim_{\mu\to 1} \|\pi_\mu(W(\bar{f})) - \pi_\mu(W(f)^*)\| \\
&= \lim_{\mu\to 1} \|\pi_\mu(W(\bar{f}) - W(f)^*)\| = \|\pi_1(W(\bar{f}) - W(f)^*)\| \\
&= \|\pi_1(W(\bar{f})) - \pi_1(W(f))^*\| = \|\bar{f} - f^*\| = 0
\end{aligned}$$

by Theorem 1 and Theorem 2, since $W(\bar{f}) - W(f)^*$ is in A and π_1 is a *-homomorphism. (Note that W and W_h are not *-preserving except for $h = 0$.) So condition (4) is verified.

Now it remains to show condition (5). First we get some basic result on the limit behaviour of $(ih)^{-1}[\pi_\mu(\boldsymbol{a}), \pi_\mu(\boldsymbol{b})]$ for generators $\boldsymbol{a}$, $\boldsymbol{b} \in \{\boldsymbol{\alpha}, \bar{\boldsymbol{\alpha}}, \boldsymbol{\gamma}, \bar{\boldsymbol{\gamma}}\}$ of $\boldsymbol{A}$.

Recall that the Poisson 2-tensor π on $SU(2)$ is given by

$$\pi\left(\begin{pmatrix}\alpha & -\bar{\gamma}\\ \gamma & \bar{\alpha}\end{pmatrix}\right) = 2^{-1}i\begin{bmatrix}\alpha & 0\\ 0 & -\bar{\alpha}\end{bmatrix} \wedge \begin{bmatrix}-c\alpha & -\bar{\gamma}\\ \gamma & -c\bar{\alpha}\end{bmatrix}$$

(up to a constant factor) where $c = \dfrac{1-|\alpha|^2}{|\alpha|^2} = \dfrac{|\gamma|^2}{|\alpha|^2}$ [**Sh2**]. Let us make explicit the brackets of the generators α, $\bar{\alpha}$, γ, and $\bar{\gamma}$ of $C(SU(2))^\infty$. We have

$$\{\alpha,\gamma\} = \pi\left(\begin{pmatrix}\alpha & -\bar{\gamma}\\ \gamma & \bar{\alpha}\end{pmatrix}\right)\cdot(d\alpha\wedge d\gamma) = \frac{1}{2}i\gamma\alpha,$$

$$\{\alpha,\bar{\alpha}\} = \pi\left(\begin{pmatrix}\alpha & -\bar{\gamma}\\ \gamma & \bar{\alpha}\end{pmatrix}\right)\cdot(d\alpha\wedge d\bar{\alpha}) = -i\gamma\gamma^*,$$

$$\{\alpha,\bar{\gamma}\} = \pi\left(\begin{pmatrix}\alpha & -\bar{\gamma}\\ \gamma & \bar{\alpha}\end{pmatrix}\right)\cdot(d\alpha\wedge d\bar{\gamma}) = \frac{1}{2}i\bar{\gamma}\alpha,$$

and

$$\{\gamma,\bar{\gamma}\} = \pi\left(\begin{pmatrix}\alpha & -\bar{\gamma}\\ \gamma & \bar{\alpha}\end{pmatrix}\right)\cdot(d\gamma\wedge d\bar{\gamma}) = 0.$$

The others can be gotten from these three since the Poisson bracket on $SU(2)$ commutes with the conjugation and is alternating.

For each pair $(\boldsymbol{a},\boldsymbol{b})$ of generators $\boldsymbol{\alpha}$, $\bar{\boldsymbol{\alpha}}$, $\boldsymbol{\gamma}$, and $\bar{\boldsymbol{\gamma}}$, we fix an element $\psi(\boldsymbol{a},\boldsymbol{b})$ in $\boldsymbol{\mathcal{A}}$ satisfying

$$\pi_1(\psi(\boldsymbol{a},\boldsymbol{b})) = \{\pi_1(\boldsymbol{a}),\pi_1(\boldsymbol{b})\}.$$

(For the convenience of presentation, we choose the obvious lifts $\psi(\boldsymbol{a},\boldsymbol{b})$ of $\{\pi_1(\boldsymbol{a}),\pi_1(\boldsymbol{b})\}$ suggested by the above formulae for brackets of generators in $C(SU(2))^\infty$, although the choice of $\psi(\boldsymbol{a},\boldsymbol{b})$ does not affect the following claim, Equation 11, which we need later.) For example, we set $\psi(\boldsymbol{\alpha},\boldsymbol{\gamma}) = \frac{1}{2}i\boldsymbol{\gamma}\boldsymbol{\alpha}$.

We claim that

$$\lim_{h\to 0}\left\|(ih)^{-1}[\pi_\mu(\boldsymbol{a}),\pi_\mu(\boldsymbol{b})] - \pi_\mu(\psi(\boldsymbol{a},\boldsymbol{b}))\right\| = 0 \tag{11}$$

for any pair $(\boldsymbol{a},\boldsymbol{b})$ of generators $\boldsymbol{\alpha}$, $\bar{\boldsymbol{\alpha}}$, $\boldsymbol{\gamma}$, and $\bar{\boldsymbol{\gamma}}$.

In fact, by noting that

$$\lim_{h\to 0}\frac{2i(1-\mu)}{ih} = 1$$

and using the commutation relations in Equation 1, we have

$$\begin{aligned}
&\lim_{h\to 0}\left\|(ih)^{-1}[\pi_\mu(\boldsymbol{\alpha}),\pi_\mu(\boldsymbol{\gamma})]-\pi_\mu\left(\frac{1}{2}i\boldsymbol{\gamma}\boldsymbol{\alpha}\right)\right\| \\
&=\lim_{\mu\to 1}\left\|(2i(1-\mu))^{-1}\pi_\mu([\boldsymbol{\alpha},\boldsymbol{\gamma}])-\pi_\mu\left(\frac{1}{2}i\boldsymbol{\gamma}\boldsymbol{\alpha}\right)\right\| \\
&=\lim_{\mu\to 1}\left\|(2i(1-\mu))^{-1}\pi_\mu(\boldsymbol{\alpha}\boldsymbol{\gamma}-\boldsymbol{\gamma}\boldsymbol{\alpha})-\pi_\mu\left(\frac{1}{2}i\boldsymbol{\gamma}\boldsymbol{\alpha}\right)\right\| \\
&=\lim_{\mu\to 1}\left\|(2i(1-\mu))^{-1}\pi_\mu((\nu-1)\boldsymbol{\gamma}\boldsymbol{\alpha})-\frac{1}{2}i\pi_\mu(\boldsymbol{\gamma}\boldsymbol{\alpha})\right\| \\
&=\lim_{\mu\to 1}\left\|-(2i)^{-1}\pi_\mu(\boldsymbol{\gamma}\boldsymbol{\alpha})-\frac{1}{2}i\pi_\mu(\boldsymbol{\gamma}\boldsymbol{\alpha})\right\|=0.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\lim_{h\to 0}\|(ih)^{-1}[\pi_\mu(\boldsymbol{\alpha}),\pi_\mu(\boldsymbol{\alpha}^*)]-\pi_\mu(-i\boldsymbol{\gamma}\boldsymbol{\gamma}^*)\| \\
&=\lim_{\mu\to 1}\|(2i(1-\mu))^{-1}\pi_\mu([\boldsymbol{\alpha},\boldsymbol{\alpha}^*])-\pi_\mu(-i\boldsymbol{\gamma}\boldsymbol{\gamma}^*)\| \\
&=\lim_{\mu\to 1}\|(2i(1-\mu))^{-1}\pi_\mu(\boldsymbol{\alpha}\boldsymbol{\alpha}^*-\boldsymbol{\alpha}^*\boldsymbol{\alpha})+\pi_\mu(i\boldsymbol{\gamma}\boldsymbol{\gamma}^*)\| \\
&=\lim_{\mu\to 1}\|(2i(1-\mu))^{-1}\pi_\mu((1-\nu^2)\boldsymbol{\gamma}\boldsymbol{\gamma}^*)+\pi_\mu(i\boldsymbol{\gamma}\boldsymbol{\gamma}^*)\| \\
&=\lim_{\mu\to 1}\|(2i)^{-1}(1+\mu)\pi_\mu(\boldsymbol{\gamma}\boldsymbol{\gamma}^*)+\pi_\mu(i\boldsymbol{\gamma}\boldsymbol{\gamma}^*)\| \\
&=\lim_{\mu\to 1}\|\pi_\mu(((2i)^{-1}(1+\mu)+i)\boldsymbol{\gamma}\boldsymbol{\gamma}^*)\|=0,
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{h\to 0}\left\|(ih)^{-1}[\pi_\mu(\boldsymbol{\alpha}),\pi_\mu(\boldsymbol{\gamma}^*)]-\pi_\mu\left(\frac{1}{2}i\boldsymbol{\gamma}^*\boldsymbol{\alpha}\right)\right\| \\
&=\lim_{\mu\to 1}\left\|(2i(1-\mu))^{-1}\pi_\mu([\boldsymbol{\alpha},\boldsymbol{\gamma}^*])-\frac{1}{2}i\pi_\mu(\boldsymbol{\gamma}^*\boldsymbol{\alpha})\right\| \\
&=\lim_{\mu\to 1}\left\|(2i(1-\mu))^{-1}\pi_\mu(\boldsymbol{\alpha}\boldsymbol{\gamma}^*-\boldsymbol{\gamma}^*\boldsymbol{\alpha})-\frac{1}{2}i\pi_\mu(\boldsymbol{\gamma}^*\boldsymbol{\alpha})\right\| \\
&=\lim_{\mu\to 1}\left\|(2i(1-\mu))^{-1}\pi_\mu((\nu-1)\boldsymbol{\gamma}^*\boldsymbol{\alpha})-\frac{1}{2}i\pi_\mu(\boldsymbol{\gamma}^*\boldsymbol{\alpha})\right\| \\
&=\lim_{\mu\to 1}\left\|-(2i)^{-1}\pi_\mu(\boldsymbol{\gamma}^*\boldsymbol{\alpha})-\frac{1}{2}i\pi_\mu(\boldsymbol{\gamma}^*\boldsymbol{\alpha})\right\|=0.
\end{aligned}$$

Clearly Equation 11 holds for the pair $(\boldsymbol{\gamma},\bar{\boldsymbol{\gamma}})$ since $[\gamma,\bar{\gamma}]=0$ in $C(S_\mu U(2))$ for each $\mu\neq 1$ and $\psi(\boldsymbol{\gamma},\bar{\boldsymbol{\gamma}})=0$. Now Equqtion 11 holds for other pairs of generators, too, since all the Poisson brackets involved here commute with the adjoint operation and are alternating. So we get Equation 11 verified.

Consider the universal Poisson algebras $\mathcal{P}(S)$ with four free generators in $S=\{\tilde{\alpha},\tilde{\gamma},\tilde{\alpha}^*,\tilde{\gamma}^*\}$. Let ϕ (resp. ϕ_μ) be the Poisson algebra homomorphism from $\mathcal{P}(S)$ to the Poisson algebra $\boldsymbol{A}$ (resp. $C(S_\mu U(2))^\infty$) extending the map sending $\tilde{\alpha},\tilde{\gamma},\tilde{\alpha}^*,\tilde{\gamma}^*$ to $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, $\boldsymbol{\alpha}^*$, $\boldsymbol{\gamma}^*$, (resp. α, γ, α^*, γ^*) respectively. We note that

$$\phi_\mu=\pi_\mu\circ\phi \tag{12}$$

on the subalgebra generated by S in $\mathcal{P}(S)$ for all μ since π_μ, ϕ_μ, and ϕ are algebra homomorphisms, and on $\mathcal{P}(S)$ for $\mu<1$ since the Poisson brackets on $\boldsymbol{A}$ and $C(S_\mu U(2))$ with $\mu<1$ come from the algebra commutator bracket and hence π_μ, ϕ_μ, and ϕ are Poisson algebra homomorphisms. For each monomial $\boldsymbol{x}$ in $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, $\boldsymbol{\alpha}^*$, and $\boldsymbol{\gamma}^*$, we fix a unique monomial $\tilde{x}$ in elements of S such that $\phi(\tilde{x})=\boldsymbol{x}$. Let $\boldsymbol{x}$, $\boldsymbol{y}$ be monomials in $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, $\boldsymbol{\alpha}^*$, and $\boldsymbol{\gamma}^*$ with $a=\tilde{x}$, and $b=\tilde{y}$. Then

$$\begin{aligned}
&\{\pi_\mu(\eta\boldsymbol{x}),\pi_\mu(\zeta\boldsymbol{y})\}_\mu=\eta(\mu)\zeta(\mu)\{\pi_\mu(\phi(a)),\pi_\mu(\phi(b))\}_\mu\\
&=\eta(\mu)\zeta(\mu)\{\phi_\mu(a),\phi_\mu(b)\}_\mu=\eta(\mu)\zeta(\mu)\phi_\mu(\{a,b\})\\
&=\eta(\mu)\zeta(\mu)\phi_\mu(p(a,b))=\eta(\mu)\zeta(\mu)\phi_\mu\left(\sum_{s,t}s_1s_2...s_k\{s_0,t_0\}t_1t_2..t_l\right)\\
&=\eta(\mu)\zeta(\mu)\sum_{s,t}\phi_\mu(s_1...s_k)\{\phi_\mu(s_0),\phi_\mu(t_0)\}_\mu\phi_\mu(t_1...t_l)\\
&=\eta(\mu)\zeta(\mu)\sum_{s,t}\pi_\mu(\phi(s_1...s_k))\{\pi_\mu(\phi(s_0)),\pi_\mu(\phi(t_0))\}_\mu\pi_\mu(\phi(t_1...t_l))\\
&=\eta(\mu)\zeta(\mu)\sum_{s,t}\pi_\mu(\boldsymbol{s})\{\pi_\mu(\boldsymbol{s}_0),\pi_\mu(\boldsymbol{t}_0)\}_\mu\pi_\mu(\boldsymbol{t})
\end{aligned}$$

by the fact that ϕ_μ is a Poisson algebra homomorphism for any μ, where s_i, t_j are in S as defined in Equation 10, $\{\cdot,\cdot\}_\mu$ is the Poisson bracket on $\mathcal{A}_h=C(S_\mu U(2))^\infty$, $\boldsymbol{s}=\phi(s_1...s_k)$ and $\boldsymbol{t}=\phi(t_1...t_l)$ are monomials in $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, $\boldsymbol{\alpha}^*$, and $\boldsymbol{\gamma}^*$ of $\boldsymbol{A}$, and $\boldsymbol{s}_0=\phi(s_0)$ and $\boldsymbol{t}_0=\phi(t_0)$ are from $\{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\alpha}^*,\boldsymbol{\gamma}^*\}$. In particular, we have

$$\begin{aligned}
&\{\pi_1(\eta\boldsymbol{x}),\pi_1(\zeta\boldsymbol{y})\}_1\\
&=\eta(1)\zeta(1)\sum_{s,t}\pi_1(\boldsymbol{s})\{\pi_1(\boldsymbol{s}_0),\pi_1(\boldsymbol{t}_0)\}_1\pi_1(\boldsymbol{t})\\
&=\eta(1)\zeta(1)\sum_{s,t}\pi_1(\boldsymbol{s})\pi_1(\psi(\boldsymbol{s}_0,\boldsymbol{t}_0))\pi_1(\boldsymbol{t})
\end{aligned}$$

$$=\pi_1\left(\sum_{s,t}\eta(1)\zeta(1)\boldsymbol{s}\psi(\boldsymbol{s}_0,\boldsymbol{t}_0)\boldsymbol{t}\right). \tag{13}$$

Recall that each $\boldsymbol{\omega}_{ik}^n$ is a finite sum $\sum \eta \boldsymbol{x}$ of monomials $\boldsymbol{x}$ in $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, $\boldsymbol{\alpha}^*$, and $\boldsymbol{\gamma}^*$ with coefficients $\eta \in C([\epsilon, 1]) \subset \boldsymbol{A}$. So any two elements f and g of $\mathcal{A}_0 = C(SU(2))^\infty$ are now written as finite sums $\sum_i \pi_1(\eta_i \boldsymbol{x}_i)$ and $\sum_j \pi_1(\zeta_j \boldsymbol{y}_j)$ with η_i, $\zeta_j \in C([\epsilon, 1])$ and $\boldsymbol{x}_i, \boldsymbol{y}_j$ monomials.

Before we proceed to finish the proof, we observe that if elements $\boldsymbol{a}, \boldsymbol{b}$ of $\boldsymbol{\mathcal{A}}$ satisfy $\pi_1(\boldsymbol{a}) = \pi_1(\boldsymbol{b})$ then

$$\begin{aligned}\lim_{h \to 0} \|W_h(\pi_1(\boldsymbol{a})) - \pi_\mu(\boldsymbol{b})\| &= \lim_{\mu \to 1} \|\pi_\mu(W(\pi_1(\boldsymbol{a})) - \boldsymbol{b})\| \\ &= \|\pi_1(W(\pi_1(\boldsymbol{a})) - \boldsymbol{b})\| = \|\pi_1(\boldsymbol{a}) - \pi_1(\boldsymbol{b})\| = 0\end{aligned}$$

by Theorem 1 and Theorem 2. With this in mind, we get, for $f = \sum_i \pi_1(\eta_i \boldsymbol{x}_i)$ and $g = \sum_j \pi_1(\zeta_j \boldsymbol{y}_j)$,

$$\lim_{h \to 0} \left\| W_h(\{f, g\}) - \pi_\mu \left(\sum_{i,j} \sum_{s,t} \eta_i \zeta_j \boldsymbol{s} \psi(\boldsymbol{s}_0, \boldsymbol{t}_0) \boldsymbol{t} \right) \right\| = 0$$

because

$$\begin{aligned}W_h(\{f, g\}) &= W_h \left(\sum_{i,j} \{\pi_1(\eta_i \boldsymbol{x}_i), \pi_1(\zeta_i \boldsymbol{y}_i)\} \right) \\ &= W_h \left(\pi_1 \left(\sum_{i,j} \sum_{s,t} \eta_i(1) \zeta_j(1) \boldsymbol{s} \psi(\boldsymbol{s}_0, \boldsymbol{t}_0) \boldsymbol{t} \right) \right)\end{aligned}$$

by Equation 13. Here, in each summand, $\boldsymbol{s}$, $\boldsymbol{s}_0$, $\boldsymbol{t}$, $\boldsymbol{t}_0$ all depend on the corresponding pair (i, j). Now in order to prove condition (5), we only need to show

$$\lim_{h \to 0} \left\| (ih)^{-1} \{W_h(f), W_h(g)\}_\mu - \pi_\mu \left(\sum_{i,j} \sum_{s,t} \eta_i(\mu) \zeta_j(\mu) \boldsymbol{s} \psi(\boldsymbol{s}_0, \boldsymbol{t}_0) \boldsymbol{t} \right) \right\| = 0,$$

which is verified in

$$\lim_{h\to 0}\left\|(ih)^{-1}\{W_h(f),W_h(g)\}_\mu - \pi_\mu\left(\sum_{i,j}\sum_{s,t}\eta_i\zeta_j \boldsymbol{s}\psi(\boldsymbol{s}_0,\boldsymbol{t}_0)\boldsymbol{t}\right)\right\|$$

$$= \lim_{h\to 0}\left\|(ih)^{-1}\sum_{i,j}\{\pi_\mu(\eta_i\boldsymbol{x}_i),\pi_\mu(\zeta_j\boldsymbol{y}_j)\}_\mu\right.$$

$$\left.-\sum_{i,j}\sum_{s,t}\eta_i(\mu)\zeta_j(\mu)\pi_\mu(\boldsymbol{s})\pi_\mu(\psi(\boldsymbol{s}_0,\boldsymbol{t}_0))\pi_\mu(\boldsymbol{t})\right\|$$

$$= \lim_{h\to 0}\left\|(ih)^{-1}\sum_{i,j}\eta_i(\mu)\zeta_j(\mu)\sum_{s,t}\pi_\mu(\boldsymbol{s})\{\pi_\mu(\boldsymbol{s}_0),\pi_\mu(\boldsymbol{t}_0)\}_\mu\pi_\mu(\boldsymbol{t})\right.$$

$$\left.-\sum_{i,j}\sum_{s,t}\eta_i(\mu)\zeta_j(\mu)\pi_\mu(\boldsymbol{s})\pi_\mu(\psi(\boldsymbol{s}_0,\boldsymbol{t}_0))\pi_\mu(\boldsymbol{t})\right\|$$

$$= \lim_{h\to 0}\left\|\sum_{i,j}\sum_{s,t}\eta_i(\mu)\zeta_j(\mu)\pi_\mu(\boldsymbol{s})((ih)^{-1}\{\pi_\mu(\boldsymbol{s}_0),\pi_\mu(\boldsymbol{t}_0)\}_\mu\right.$$

$$\left.-\pi_\mu(\psi(\boldsymbol{s}_0,\boldsymbol{t}_0)))\pi_\mu(\boldsymbol{t})\right\| = 0$$

by Equation 11.

Theorem 3. *The Weyl transformations $W_h : C(SU(2))^\infty \longrightarrow C(S_\mu U(2))^\infty$ form a C^*-algebraic deformation quantization of the Poisson algebra $C(SU(2))^\infty$ of regular functions on the Poisson Lie group $SU(2)$.*

It is not hard to check that the above proof works well for a quite general class of transformations, and in particular, if we replace $\boldsymbol{\omega}^n_{ik}$'s by $\boldsymbol{u}^n_{ik}$'s in defining the Weyl transformations, Theorem 3 still holds. The same technique used in this paper seems to work for other Poisson Lie groups, especially the twisted $SU(n)$'s, on which the author is currently investigating.

We remark that this Weyl deformation quantization of Poisson $SU(2)$ is not a "leaf-preserving" quantization (with respect to the canonical symplectic foliation [**We**] of the Poisson $SU(2)$) in contrast with the deformation quantization $\mathcal{W}_h : C^\infty(SU(2)) \longrightarrow C(S_\mu U(2))$ found in [**Sh1**], which is leaf-preserving in the sense that if $f = g$ on a symplectic leaf L of $SU(2)$ then $\pi_L(\mathcal{W}_h(f)) = \pi_L(\mathcal{W}_h(g))$ where π_L is the irreducible representation of the C*-algebra $C(S_\mu U(2))$ naturally associated with the leaf L [**V-So**]. In fact, we can actually find explicitly a regular function f on $SU(2)$ vanishing on a specific symplectic leaf L such that $\pi_L(W_h(f)) \neq 0$ for all $h \neq 0$. For example,

$$f = \alpha\bar{\alpha} - \gamma\bar{\gamma} + 2\bar{\gamma}^2 - 1$$

is a regular function on $SU(2)$ vanishing on the symplectic leaf

$$L_0 = \left\{ \begin{pmatrix} \alpha & -\bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix} \in SU(2) : |\alpha| < 1 \text{ and } \gamma > 0 \right\}.$$

But

$$\pi_{L_0}(W_h(f)) = (1-\mu^2) \sum_{k \in \mathrm{N}} \mu^{2(k-1)} e_{k,k} \neq 0$$

for all $h \neq 0$. Finally, we note that this example is still valid when the Weyl transformation is defined by using $\boldsymbol{u}^n_{ik}$'s instead of $\boldsymbol{\omega}^n_{ik}$'s, so the unitarized version is still not leaf-preserving.

References

[B] A. Bauval, *Quantum group- and Poisson- deformation of $SU(2)$*, preprint.

[BFFLS] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, *Deformation theory and quantization*, I, II, Ann. Physics, **110** (1978), 61-110, 111-151.

[C1] A. Connes, *Non-commutative differential geometry*, I and II, Publ. I.H.E.S., **62** (1985), 257-360.

[C2] ———, *A survey of foliation and operator algebras*, Proc. Symp. Pure Math. Vol. 38, Part I, AMS, Providence, 1982, 521-628.

[Dr] V.G. Drinfeld, *Quantum groups*, Proc. I.C.M. Berkeley 1986, Vol. 1, 789-820, Amer. Math. Soc., Providence, 1987.

[Du] M. Dubois-Violette, *On the theory of quantum groups*, Lett. Math. Phys., **19** (1990), 121-126.

[Ge] M. Gerstenhaber, *On the deformation of rings and algebras*, Ann. Math., **79** (1964), 59-103.

[Gi] A. Giaquinto, *Deformation Methods in quantum groups*, preprint.

[K] H.T. Koelink, *On *-representations of the Hopf *-algebra associated with the quantum group $U_q(n)$*, Compositio Mathematica, **77** (1991), 199-231.

[Lu-We] J.H. Lu and A. Weinstein, *Poisson Lie groups, dressing transformations and Bruhat decompositions*, J. Diff. Geom., **31** (1990), 501-526.

[N] G. Nagy, *On the Haar measure of the quantum $SU(N)$ group*, Comm. Math. Phys., **153** (1993), 217-228.

[P] G.K. Pedersen, *C^*-algebras and their Automorphism Groups*, Academic Press, New York, 1979.

[R1] M.A. Rieffel, *Deformation quantization and operator algebras*, in "Proc. Symp.Pure Math., Vol. 51", AMS, Providence, 1990, 411-423.

[R2] ———, *Deformation quantization of Heisenberg manifolds*, Comm. Math. Phys., **122** (1989), 531-562.

[R3] ———, *Lie group convolution algebras as deformation quantization of linear Poisson structures*, Amer. J. Math., **112** (1990), 657-686.

[R4] ———, *Continuous fields of C^*-algebras coming from group cocycles and actions*, Math. Ann., **283** (1989), 631-643.

[R5] ———, *Deformation quantization for actions of* $\mathbb{R}^d$, Memoirs AMS, **106(506)** (1993).

[Sh1] A.J.L. Sheu, *Quantization of the Poisson* $SU(2)$ *and its Poisson homogeneous space - the* 2*-sphere*, Comm. Math. Phys., **135** (1991), 217-232.

[Sh2] ———, *Quantum Poisson* $SU(2)$ *and quantum Poisson spheres*, in "Deformation Theory and Quantum Groups with Applications to Mathematical Physics", M. Gerstenhaber and J.D. Stasheff (Eds.), Contemporary Math. Vol. 134, Amer. Math. Soc., Providence, 1992, 247-258.

[Sh3] ———, *Some examples of singular foliation C^*-algebras*, in "Current Topics in Operator Algebras", H. Araki et al (Eds.), World Scientific, Singapore, 1991, 326-340.

[Sh4] ———, *Leaf-preserving quantizations of Poisson* $SU(2)$ *are not coalgebra homomorphisms*, Comm Math Phys., **172** (1995), 287-292.

[V-So] L.L. Vaksman and Ya.S. Soibelman, *Algebra of functions on the quantum group* $SU(2)$, Func. Anal. Appl., **22** (1988), 170-181.

[We] A. Weinstein, *The local structure of Poisson manifolds*, J. Diff. Geom., **18** (1983), 523-557.

[We-X] A. Weinstein and P. Xu, *Extensions of symplectic groupoids and quantization*, J. Reine. Angew. Math., **417** (1991), 159-189.

[Wo1] S.L. Woronowicz, *Twisted* $SU(2)$ *group: an example of a non-commutative differential calculus*, Publ. RIMS., **23** (1987), 117-181.

[Wo2] ———, *Compact matrix pseudogroups*, Comm. Math. Phys., **111** (1987), 613-665.

Received September 1, 1993. The author was partially supported by NSF-Grant DMS-9002895.

University of Kansas
Lawrence, KS 66045-2142
E-mail address: sheu@kuhub.cc.ukans.edu