PACIFIC JOURNAL OF MATHEMATICS
Vol. 173, No. 1, 1996

TRACE IDEAL CRITERIA FOR TOEPLITZ AND HANKEL OPERATORS ON THE WEIGHTED BERGMAN SPACES WITH EXPONENTIAL TYPE WEIGHTS

PENG LIN AND RICHARD ROCHBERG

Let $\varphi : \mathbb{D} \to \mathbb{R}$ be a subharmonic function and let $AL^2_\varphi(\mathbb{D})$ denote the closed subspace of $L^2(\mathbb{D},\, e^{-2\varphi}dA)$ consisting of analytic functions in the unit disk $\mathbb{D}$. For a certain class of subharmonic φ, the necessary and sufficient conditions are obtained for the Toeplitz operator T_μ on $AL^2_\varphi(\mathbb{D})$ and the Hankel operator H_b on $AL^2_\varphi(\mathbb{D})$ in order that they belong to the Schatten ideal S_p.

1. Introduction.

Let dA denote the area measure for the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. Let $L^2(\mathbb{D})$ denote $L^2(\mathbb{D},\, dA)$ and let $L^\infty(\mathbb{D})$ denote $L^\infty(\mathbb{D},\, dA)$. Let $\varphi : \mathbb{D} \to \mathbb{R}$ be a subharmonic function. Let $L^\infty_\varphi(\mathbb{D})$ be the space of all measurable functions f on $\mathbb{D}$ such that $e^{-\varphi}f \in L^\infty(\mathbb{D})$ and let $H^\infty_\varphi(\mathbb{D})$ denote the subspace of $L^\infty_\varphi(\mathbb{D})$ consisting of analytic functions. Let $L^2_\varphi(\mathbb{D})$ be the Hilbert space of all measurable functions f on $\mathbb{D}$ such that $\|f\|_{L^2_\varphi} =: (\int_{\mathbb{D}} |f|^2 e^{-2\varphi}\, dA)^{1/2} < \infty$. The inner product of $L^2_\varphi(\mathbb{D})$ is given by $\langle f,\, g\rangle_{L^2_\varphi} = \int_{\mathbb{D}} f\bar{g}e^{-2\varphi}\, dA$ for $f,\, g \in L^2_\varphi(\mathbb{D})$. Let $AL^2_\varphi(\mathbb{D})$ denote the closed subspace of $L^2_\varphi(\mathbb{D})$ consisting of analytic functions. Let P be the orthogonal projection from $L^2_\varphi(\mathbb{D})$ onto $AL^2_\varphi(\mathbb{D})$, which is given by $Pf(z) = \int_{\mathbb{D}} K(z,w)f(w)e^{-2\varphi(w)}\, dA(w)$, where $K(z,w)$ is the reproducing kernel of $AL^2_\varphi(\mathbb{D})$. For $b \in L^2(\mathbb{D})$, the Hankel operator H_b on $AL^2_\varphi(\mathbb{D})$ is defined on the dense set $H^\infty_\varphi(\mathbb{D})$ of $AL^2_\varphi(\mathbb{D})$ (for certain class of subharmonic φ) by

$$H_b f = bf - P(bf).$$

For a finite positive Borel measure μ on $\mathbb{D}$, the Toeplitz operator T_μ on $AL^2_\varphi(\mathbb{D})$ is defined by

$$T_\mu f(z) = \int_{\mathbb{D}} K(z,w)f(w)e^{-2\varphi(w)}\, d\mu(w).$$

The purpose of this paper is for a certain class of subharmonic φ to prove necessary and sufficient conditions on b (respectively on μ) in order that the Hankel operator H_b (respectively the Toepolitz operator T_μ) on $AL^2_\varphi(\mathbb{D})$ belongs to a Schatten ideal S_p.

In [**Lu1**], Luecking obtained the trace ideal criteria for the Toplitz operators on the (standard) weighted Bergman spaces. In [**Lu2**], he considered the boundedness, compactness and the Schatten class properties of the Hankel operators on the Bergman spaces of the unit disk $\mathbb{D}$ with the symbol functions in $L^2(\mathbb{D})$.

In [**LR**], we studied the boundedness and compactness of the Hankel operator H_b on the weighted space $AL^2_\varphi(\mathbb{D})$ with $b \in L^2(\mathbb{D})$ for a certain class of subharmonic φ.

In the present paper, we will continue to study the Hankel operator H_b on $AL^2_\varphi(\mathbb{D})$ and we will also consider the Toepolitz operator T_μ on $AL^2_\varphi(\mathbb{D})$. We will still concentrate on the same class of subharmonic φ as in [**LR**]. The typical examples of our weight $e^{-2\varphi}$ are $(1-|z|^2)^A$, $A > 0$ (which corresponds to the weights for the standard weighted Bergman spaces $A^{2,\alpha}$ for $\alpha > 0$) and $(1-|z|^2)^A \exp\{-B/(1-|z|^2)^\alpha\}$, $A \geq 0$, $B > 0$, $\alpha > 0$. For the Toeplitz operator T_μ on $AL^2_\varphi(\mathbb{D})$, we will give conditions on the finite positive Borel measure μ on $\mathbb{D}$ in order that T_μ be bounded, compact and in S_p respectively. For the Hankel operator H_b on $AL^2_\varphi(\mathbb{D})$, we will give conditions on the function $b \in L^2(\mathbb{D})$ in order that H_b belong to S_p.

The paper is arranged as follows. In Section 2 we recall some results about the Carleson measures on $AL^2_\varphi(\mathbb{D})$. In Section 3, we consider the Toeplitz operator T_μ on $AL^2_\varphi(\mathbb{D})$ for finite positive Borel measure μ on $\mathbb{D}$. In Section 4, by using the results obtained in Section 3, we prove the trace ideal criteria for the Hankel operator H_b on $AL^2_\varphi(\mathbb{D})$ for a certain class of subharmonic φ.

Throughout this paper, we will use the letter C to denote constants and they may change from line to line.

2. Carleson measures on $AL^2_\varphi(\mathbb{D})$.

Let μ be a locally finite nonnegative Borel measure on the unit disk $\mathbb{D}$, dA be the area measure on $\mathbb{D}$ and $\varphi : \mathbb{D} \to \mathbb{R}$ be subharmonic function. Let $L^2_{\varphi,\mu}(\mathbb{D})$ be the space of all measurable functions f on $\mathbb{D}$ such that

$$\|f\|_{L^2_{\varphi,\mu}} = \left(\int_{\mathbb{D}} |f|^2 e^{-2\varphi}\, d\mu\right)^{1/2} < \infty.$$

Let $L^2_\varphi(\mathbb{D})$ denote $L^2_{\varphi,dA}(\mathbb{D})$ and $AL^2_\varphi(\mathbb{D})$ be the closed subspace of $L^2_\varphi(\mathbb{D})$ consisting of analytic functions.

Definition 2.1. μ is called a Carleson measure on $AL^2_\varphi(\mathbb{D})$ if the imbedding operator $J : AL^2_\varphi(\mathbb{D}) \to L^2_{\varphi,\mu}(\mathbb{D})$ is bounded.

Definition 2.2. μ is called a vanishing Carleson measure on $AL^2_\varphi(\mathbb{D})$ if the imbedding operator $J : AL^2_\varphi(\mathbb{D}) \to L^2_{\varphi,\mu}(\mathbb{D})$ is compact.

Definition 2.3. For real valued function $\varphi \in C^2(\mathbb{D})$ with $\Delta\varphi > 0$, let $\tau(z) = (\Delta\varphi(z))^{-1/2}$. We say that $\varphi \in \mathcal{D}$ if the following conditions are satisfied.

(1) There exists a constant $C_1 > 0$ such that $|\tau(z) - \tau(\xi)| \le C_1|z - \xi|$ for $z, \xi \in \mathbb{D}$.

(2) There exists a constant $C_2 > 0$ such that $\tau(z) \le C_2(1 - |z|)$ for $z \in \mathbb{D}$.

(3) There exist constants $0 < t < 1$ and $a > 0$ such that $\tau(z) \le \tau(\xi) + t|z - \xi|$ for $|z - \xi| > a\tau(\xi)$.

Some typical examples of functions in class $\mathcal{D}$ are as follows:

(i) $\varphi_1(z) = -\frac{A}{2}\log(1 - |z|^2)$, $A > 0$. The corresponding weight $e^{-2\varphi_1}$ is the standard weight $(1 - |z|^2)^A$ for $A > 0$.

(ii) $\varphi_2(z) = \frac{1}{2}(-A\log(1-|z|^2)+B/(1-|z|^2)^\alpha)$, $A \ge 0$, $B > 0$, $\alpha > 0$. The corresponding weight $e^{-2\varphi_2}$ is the exponential weight

$$(1 - |z|^2)^A \exp\{-B/(1 - |z|^2)^\alpha\}, \ A \ge 0,\ B > 0,\ \alpha > 0.$$

(iii) $\varphi_1 + h$ and $\varphi_2 + h$, where φ_1 and φ_2 are as in (i) and (ii) respectively, and $h \in C^2(\mathbb{D})$ can be any harmonic function on $\mathbb{D}$.

The following notation will be frequently used:

$$m_\varphi = \frac{\min(C_1^{-1}, C_2^{-1})}{4}$$

where C_1 and C_2 are the constants of φ in Definition 2.3.

For $\varphi \in \mathcal{D}$, we have the following theorem about the Carleson measure on $AL^2_\varphi(\mathbb{D})$.

Theorem 2.4. *Let $\varphi \in \mathcal{D}$. Then μ is a Carleson measure on $AL^2_\varphi(\mathbb{D})$ if and only if there exists a constant $\alpha \in (0, m_\varphi)$ such that*

$$\sup_{z\in\mathbb{D}} \frac{1}{\tau(z)^2}\mu\{\xi \in \mathbb{D} : |\xi - z| \le \alpha\tau(z)\} < \infty. \tag{2.1}$$

Proof. The sufficiency was proved by Oleinik [**O**] under the condition (1) and (2) of Definition 2.3 for any $\alpha \in (0, m_\varphi)$. For the necessity, see [**LR**]. □

The following theorem is about the vanishing Carleson measures on $AL^2_\varphi(\mathbb{D})$.

Theorem 2.5. *Let $\varphi \in \mathcal{D}$. Then μ is a vanishing Carleson measure on $AL^2_\varphi(\mathbb{D})$ if and only if there exists a constant $\alpha \in (0, m_\varphi)$ such that*

$$\lim_{r\to 1} \sup_{r\le|z|<1} \frac{1}{\tau(z)^2}\mu\{\xi \in \mathbb{D} : |\xi - z| \le \alpha\tau(z)\} = 0.$$

Proof. For the sufficiency, see [**O**]. For the necessity, see [**LR**]. □

In this paper, we will use the equivalent discrete form of condition (2.1) in Theorem 2.4. In order to get the equivalent condition of (2.1) in discrete form, we need some notations and a covering lemma.

Throughout this paper, we will always use the following notations: $\tau(z) = (\Delta\varphi(z))^{-1/2}$, for any constant $\alpha > 0$, $D(\alpha\tau(z)) = D(z, \alpha\tau(z))$ denotes the Euclidean disk in $\mathbb{C}$ with center z and radius $\alpha\tau(z)$.

Lemma 2.6 ([**O**]). *Let $\varphi \in \mathcal{D}$ and let $\alpha \in (0, m_\varphi)$. Then there exists a sequence of points $\{z_j\} \subset \mathbb{D}$, such that the following conditions are satisfied:*

(1) $z_j \notin D(\alpha\tau(z_k))$, $j \neq k$.

(2) $\bigcup_j D(\alpha\tau(z_j)) = \mathbb{D}$.

(3) $\tilde{D}(\alpha\tau(z_j)) \subset D(3\alpha\tau(z_j))$, *where* $\tilde{D}(\alpha\tau(z_j)) = \bigcup_{z\in D(\alpha\tau(z_j))} D(\alpha\tau(z))$, $\quad j = 1, 2, \ldots$.

(4) $\{D(3\alpha\tau(z_j))\}$ *is a covering of* $\mathbb{D}$ *of finite multiplicity* N.

Definition 2.7. A covering $\{D(\alpha\tau(z_j))\}$ of $\mathbb{D}$ is called a τ-covering of $\mathbb{D}$ if it satisfies all the conditions in Lemma 2.6.

Theorem 2.8. *Let $\varphi \in \mathcal{D}$. Then μ is a Carleson measure on $AL^2_\varphi(\mathbb{D})$ if and only if there exists a constant $\alpha \in (0, m_\varphi)$ such that for every τ-covering $\{D(\alpha\tau(z_j))\}$ of $\mathbb{D}$,*

$$\sup_j \frac{\mu(D(\alpha\tau(z_j)))}{|D(\alpha\tau(z_j))|} < \infty.$$

Proof. The necessity follows from Theorem 2.4 immediately. The sufficiency follows from the proof of the sufficiency of Theorem 2.4 (see [**O**]). □

3. Toeplitz operators on $AL^2_\varphi(\mathbb{D})$.

Let μ be a finite positive Borel measure on $\mathbb{D}$ and let $K(z,w)$ be the reproducing kernel of $AL^2_\varphi(\mathbb{D})$. The Toeplitz operator T_μ on $AL^2_\varphi(\mathbb{D})$ is defined by

$$T_\mu f(z) = \int_{\mathbb{D}} K(z,w)f(w)e^{-2\varphi(w)}\, d\mu(w).$$

Recall that $J : AL^2_\varphi(\mathbb{D}) \to L^2_{\varphi,\mu}(\mathbb{D})$ is the imbedding operator. By direct computation one can check that for g, $h \in AL^2_\varphi(\mathbb{D})$,

$$\langle Jg,\, Jh\rangle_{L^2_{\varphi,\mu}} = \langle T_\mu g,\, h\rangle_{L^2_\varphi}.$$

Thus $T_\mu = J^*J$. Then the next two theorems about the Toeplitz operator T_μ on $AL^2_\varphi(\mathbb{D})$ follow immediately from Theorem 2.4 and Theorem 2.5.

Theorem 3.1. *Let $\varphi \in \mathcal{D}$. Then the Toeplitz operator T_μ is bounded on $AL^2_\varphi(\mathbb{D})$ if and only if there exists a constant $\alpha \in (0, m_\varphi)$ such that*

$$\sup_{z\in\mathbb{D}} \frac{\mu(D(\alpha\tau(z)))}{|D(\alpha\tau(z))|} < \infty.$$

Theorem 3.2. *Let $\varphi \in \mathcal{D}$. Then the Toeplitz operator T_μ is a compact operator from $AL^2_\varphi(\mathbb{D})$ to $L^2_\varphi(\mathbb{D})$ if and only if there exists a constant $\alpha \in (0, m_\varphi)$ such that*

$$\lim_{r\to 1} \sup_{r\le|z|<1} \frac{\mu(D(\alpha\tau(z)))}{|D(\alpha\tau(z))|} = 0.$$

From Theorem 2.8 we also have

Theorem 3.3. *Let $\varphi \in \mathcal{D}$. Then the Toeplitz operator T_μ is bounded on $AL^2_\varphi(\mathbb{D})$ if and only if there exists a constant $\alpha \in (0, m_\varphi)$ such that for every τ-covering $\{D(\alpha\tau(z_j))\}$ of $\mathbb{D}$,*

$$\sup_{j} \frac{\mu(D(\alpha\tau(z_j)))}{|D(\alpha\tau(z_j))|} < \infty.$$

In the rest of this section, we will characterize those finite positive Borel measure μ for which the Toeplitz operator T_μ on $AL^2_\varphi(\mathbb{D})$ belongs to the Schatten ideal S_p.

The Schatten ideal S_p consists of all the operators T on Hilbert space for which the singular numbers $s_n(T)$ form a sequence belonging to l^p. The singular numbers of the operator T are defined by

$$s_n = s_n(T) = \inf\{\|T-K\| : \text{rank } K \le n\}.$$

We denote $|T|_p = (\sum_{n=1}^\infty s_n^p)^{1/p}$. For $p \ge 1$ the quantity $|T|_p$ is a norm, while for $0 < p < 1$ we have the following inequality

$$|T+S|_p^p \le |T|_p^p + |S|_p^p.$$

We refer to [**GK**] and [**S**] for more information about S_p.

First we consider the case $1 \le p < \infty$.

Theorem 3.4. *Let $1 \le p < \infty$ and let $\varphi \in \mathcal{D}$. Then T_μ belongs to S_p if and only if there exists a constant $\alpha \in (0, m_\varphi)$ such that for every τ-covering $\{D(\alpha\tau(z_j))\}$ of $\mathbb{D}$,*

$$\sum_j \left(\frac{\mu(D(\alpha\tau(z_j)))}{|D(\alpha\tau(z_j))|}\right)^p < \infty.$$

We will prove the sufficiency first. We need a lemma.

Lemma 3.5. *Let $\varphi \in \mathcal{D}$. Then we have*

$$K(z,z)e^{-2\varphi(z)} \sim (\tau(z))^{-2} = \Delta\varphi(z), \ z \in \mathbb{D}.$$

By the relation $\sim$ we mean that the ratio of the two expressions is bounded above and below by absolute positive constants.

Proof. For any $z \in \mathbb{D}$, let $L_z f = f(z)$ be the point evaluation on $AL^2_\varphi(\mathbb{D})$. It is well known that

$$K(z,z) = \|L_z\|^2.$$

The point evaluation L_z can be regarded as an imbedding operator from $AL^2_\varphi(\mathbb{D})$ to $L^2_{\varphi, e^{2\varphi}\delta_z}(\mathbb{D})$, where δ_z is the Dirac measure at the point z. Then by Theorem 2.4 and the estimate of the norm of the imbedding operator (obtained in the proof of Theorem 2.4, see [**O**] and [**LR**]), we have

$$\begin{aligned}\|L_z\|^2 &\sim \sup_{w\in\mathbb{D}} \frac{1}{\tau(w)^2}\int_{D(\alpha\tau(w))} e^{2\varphi(\xi)}\delta_z(\xi), \quad \text{for some} \quad \alpha \in (0, m_\varphi)\\ &\sim \frac{1}{\tau(z)^2}e^{2\varphi(z)}\end{aligned}$$

where we use the fact that $\tau(w) \sim \tau(z)$ whenever $|z-w| < m_\varphi \tau(w)$, which follows easily from condition (1) of Definition 2.3. Thus

$$K(z,z)e^{-2\varphi(z)} \sim (\tau(z))^{-2} = \Delta\varphi(z), \quad z \in \mathbb{D}.$$

This finishes the proof of Lemma 3.5. □

Proof of the Sufficiency of Theorem 3.4. Let $\{e_n\}$ be any orthonormal set in $AL^2_\varphi(\mathbb{D})$. For any $n \ge 1$,

$$\langle T_\mu e_n, e_n\rangle_{L^2_\varphi} = \int_{\mathbb{D}} |e_n(z)|^2 e^{-2\varphi(z)}\,d\mu(z).$$

Since μ is a finite positive Borel measure on $\mathbb{D}$, it follows that

$$\begin{aligned}\sum_n |\langle T_\mu e_n, e_n\rangle_{L^2_\varphi}| &= \int_{\mathbb{D}} \sum_n |e_n(z)|^2 e^{-2\varphi(z)}\,d\mu(z)\\ &\le \int_{\mathbb{D}} K(z,z)e^{-2\varphi(z)}\,d\mu(z).\end{aligned}$$

Let $\{D(\alpha\tau(z_j))\}$ be a τ-covering of $\mathbb{D}$ with $\alpha \in (0, m_\varphi)$. Then

$$\begin{aligned}\sum_n |\langle T_\mu e_n, e_n\rangle_{L^2_\varphi}| &\le \int_{\mathbb{D}} K(z,z)e^{-2\varphi(z)}\, d\mu(z) \\ &\le \sum_j \int_{D(\alpha\tau(z_j))} K(z,z)e^{-2\varphi(z)}\, d\mu(z) \\ &\le C\sum_j \int_{D(\alpha\tau(z_j))} (\tau(z))^{-2}\, d\mu(z)\end{aligned}$$

where the last inequality is by Lemma 3.5. As we pointed out before, $\tau(z) \sim \tau(w)$ whenever $|z-w| < m_\varphi \tau(w)$. Thus we have

$$\begin{aligned}\sum_n |\langle T_\mu e_n, e_n\rangle_{L^2_\varphi}| &\le C\sum_j (\tau(z_j))^{-2} \int_{D(\alpha\tau(z_j))} d\mu(z) \\ &= C\sum_j \frac{\mu(D(\alpha\tau(z_j)))}{|D(\alpha\tau(z_j))|}.\end{aligned}$$

Therefore

$$T_\mu \in S_1 \text{ if } \sum_j \frac{\mu(D(\alpha\tau(z_j)))}{|D(\alpha\tau(z_j))|} < \infty.$$

On the other hand, by Theorem 3.3 we have

$$T_\mu \in S_\infty \text{ if } \sup_j \frac{\mu(D(\alpha\tau(z_j)))}{|D(\alpha\tau(z_j))|} < \infty.$$

It then follows by interpolation that

$$T_\mu \in S_p \text{ if } \sum_j \left(\frac{\mu(D(\alpha\tau(z_j)))}{|D(\alpha\tau(z_j))|}\right)^p < \infty.$$

This completes the proof of the sufficiency. □

To prove the necessity, we need two more lemmas.

Lemma 3.6. *Let $\varphi \in \mathcal{D}$ and let*

$$k_w(z) = K(z,w)(K(w,w))^{-1/2}.$$

Then there exists a small constant $\alpha_0 \in (0, m_\varphi)$ such that

$$|k_w(z)|^2 \sim K(z,z) \quad \textit{whenever} \quad |z-w| < \alpha_0\tau(w).$$

Proof. For any fixed $w_0 \in \mathbb{D}$, consider the subspace $AL^2_\varphi(\mathbb{D}, w_0)$ which is defined by

$$AL^2_\varphi(\mathbb{D}, w_0) = \left\{ f \in AL^2_\varphi(\mathbb{D}) : f(w_0) = 0 \right\}.$$

Note that we have the decomposition

$$AL^2_\varphi(\mathbb{D}) = AL^2_\varphi(\mathbb{D}, w_0) \oplus \mathcal{L}_{w_0} \tag{3.1}$$

where $\mathcal{L}_{w_0}$ is the one-dimensional subspace spanned by the function

$$k_{w_0}(z) = K(z, w_0)(K(w_0, w_0))^{-1/2}.$$

We denote by $K_{w_0}(z, w)$ the reproducing kernel of $AL^2_\varphi(\mathbb{D}, w_0)$. From (3.1) we obtain

$$K(z, z) = K_{w_0}(z, z) + |k_{w_0}(z)|^2. \tag{3.2}$$

Hence we always have

$$|k_{w_0}(z)|^2 \leq K(z, z). \tag{3.3}$$

Now we need to prove the reverse inequality. By (3.2) we only need to show that there exist constants $0 < \alpha_0 < m_\varphi$ and $0 < \delta_0 < 1$ such that

$$K_{w_0}(z, z) \leq \delta_0 K(z, z) \quad \text{whenever} \quad |z - w_0| < \alpha_0 \tau(w_0). \tag{3.4}$$

Let us consider the operator

$$(S_{w_0} f)(z) = f(z)(z - w_0)^{-1}.$$

It is easy to check that S_{w_0} maps $AL^2_\varphi(\mathbb{D}, w_0)$ into $AL^2_\varphi(\mathbb{D})$. Let $V^z_{w_0} : \mathbb{C} \to \mathbb{C}$ be defined by $V^z_{w_0}(\xi) = (z - w_0)\xi$. Then the point evaluation $U^z_{w_0} f = f(z)$ on $AL^2_\varphi(\mathbb{D}, w_0)$ can be represented as

$$U^z_{w_0} = V^z_{w_0} L_z S_{w_0}$$

where L_z is the point evaluation on $AL^2_\varphi(\mathbb{D})$. Hence

$$\|U^z_{w_0}\| \leq \|V^z_{w_0}\| \, \|L_z\| \, \|S_{w_0}\|. \tag{3.5}$$

When $|z - w_0| < \alpha_0 \tau(w_0)$, where $\alpha_0 \in (0, m_\varphi)$ will be chosen later, it is obvious that

$$\|V^z_{w_0}\| \leq \alpha_0 \tau(w_0). \tag{3.6}$$

To estimate the norm of S_{w_0}, let us take a small $\alpha_1 \in (0,\, m_\varphi)$. The choice of α_1 will be made precise later. For any $f \in AL^2_\varphi(\mathbb{D}, w_0)$, let

$$g(z) = f(z)(z-w_0)^{-1} = (S_{w_0}f)(z).$$

Then $g \in AL^2_\varphi(\mathbb{D})$ since S_{w_0} maps $AL^2_\varphi(\mathbb{D}, w_0)$ into $AL^2_\varphi(\mathbb{D})$. For this g we have

$$\|g\|^2_{L^2_\varphi} = \int_{D(\alpha_1\tau(w_0))} |g(z)|^2 e^{-2\varphi(z)}\,dA(z) + \int_{\mathbb{D}\setminus D(\alpha_1\tau(w_0))} |g(z)|^2 e^{-2\varphi(z)}\,dA(z). \tag{3.7}$$

By the reproducing property we have

$$g(z) = \int_{\mathbb{D}} K(z,w)g(w)e^{-2\varphi(w)}\,dA(w).$$

It then follows

$$\begin{aligned}\int_{D(\alpha_1\tau(w_0))} &|g(z)|^2 e^{-2\varphi(z)}\,dA(z)\\ &\le \int_{D(\alpha_1\tau(w_0))} \|K(z,\cdot)\|^2_{L^2_\varphi}\|g\|^2_{L^2_\varphi} e^{-2\varphi(z)}\,dA(z)\\ &= \int_{D(\alpha_1\tau(w_0))} K(z,z)e^{-2\varphi(z)}\,dA(z)\cdot\|g\|^2_{L^2_\varphi}.\end{aligned} \tag{3.8}$$

By using Lemma 3.5, we obtain

$$\begin{aligned}\int_{D(\alpha_1\tau(w_0))} K(z,z)e^{-2\varphi(z)}\,dA(z) &\le C\int_{D(\alpha_1\tau(w_0))} (\tau(z))^{-2}\,dA(z)\\ &\le C(\tau(w_0))^{-2}\int_{D(\alpha_1\tau(w_0))} dA(z)\\ &\le C\alpha_1.\end{aligned} \tag{3.9}$$

The second inequlity is because $\tau(z) \sim \tau(w)$ whenever $|z-w| < m_\varphi\tau(w)$. Note that the constant C in (3.9) is independent of w_0. Now we choose a small $\alpha_1 \in (0,\, m_\varphi)$ such that $C\alpha_1 < 1$ in (3.9). Then from (3.7), (3.8) and (3.9) we obtain

$$\begin{aligned}\|g\|^2_{L^2_\varphi} &\le C\int_{\mathbb{D}\setminus D(\alpha_1\tau(w_0))} |g(z)|^2 e^{-2\varphi(z)}\,dA(z)\\ &= C\int_{\mathbb{D}\setminus D(\alpha_1\tau(w_0))} \left|\frac{f(z)}{z-w_0}\right|^2 e^{-2\varphi(z)}\,dA(z)\\ &\le C(\tau(w_0))^{-2}\|f\|^2_{L^2_\varphi}.\end{aligned}$$

It then follows that

$$\|S_{w_0}\| \le C(\tau(w_0))^{-1} \tag{3.10}$$

where C is independent of w_0.

Now from (3.5), (3.6) and (3.10) we obtain, for $|z - w_0| < \alpha_0\tau(w_0)$, that

$$\begin{aligned}\|U^z_{w_0}\| &\le C\alpha_0\tau(w_0)(\tau(w_0))^{-1}\|L_z\| \\ &= C\alpha_0\|L_z\|.\end{aligned} \tag{3.11}$$

Choose $\alpha_0 \in (0, m_\varphi)$ such that $C\alpha_0 < 1$ in (3.11). Since $\|U^z_{w_0}\|^2 = K_{w_0}(z,z)$ and $\|L_z\|^2 = K(z,z)$, we have

$$K_{w_0}(z,z) \le \delta_0 K(z,z) \quad \text{whenever} \quad |z - w_0| < \alpha_0\tau(w_0)$$

where $\delta_0 = (C\alpha_0)^2 < 1$ is independent of w_0. This completes the proof of (3.4) and of Lemma 3.6. □

We will always let $k_w(z) = K(z,w)(K(w,w))^{-1/2}$, which is the normalized reproducing kernel of $AL^2_\varphi(\mathbb{D})$.

Lemma 3.7. *Let $\varphi \in \mathcal{D}$ and let $\{D(\alpha\tau(z_j))\}$ be a τ-covering of $\mathbb{D}$ with $0 < \alpha < m_\varphi$. Then for every orthonormal sequence $\{e_j\}$ in $AL^2_\varphi(\mathbb{D})$, the operator A taking e_j to $k_{z_j}(z)$ is bounded.*

Proof. It is required to show

$$\left\| A\left(\sum_j c_j e_j\right)\right\|_{L^2_\varphi} \le C\left(\sum_j |c_j|^2\right)^{1/2}.$$

For any $g \in AL^2_\varphi(\mathbb{D})$, we have

$$\begin{aligned}\left|\left\langle A\left(\sum_j c_j e_j\right), g\right\rangle_{L^2_\varphi}\right| &= \left|\left\langle \sum_j c_j k_{z_j}, g\right\rangle_{L^2_\varphi}\right| \\ &= \left|\sum_j c_j \left\langle K(\cdot, z_j)(K(z_j,z_j))^{-1/2}, g\right\rangle_{L^2_\varphi}\right| \\ &= \left|\sum_j c_j (K(z_j,z_j))^{-1/2}\bar{g}(z_j)\right|\end{aligned}$$

$$\leq \left(\sum_j |c_j|^2\right)^{1/2} \left(\sum_j |g(z_j)|^2 (K(z_j, z_j))^{-1}\right)^{1/2}$$
$$\leq \left(\sum_j |c_j|^2\right)^{1/2} \left(\int_{\mathbb{D}} |g|^2 e^{-2\varphi}\, d\mu_0\right)^{1/2}$$

where μ_0 is the discrete measure defined by

$$\mu_0(\{z_j\}) = (K(z_j, z_j))^{-1} e^{2\varphi(z_j)}, \quad j = 1, 2, \cdots .$$

For μ_0 we have

$$\begin{aligned}\sup_j \frac{\mu_0(D(\alpha\tau(z_j)))}{|D(\alpha\tau(z_j))|} &= \sup_j \frac{(K(z_j, z_j))^{-1} e^{2\varphi(z_j)}}{|D(\alpha\tau(z_j))|} \\ &\leq C \sup_j \frac{\tau(z_j)^2}{|D(\alpha\tau(z_j))|} \quad \text{(by Lemma 3.5)} \\ &= C \sup_j \frac{\tau(z_j)^2}{\pi\alpha^2\tau(z_j)^2} < \infty.\end{aligned}$$

Thus by Theorem 2.8 we have

$$\left(\int_{\mathbb{D}} |g|^2 e^{-2\varphi}\, d\mu_0\right)^{1/2} \leq C\|g\|_{L^2_\varphi}.$$

This completes the proof of Lemma 3.7. □

Proof of the Necessity of Theorem 3.4. From p. 94 of [**GK**] a necessary condition for an operator T on a Hilbert space to be in S_p is that $\sum_j |\langle Te_j, e_j\rangle|^p < \infty$ for any orthonormal set $\{e_j\}$. If T is in S_p then so is A^*TA for any bounded operator A. If we choose A as in Lemma 3.7, then the necessary condition $\sum_j |\langle A^*T_\mu Ae_j, e_j\rangle_{L^2_\varphi}|^p < \infty$ becomes $\sum_j |\langle T_\mu k_{z_j}, k_{z_j}\rangle_{L^2_\varphi}|^p < \infty$. But

$$\begin{aligned}|\langle T_\mu k_{z_j}, k_{z_j}\rangle_{L^2_\varphi}|^p &= \left(\int_{\mathbb{D}} |k_{z_j}(z)|^2 e^{-2\varphi(z)}\, d\mu(z)\right)^p \\ &\geq \left(\int_{D(\alpha_0\tau(z_j))} |k_{z_j}(z)|^2 e^{-2\varphi(z)}\, d\mu(z)\right)^p\end{aligned}$$

where α_0 is chosen as in Lemma 3.6. By Lemma 3.6 and Lemma 3.5 we have

$$\begin{aligned}\left(\int_{D(\alpha_0\tau(z_j))} |k_{z_j}(z)|^2 e^{-2\varphi(z)}\,d\mu(z)\right)^p &\geq C\left(\int_{D(\alpha_0\tau(z_j))} |K(z,z)|^2 e^{-2\varphi(z)}\,d\mu(z)\right)^p \\ &\geq C\left(\int_{D(\alpha_0\tau(z_j))} (\tau(z))^{-2}\,d\mu(z)\right)^p \\ &\geq C\left((\tau(z_j))^{-2}\int_{D(\alpha_0\tau(z_j))} d\mu(z)\right)^p \\ &= C\left(\frac{\mu(D(\alpha\tau(z_j)))}{|D(\alpha\tau(z_j))|}\right)^p.\end{aligned}$$

The last inequality is because $\tau(z) \sim \tau(w)$ whenever $|z-w| < m_\varphi \tau(w)$. Therefore

$$\sum_j \left(\frac{\mu(D(\alpha\tau(z_j)))}{|D(\alpha\tau(z_j))|}\right)^p < \infty.$$

This completes the proof of the necessity and of Theorem 3.4. □

For the case $0 < p < 1$, we have a sufficient condition.

Theorem 3.8. *Let $0 < p < 1$ and let $\varphi \in \mathcal{D}$. If there exists a constant $\alpha \in (0, m_\varphi)$ such that for every τ-covering $\{D(\alpha\tau(z_j))\}$ of $\mathbb{D}$,*

$$\sum_j \left(\frac{\mu(D(\alpha\tau(z_j)))}{|D(\alpha\tau(z_j))|}\right)^p < \infty,$$

then T_μ belongs to S_p.

Proof. We only need to consider the case $0 < p \leq 1/2$ because the results for $1/2 < p < 1$ can follow by interpolation.

Since $T_\mu = J^*J$, where $J : AL^2_\varphi(\mathbb{D}) \to L^2_{\varphi,\mu}(\mathbb{D})$ is the imbedding operator, we have $|T_\mu|_p^p = |J|_{2p}^{2p}$. Let $\{D(\alpha_0(z_j))\}$ be a τ-covering of $\mathbb{D}$ where $0 < \alpha_0 < m_\varphi$ is chosen as in Lemma 3.6 and let $\{\sigma_j\}$ be a partition of unity subordinate to the covering $\{D(\alpha_0\tau(z_j))\}$. Then for any $f \in AL^2_\varphi(\mathbb{D})$ we have

$$f = \sum_j \sigma_j f.$$

We introduce the following operators

$$J_j : AL^2_\varphi(\mathbb{D}) \to L^2_{\varphi,\mu}(D(\alpha_0\tau(z_j))); \quad J_j f = \sigma_j f$$

and the natural imbeddings

$$I_j : L^2_{\varphi,\mu}(D(\alpha_0\tau(z_j))) \to L^2_{\varphi,\mu}(\mathbb{D}); \quad I_j g = g.$$

We then have

$$J = \sum_j I_j J_j.$$

Since $0 < 2p \le 1$, we have

$$|J|_{2p}^{2p} \le \sum_j |I_j J_j|_{2p}^{2p} \le \sum_j |J_j|_{2p}^{2p}.$$

Now everything reduces to an estimate of the norm $|J_j|_{2p}$.

By (3.1) we have the orthogonal decomposition

$$AL^2_\varphi(\mathbb{D}) = AL^2_\varphi(\mathbb{D}, z_j) \oplus \mathcal{L}_{z_j}$$

where $AL^2_\varphi(\mathbb{D}, z_j) = \left\{ f \in AL^2_\varphi(\mathbb{D}) : f(z_j) = 0 \right\}$ and $\mathcal{L}_{z_j}$ is the one-dimensional subspace spanned by the function $k_{z_j}(z) = K(z, z_j)(K(z_j, z_j))^{-1/2}$. Set

$$\begin{aligned} J_j^{(1)} &= J_j|_{AL^2_\varphi(\mathbb{D}, z_j)} : \ AL^2_\varphi(\mathbb{D}, z_j) \to L^2_{\varphi,\mu}(D(\alpha_0\tau(z_j))), \\ J_j^{(2)} &= J_j|_{\mathcal{L}_{z_j}} : \ \mathcal{L}_{z_j} \to L^2_{\varphi,\mu}(D(\alpha_0\tau(z_j))). \end{aligned}$$

It is clear that $J_j = J_j^{(1)} + J_j^{(2)}$. Hence for $0 < 2p \le 1$,

$$|J_j|_{2p}^{2p} \le |J_j^{(1)}|_{2p}^{2p} + |J_j^{(2)}|_{2p}^{2p}. \tag{3.12}$$

Since $J_j^{(2)}$ is a rank one operator, we have

$$|J_j^{(2)}|_{2p} = |J_j^{(2)}|_2 \le \left(\int_{D(\alpha_0\tau(z_j))} K(z,z) e^{-2\varphi(z)}\, d\mu(z) \right)^{1/2}. \tag{3.13}$$

To estimate $|J_j^{(1)}|_{2p}$ we consider the division operator

$$S_j : AL^2_\varphi(\mathbb{D}, z_j) \to AL^2_\varphi(\mathbb{D}); \quad (S_j f)(z) = f(z)(z - z_j)^{-1}$$

and the multiplication operator

$$T_j : L^2_{\varphi,\mu}(D\alpha_0\tau(z_j)) \to L^2_{\varphi,\mu}(D(\alpha_0\tau(z_j))); \quad (T_j f)(z) = f(z)(z - z_j).$$

The operator $J_j^{(1)}$ admits a decomposition $J_j^{(1)} = T_j J_j S_j$. Hence

$$|J_j^{(1)}|_{2p} \le \|T_j\|\, \|S_j\|\, |J_j|_{2p}. \tag{3.14}$$

Then as in the proof of Lemma 3.6, there exists a constant $0 < \delta_0 < 1$ such that

$$\|T_j\| \, \|S_j\| < \delta_0.$$

Thus from (3.12), (3.13) and (3.14) we obtain

$$|J_j|_{2p}^{2p} \leq (1 - \delta_0^{2p})^{-1} \left(\int_{D(\alpha_0 \tau(z_j))} K(z,z) e^{-2\varphi(z)} \, d\mu(z) \right)^p .$$

Then by Lemma 3.5 and the fact that $\tau(z) \sim \tau(w)$ whenever $|z - w| < m_\varphi \tau(w)$, we obtain

$$|J_j|_{2p}^{2p} \leq C \left(\frac{\mu(D(\alpha_0 \tau(z_j)))}{|D(\alpha_0 \tau(z_j))|} \right)^p .$$

Thus

$$|T_\mu|_p^p = |J|_{2p}^{2p} \leq \sum_j |J_j|_{2p}^{2p} \leq C \sum_j \left(\frac{\mu(D(\alpha_0 \tau(z_j)))}{|D(\alpha_0 \tau(z_j))|} \right)^p < \infty.$$

This completes the proof. $\square$

4. Schatten class Hankel operators on $AL^2_\varphi(\mathbb{D})$.

In [**LR**] we studied the boundedness and compactness of the Hankel operator H_b on $AL^2_\varphi(\mathbb{D})$ with $b \in L^2(\mathbb{D})$. We restate our main results in [**LR**] here for convenient reference.

Theorem 4.1 ([**LR**]). *Let $\varphi \in \mathcal{D}$ and suppose that $H^\infty_\varphi(\mathbb{D})$ is dense in $AL^2_\varphi(\mathbb{D})$. Let $b \in L^2(\mathbb{D})$ and let H_b be defined on $H^\infty_\varphi(\mathbb{D})$ by $H_b f = bf - P(bf)$. Then the following are equivalent.*
(1) H_b is bounded in the L^2_φ norm.
(2) The function $F_\alpha(z)$ defined by

$$F_\alpha(z)^2 = \inf \left\{ \frac{1}{|D(\alpha\tau(z))|} \int_{D(\alpha\tau(z))} |b - h|^2 \, dA : \; h \;\; \textit{analytic in} \;\; D(\alpha\tau(z)) \right\}$$

is bounded for some $\alpha \in (0, m_\varphi)$.
(3) b admits a decomposition $b = b_1 + b_2$ where $b_2 \in C^1(\mathbb{D})$ and satisfies

$$\frac{\bar{\partial} b_2}{(\Delta\varphi)^{1/2}} \in L^\infty(\mathbb{D}),$$

while b_1 satisfies the following condition: the function $G_\alpha(z)$ defined by

$$G_\alpha(z)^2 = \frac{1}{|D(\alpha\tau(z))|}\int_{D(\alpha\tau(z))} |b_1|^2\, dA$$

is bounded for some $\alpha \in (0, m_\varphi)$.

Theorem 4.2 ([**LR**]). *Let $\varphi \in \mathcal{D}$ and suppose that $H^\infty_\varphi(\mathbb{D})$ is dense in $AL^2_\varphi(\mathbb{D})$. Let $b \in L^2(\mathbb{D})$ and let H_b be defined on $H^\infty_\varphi(\mathbb{D})$ by $H_b f = bf - P(bf)$. Then the following are equivalent.*
(1) H_b is (extends to) a compact operator from $AL^2_\varphi(\mathbb{D})$ to $L^2_\varphi(\mathbb{D})$.
(2) The function $F_\alpha(z)$ defined by

$$F_\alpha(z)^2 = \inf\left\{\frac{1}{|D(\alpha\tau(z))|}\int_{D(\alpha\tau(z))} |b-h|^2\, dA :\ h \ \ \text{analytic in} \ \ D(\alpha\tau(z))\right\}$$

tends to zero as $|z| \to 1$ for some $\alpha \in (0, m_\varphi)$.
(3) b admits a decomposition $b = b_1 + b_2$ with $b_2 \in C^1(\mathbb{D})$ so that

$$\frac{\bar\partial b_2(z)}{(\Delta\varphi(z))^{1/2}} \to 0 \quad \text{as } |z| \to 1,$$

and for some $\alpha \in (0, m_\varphi)$, $G_\alpha(z) \to 0$ as $|z| \to 1$, where the function $G_\alpha(z)$ is defined by

$$G_\alpha(z)^2 = \frac{1}{|D(\alpha\tau(z))|}\int_{D(\alpha\tau(z))} |b_1|^2\, dA.$$

Remark 4.3. If $\varphi \in \mathcal{D}$ is a radial function, one can show that $H^\infty_\varphi(\mathbb{D})$ is dense in $AL^2_\varphi(\mathbb{D})$. So, at least for radial function $\varphi \in \mathcal{D}$, the assumption that $H^\infty_\varphi(\mathbb{D})$ is dense in $AL^2_\varphi(\mathbb{D})$ is satisfied in Theorem 4.1 and Theorem 4.2.

Now we consider the membership of H_b in the Schatten classes S_p. We have the following theorem.

Theorem 4.4. *Let $\varphi \in \mathcal{D}$ and suppose that $H^\infty_\varphi(\mathbb{D})$ is dense in $AL^2_\varphi(\mathbb{D})$. Let $1 \le p < \infty$ and let $b \in L^2(\mathbb{D})$. Assume that H_b is bounded in the L^2_φ norm. Then the following are equivalent.*
(1) H_b belongs to S_p.
(2) $F_\alpha(z) \in L^p(\mathbb{D}, \Delta\varphi dA)$ for some $\alpha \in (0, m_\varphi)$, where the function $F_\alpha(z)$ is defined by

$$F_\alpha(z)^2 = \inf\left\{\frac{1}{|D(\alpha\tau(z))|}\int_{D(\alpha\tau(z))} |b-h|^2\, dA :\ h \ \ \text{analytic in} \ \ D(\alpha\tau(z))\right\}.$$

(3) *b admits a decomposition $b = b_1 + b_2$ where b_1 satisfies*

$$G_\alpha(z) = \left(\frac{1}{|D(\alpha\tau(z))|} \int_{D(\alpha\tau(z))} |b_1|^2 \, dA \right)^{1/2} \in L^p(\mathbb{D}, \Delta\varphi dA)$$

for some $\alpha \in (0, m_\varphi)$, and $\bar\partial b_2/(\Delta\varphi)^{1/2}$ satisfies the same condition as b_1.

By using a similar argument of [**Lu2**], one can show that Theorem 4.4 is equivalent to the following theorem.

Theorem 4.4$'$. *Let $\varphi \in \mathcal{D}$ and suppose that $H^\infty_\varphi(\mathbb{D})$ is dense in $AL^2_\varphi(\mathbb{D})$. Let $1 \le p < \infty$ and let $b \in L^2(\mathbb{D})$. Assume that H_b is bounded in the L^2_φ norm. Then the following are equivalent.*
(1$'$) *H_b belongs to S_p.*
(2$'$) *There exists a constant $\alpha \in (0, m_\varphi)$ such that for every τ-covering $\{D(\alpha\tau(z_j))\}$ of $\mathbb{D}$,*

$$\sum_j F_\alpha(z_j)^p < \infty.$$

(3$'$) *There exist a constant $\alpha \in (0, m_\varphi)$ and a decomposition $b = b_1 + b_2$ such that for every τ-covering $\{D(\alpha\tau(z_j))\}$ of $\mathbb{D}$,*

$$\sum_j \left(\frac{1}{|D(\alpha\tau(z_j))|} \int_{D(\alpha\tau(z_j))} |b_1|^2 \, dA \right)^{p/2} < \infty,$$

and the same holds with $\bar\partial b_2/(\Delta\varphi)^{1/2}$ in place of b_1.

So we only need to prove Theorem 4.4$'$.

Proof of Theorem 4.4$'$. First we prove (1$'$) $\Rightarrow$ (2$'$). Let $\{D(\alpha_0\tau(z_j))\}$ be a τ-covering of $\mathbb{D}$, where $0 < \alpha_0 < m_\varphi$ is chosen as in Lemma 3.6. Since $H^\infty_\varphi(\mathbb{D})$ is dense in $AL^2_\varphi(\mathbb{D})$ and convergence in $AL^2_\varphi(\mathbb{D})$ implies uniform convergence on compactra, it is easy to see that for each z_j $(j = 1, 2, \cdots)$ there exists $\tilde{k}_{z_j}(z) \in H^\infty_\varphi(\mathbb{D})$ $(j = 1, 2, \cdots)$ satisfying the following conditions:

$$\|\tilde{k}_{z_j} - k_{z_j}\|_{L^2_\varphi} < \frac{1}{2^j} \tag{4.1}$$

and

$$|\tilde{k}_{z_j}(z)|^2 e^{-2\varphi(z)} \ge C(\tau(z_j))^{-2} \quad \text{whenever} \quad |z - z_j| < \alpha_0\tau(z_j). \tag{4.2}$$

Let $\{e_j\}$ be an orthonormal sequence in $AL^2_\varphi(\mathbb{D})$ and let $\tilde{A}$ be the operator taking e_j to $\tilde{k}_{z_j}(z)$. We have

$$\tilde{A}e_j = \tilde{k}_{z_j}(z) = k_{z_j}(z) + (\tilde{k}_{z_j}(z) - k_{z_j}(z)) = Ae_j + Ee_j, \quad (j = 1, 2, \cdots)$$

where $Ae_j = k_{z_j}(z)$ is a bounded operator by Lemma 3.7 and $Ee_j = \tilde{k}_{z_j}(z) - k_{z_j}(z)$. To prove $\tilde{A}$ is bounded, we only need to show that E is bounded. By using (4.1) we have

$$\begin{aligned}
\left\| E\left(\sum_j c_j e_j\right)\right\|_{L^2_\varphi} &= \left\|\sum_j c_j(\tilde{k}_{z_j} - k_{z_j})\right\|_{L^2_\varphi} \\
&\le \sum_j |c_j| \|\tilde{k}_{z_j} - k_{z_j}\|_{L^2_\varphi} \\
&\le \sum_j |c_j| \frac{1}{2^j} \\
&\le \left(\sum_j |c_j|^2\right)^{1/2} \left(\sum_j \frac{1}{2^{2j}}\right)^{1/2} \\
&= C\left(\sum_j |c_j|^2\right)^{1/2}.
\end{aligned}$$

Thus E is bounded, so is $\tilde{A}$.

From p. 94 of [**GK**] a necessary condition for an operator T on a Hilbert space to be in S_p is that $\sum_j |\langle Te_j, e_j\rangle|^p < \infty$ for any orthonormal sequence $\{e_j\}$. We apply this to $B^*H_b\tilde{A}$ where $\tilde{A}$ is as above and $Be_j = a_j \chi_{D(\alpha_0\tau(z_j))} H_b(\tilde{k}_{z_j})$ with $a_j = (\int_{D(\alpha_0\tau(z_j))} |H_b(\tilde{k}_{z_j})|^2 e^{-2\varphi}\, dA)^{-1/2}$. By the finite multiplicity property of the τ-covering, it is easy to see that B is bounded. Since $H_b \in S_p$, we have $B^*H_b\tilde{A} \in S_p$. Thus

$$\begin{aligned}
&\sum_j \left|\left\langle B^* H_b \tilde{A} e_j, e_j\right\rangle_{L^2_\varphi}\right|^p \\
&\quad = \sum_j \left| a_j \left\langle H_b \tilde{k}_{z_j}, \chi_{D(\alpha_0\tau(z_j))} H_b \tilde{k}_{z_j}\right\rangle_{L^2_\varphi}\right|^p \\
&\quad = \sum_j \left(\int_{D(\alpha_0\tau(z_j))} |H_b(\tilde{k}_{z_j})|^2 e^{-2\varphi}\, dA\right)^{p/2} \\
&\quad = \sum_j \left(\int_{D(\alpha_0\tau(z_j))} \left| b\tilde{k}_{z_j} - P(b\tilde{k}_{z_j})\right|^2 e^{-2\varphi}\, dA\right)^{p/2} \\
&\quad = \sum_j \left(\int_{D(\alpha_0\tau(z_j))} \left| b - \frac{1}{\tilde{k}_{z_j}} P(b\tilde{k}_{z_j})\right|^2 |\tilde{k}_{z_j}|^2 e^{-2\varphi}\, dA\right)^{p/2}
\end{aligned}$$

$$\geq C\sum_j \left(\frac{1}{|D(\alpha_0\tau(z_j))|} \int_{D(\alpha_0\tau(z_j))} \left| b - \frac{1}{\tilde{k}_{z_j}} P(b\tilde{k}_{z_j}) \right|^2 e^{-2\varphi}\, dA \right)^{p/2}$$
$$\geq C\sum_j F_{\alpha_0}(z_j)^p$$

where the first inequality is by (4.2). Hence (1′) implies (2′).

Now we prove (2′) $\Rightarrow$ (3′). As we point out in the proof of (2) $\Rightarrow$ (3) of Theorem 4.1 in [**LR**], the functions b_1 and b_2 produced in the proof of Theorem 3.1 in [**LR**] actually satisfy the following conditions:

$$(4.3) \qquad \frac{1}{|D(\alpha\tau(z))|} \int_{D(\alpha\tau(z))} |b_1|^2\, dA \leq C \sup\left\{F_\alpha(w)^2 : w \in D(3\alpha\tau(z))\right\}$$

and

$$(4.4) \qquad \left| \frac{\bar{\partial} b_2(z)}{(\Delta\varphi(z))^{1/2}} \right| \leq C \sup\left\{F_\alpha(w) : w \in D(3\alpha\tau(z))\right\}.$$

It is easy to verify from the definition of $F_\alpha(w)$ that

$$(4.5) \qquad \sup\left\{F_\alpha(w) : w \in D(3\alpha\tau(z))\right\} \leq CF_{4\alpha}(z).$$

If we replace α by $\alpha/5$, then from (4.3), (4.4) and (4.5) we obtain

$$(4.6) \qquad \frac{1}{|D(\alpha/5\tau(z))|} \int_{D(\alpha/5\tau(z))} |b_1|^2\, dA \leq CF_{4\alpha/5}(z)$$

and

$$(4.7) \qquad \left| \frac{\bar{\partial} b_2(z)}{(\Delta\varphi(z))^{1/2}} \right| \leq CF_{4\alpha/5}(z).$$

Let us consider the τ-covering $\{D(\alpha/5\tau(z_j))\}$ of $\mathbb{D}$. From (4.6) and the fact $F_{4\alpha/5}(z) \leq CF_\alpha(z)$, we obtain

$$\sum_j \left(\frac{1}{|D(\alpha/5\tau(z_j))|} \int_{D(\alpha/5\tau(z_j))} |b_1|^2\, dA \right)^{p/2} \leq C\sum_j F_{4\alpha/5}(z_j)^p$$
$$\leq C\sum_j F_\alpha(z_j)^p < \infty.$$

If $z \in D(\alpha/5\tau(z_j))$, then it is easy to verify from its definition that $F_{4\alpha/5\tau(z)} \leq CF_\alpha(z_j)$. Therefore, from (4.7) we obtain

$$\sum_j \left(\frac{1}{|D(\alpha/5\tau(z_j))|} \int_{D(\alpha/5\tau(z_j))} \left| \frac{\bar{\partial} b_2(z)}{(\Delta\varphi(z))^{1/2}} \right|^2 dA \right)^{p/2} \leq C\sum_j F_\alpha(z_j)^p < \infty.$$

Thus (2′) implies (3′).

Finally, let us show (3′) ⇒ (1′). Let H_b be a bounded Hankel operator and let $b = b_1 + b_2$ be as in part (3) of Theorem 4.1. The argument in the proof of the boundedness theorem of H_b in [**LR**] actually shows that for any $f \in AL^2_\varphi(\mathbb{D})$,

$$\begin{aligned} &\|H_{b_1}f\|_{L^2_\varphi} \le C\|M_{b_1}f\|_{L^2_\varphi} \\ \text{and} \qquad &\|H_{b_2}f\|_{L^2_\varphi} \le C\|M_{\bar\partial b_2/(\Delta\varphi)^{1/2}}f\|_{L^2_\varphi}, \end{aligned}$$

where $M_{b_1}f = b_1 f$ and $M_{\bar\partial b_2/(\Delta\varphi)^{1/2}}f = \bar\partial b_2/(\Delta\varphi)^{1/2}f$ are the multiplication operators. By the following equivalent definition of the singular numbers of the operator T,

$$s_n = \inf\{\|T|_W\| : \text{comdim } W = n\},$$

we know that the singular numbers for H_{b_1} and H_{b_2} are dominated by those for $M_{b_1}|_{AL^2_\varphi(\mathbb{D})}$ and $M_{\bar\partial b_2/(\Delta\varphi)^{1/2}}|_{AL^2_\varphi(\mathbb{D})}$. So, to prove $H_b \in S_p$ it suffices to show that $M_\psi : AL^2_\varphi(\mathbb{D}) \to L^2_\varphi(\mathbb{D})$ belongs to S_p for $\psi = b_1$ or $\bar\partial b_2/(\Delta\varphi)^{1/2}$. Observe that

$$\langle M_\psi f, M_\psi g\rangle_{L^2_\varphi} = \int_{\mathbb{D}} f\bar g|\psi|^2 e^{-2\varphi}\, dA = \langle T_{|\psi|^2}f, g\rangle_{L^2_\varphi}.$$

Therefore $M^*_\psi M_\psi = T_{|\psi|^2}$. Thus $M_\psi \in S_p$ if and only if $T_{|\psi|^2} \in S_{p/2}$. By Theorem 3.4 (for $p/2 \ge 1$) and Theorem 3.8 (for $0 < p/2 < 1$), the condition in (3′) is exactly what is needed to have both $T_{|b_1|^2}$ and $T_{|\bar\partial b_2/(\Delta\varphi)^{1/2}|^2}$ belong to $S_{p/2}$. Thus the corresponding multiplication operators M_{b_1} and $M_{\bar\partial b_2/(\Delta\varphi)^{1/2}}$ belong to S_p. Hence H_b belongs to S_p. This completes the proof. □

Acknowledgment. We thank the referee for several helpful suggestions.

References

[AFP] J. Arazy, S. Fisher and J. Peetre, *Hankel operators on weighted Bergman spaces*, Amer. J. Math., **110** (1988), 989-1054.

[GK] I.C. Gohberg and M.G. Krein, *Introduction to the Theory of Linear Non-selfadjoint Operators*, Nauka, Moscow, 1965; translation in Trans. Math. Monographs, Vol. 18, America Math. Society, Providence, RI, 1969.

[JPR] S. Janson, J. Peetre and R. Rochberg, *Hankel forms and the Fock space*, Rev. Mat. Iberoamericana, **3** (1987), 61-138.

[LR] Peng Lin and R. Rochberg, *Hankel operators on the weighted Bergman spaces with exponential type weights*, Integral Equations and Operator Theory, **21** (1995), 460-483.

[Lu1] D. Luecking, *Trace ideal criteria for Toeplitz operators*, J. Funct. Anal., **73** (1987), 345-368.

[Lu2] ———, *Characterizations of certain classes of Hankel operators on the Bergman spaces of the unit disk*, J. Funct. Anal., **110** (1992), 247-271.

[O] V.L. Oleinik, *Embedding theorems for weighted classes of harmonic and analytic functions*, J. Soviet Math., **9** (1978) 228-243.

[P] O.G. Parfenov, *On properties of imbedding operators of certain classes of analytic functions*, St. Petersburg Math., **3** (1992), 425-446.

[S] B. Simon, *Trace ideals and their applications*, London Math. Soc. Lecture Notes, Vol. 35, Cambridge Univ. Press, London, 1979.

[Z] Kehe Zhu, *Operator Theory in Function Spaces*, Marcel Dekker, New York, 1990.

Received August 12, 1993 and revised April 6, 1994. R. Rochberg's research was partially supported by a grant from the National Science Foundation.

WASHINGTON UNIVERSITY
ST. LOUIS, MO 63130
E-mail address: plin@math.wustl.edu
rr@math.wustl.edu