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QUANTUM AFFINE ALGEBRAS AND AFFINE HECKE ALGEBRAS

VYJAYANTHI CHARI AND ANDREW PRESSLEY

We describe a functor from the category $\mathcal{C}_m$ of finite-dimensional representations of the affine Hecke algebra of $GL(m)$ to the category $\mathcal{D}_n$ of finite-dimensional representations of affine $sl(n)$. If $m < n$, this functor is an equivalence between $\mathcal{C}_m$ and the subcategory of $\mathcal{D}_n$ consisting of those representations whose irreducible components under quantum sl(n) all occur in the m-fold tensor product of the natural representation of quantum $sl(n)$. These results are analogous to the classical Frobenius-Schur duality between the representations of general linear and symmetric groups.

1. Introduction.

One of the most beautiful results from the classical period of the representation theory of Lie groups is the correspondence, due to Frobenius and Schur, between the representations of symmetric groups and those of general or special linear groups. If V_0 is the natural irreducible $(n+1)$-dimensional representation of $SL_{n+1}(\mathbb{C})$, the symmetric group S_ℓ acts on $V_0^{\otimes \ell}$ by permuting the factors. This action obviously commutes with the action of $SL_{n+1}(\mathbb{C})$. It follows that one may associate to any right S_ℓ-module M a representation of $SL_{n+1}(\mathbb{C})$, namely

$$\mathcal{F}S(M) = M \otimes_{S_\ell} V_0^{\otimes \ell},$$

the action of $SL_{n+1}(\mathbb{C})$ on $\mathcal{F}S(M)$ being induced by its natural action on $V_0^{\otimes \ell}$. The main result of the Frobenius-Schur theory is that, if $\ell \leq n$, the assignment $M \to \mathcal{F}S(M)$ defines an equivalence from the category of finite-dimensional representations of S_ℓ to the category of finite-dimensional representations of $SL_{n+1}(\mathbb{C})$, all of whose irreducible components occur in $V_0^{\otimes \ell}$.

Around 1985, Drinfeld and Jimbo independently introduced a family of Hopf algebras $U_q(\mathfrak{g})$, depending on a parameter $q \in \mathbb{C}^\times$, associated to any symmetrizable Kac-Moody algebra $\mathfrak{g}$. Assuming that q is not a root of unity, Jimbo [7] proved an analogue of the Frobenius-Schur correspondence in which $SL_{n+1}(\mathbb{C})$ is replaced by $U_q(sl_{n+1})$, V_0 by the natural $(n+1)$-dimensional irreducible representation V of $U_q(sl_{n+1})$, and S_ℓ by its Hecke algebra $H_\ell(q^2)$.

In [**5**], Drinfeld announced an analogue of the Frobenius–Schur theory for the Yangian $Y(sl_{n+1})$, which is a "deformation" of the universal enveloping algebra of the Lie algebra of polynomial maps $\mathbb{C} \to sl_{n+1}$. The role of S_ℓ in this theory is played by the degenerate affine Hecke algebra Λ_ℓ, an algebra whose defining relations are obtained from those of the affine Hecke algebra $\widehat{H}_\ell(q^2)$ by letting $q \to 1$ in a certain non–trivial fashion.

In the same paper, Drinfeld conjectured that there should be an analogue of the Frobenius–Schur theory relating the quantum affine algebra $U_q(\widehat{sl}_{n+1})$ and $\widehat{H}_\ell(q^2)$. In this paper, we construct a functor from the category of finite-dimensional $\widehat{H}_\ell(q^2)$–modules to the category of finite–dimensional $U_q(\widehat{sl}_{n+1})$–modules W of 'type 1' (a mild spectral condition) with the property that every irreducible $U_q(sl_{n+1})$–type which occurs in W also occurs in $V^{\otimes \ell}$ (we assume that q is not a root of unity). We prove that this functor is an equivalence if $\ell \leq n$. Drinfeld's theory can be obtained from ours by taking a suitable limit $q \to 1$. Related results were obtained by Cherednik in [**4**].

We give a precise description of our functor at the level of irreducible representations, using the known parametrizations of such representations of $U_q(\widehat{sl}_{n+1})$ and of $\widehat{H}_\ell(q^2)$. Namely, in [**2**], [**3**] we showed that the finite–dimensional irreducible $U_q(\widehat{sl}_{n+1})$–modules of type 1 are in one to one correspondence with n–tuples of monic polynomials in one variable. On the other hand, Zelevinsky [**13**] and Rogawski [**12**] have given a one to one correspondence between the finite–dimensional irreducible $\widehat{H}_\ell(q^2)$–modules and the set of (unordered) collections of 'segments' of complex numbers, the sum of whose lengths is ℓ. (A segment of length k is a k–tuple of the form $(a, q^2a, \ldots, q^{2k-2}a)$, for some $a \in \mathbb{C}^\times$.) We compute explicitly the n–tuple of polynomials associated under our functor to any such collection of segments.

The affine Lie algebra $\widehat{sl}_{n+1}$ is a central extension, with one-dimensional centre, of the Lie algebra of Laurent polynomial maps $f : \mathbb{C}^\times \to sl_{n+1}$. An obvious way to construct representations of $\widehat{sl}_{n+1}$ is to pull back a representation of sl_{n+1} by the one-parameter family of homomorphisms $ev_a^0 : \widehat{sl}_{n+1} \to sl_{n+1}$ which annihilate the centre and evaluate the maps f at $a \in \mathbb{C}^\times$. In [**7**], Jimbo defined a one-parameter family of algebra homomorphisms $ev_a : U_q(\widehat{sl}_{n+1}) \to U_q(sl_{n+1})$ which are quantum analogues of the ev_a^0 (actually, ev_a takes values in an 'enlargement' of $U_q(sl_{n+1})$). On the other hand, in [**4**] Cherednik defined a one-parameter family of homomorphisms $\tilde{ev}_a : \widehat{H}_\ell(q^2) \to H_\ell(q^2)$ which are the identity on $H_\ell(q^2) \subset \widehat{H}_\ell(q^2)$. Pulling back representations of $U_q(sl_{n+1})$ (resp. $H_\ell(q^2)$) under ev_a (resp. $\tilde{ev}_a$) gives a one-parameter family of representations of $U_q(\widehat{sl}_{n+1})$ (resp. $\widehat{H}_\ell(q^2)$). We show that these 'evaluation' representations correspond to each other under our functor.

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2. Quantum Kac–Moody algebras.

Let $A = (a_{ij})$ be a symmetric generalized Cartan matrix, where the indices i, j lie in some finite set I. Thus, $a_{ij} \in \mathbf{Z}$, $a_{ii} = 2$, and $a_{ij} \leq 0$ if $i \neq j$. To A one can associate a Kac–Moody Lie algebra $\mathfrak{g}(A)$ (see [**8**]).

Let q be a non–zero complex number, assumed throughout this paper not to be a root of unity. For $n, r \in \mathbf{N}$, $n \geq r$, define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$
$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q[n-1]_q \dots [n-r+1]_q}{[r]_q[r-1]_q \dots [1]_q}.$$

2.1.

Definition. The quantum Kac–Moody algebra $U_q(\mathfrak{g}(A))$ associated to a symmetric generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ is the unital associative algebra over $\mathbb{C}$ with generators $x_i^{\pm}$, $k_i^{\pm 1}$ ($i \in I$) and the following defining relations:

$$k_i k_i^{-1} = 1 = k_i^{-1} k_i,$$
$$k_i k_j = k_j k_i,$$
$$k_i x_j^{\pm} k_i^{-1} = q^{\pm a_{ij}} x_j^{\pm},$$
$$[x_i^+, x_j^-] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}},$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_q (x_i^{\pm})^r x_j^{\pm} (x_i^{\pm})^{1-a_{ij}-r} = 0\ ,\ i \neq j.$$

It is well–known that $U_q(\mathfrak{g}(A))$ is a Hopf algebra with comultiplication Δ given on generators by

$$\Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1},$$
$$\Delta(x_i^+) = x_i^+ \otimes k_i + 1 \otimes x_i^+,$$
$$\Delta(x_i^-) = x_i^- \otimes 1 + k_i^{-1} \otimes x_i^-$$

(we shall not need the formulas for the counit and antipode of $U_q(\mathfrak{g}(A))$).

2.2.
By a representation of a quantum Kac–Moody algebra $U_q(\mathfrak{g}(A))$ we shall mean a left $U_q(\mathfrak{g}(A))$–module. A representation W is said to be of type 1 if

$$W = \bigoplus_{\mu \in \mathbf{Z}^I} W_\mu,$$

where $W_\mu = \{w \in W | k_i.w = q^{\mu(i)}w\}$. If W_μ is non–zero, then W_μ is called the weight space of W with weight μ. Restricting consideration to type 1 representations results in no essential loss of generality, for any finite–dimensional irreducible representation can be obtained by twisting a type 1 representation with a suitable automorphism of $U_q(\mathfrak{g}(A))$ (cf. [**10**]).

2.3.
Assume that $\dim(\mathfrak{g}(A)) < \infty$. A representation W of $U_q(\mathfrak{g}(A))$ is said to be highest weight with highest weight $\lambda \in \mathbf{Z}^I$ if W is generated as a $U_q(\mathfrak{g}(A))$–module by an element w_λ satisfying

$$x_i^+.w_\lambda = 0, \quad k_i.w_\lambda = q^{\lambda(i)}w_\lambda,$$

for all $i \in I$.

A weight $\lambda \in \mathbf{Z}^I$ is said to be dominant if $\lambda(i)$ is non–negative for all $i \in I$.

Proposition ([**10**]). *Assume that* $\dim(\mathfrak{g}(A)) < \infty$.
(i) *Every finite–dimensional $U_q(\mathfrak{g}(A))$–module is completely reducible.*
(ii) *Every irreducible finite–dimensional $U_q(\mathfrak{g}(A))$–module of type* 1 *is highest weight. Assigning to such a representation its highest weight defines a one to one correspondence between the set of isomorphism classes of finite–dimensional irreducible representations of type* 1 *and the set of dominant weights.*
(iii) *The finite–dimensional irreducible $U_q(\mathfrak{g}(A))$–module $V(\lambda)$ of type* 1 *and highest weight λ has the same character (in particular, the same dimension) as the irreducible $\mathfrak{g}(A)$–module of the same highest weight.*
(iv) *The multiplicities of the irreducible components in a tensor product $V(\lambda)\otimes V(\mu)$ of irreducible finite–dimensional $U_q(\mathfrak{g}(A))$–modules is the same as in the tensor product of the irreducible $\mathfrak{g}(A)$–modules of the same highest weights.*

2.4
The case of most interest to us is when A is the matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix},$$

where $i, j \in \{0, 1, \ldots, n\}$. Then $\mathfrak{g}(A)$ is the affine Lie algebra $\widehat{sl}_{n+1}$. Fix a square root $q^{1/2}$ of q. For any elements a, b of an associative algebra over $\mathbb{C}$, set

$$[a, b]_{q^{1/2}} = q^{1/2}ab - q^{-1/2}ba.$$

Since $a_{ij} = 0$ or -1 if $i \neq j$, the quantized Serre relations in $U_q(\widehat{sl}_{n+1})$ can be written

$$\begin{aligned} [x_i^{\pm}, x_j^{\pm}] &= 0 \quad \text{if } i - j \neq 0, \ \pm 1 \ (\text{mod } n), \\ [x_i^{\pm}, [x_j^{\pm}, x_i^{\pm}]_{q^{1/2}}]_{q^{1/2}} &= 0 \quad \text{if } i - j = \ \pm 1 \ (\text{mod } n). \end{aligned}$$

Deleting the 0^{th} row and column of A gives the Cartan matrix of sl_{n+1}. Thus, there is a natural Hopf algebra homomorphism from $U_q(sl_{n+1})$ to $U_q(\widehat{sl}_{n+1})$; this homomorphism is injective (this follows from Proposition 5.4 below).

If $\mathfrak{g}(A) = sl_{n+1}$, then $I = \{1, \ldots, n\}$ and so weights are identified with n–tuples of integers. It is useful to introduce the weights ϵ_i, for $1 \leq i \leq n$, defined by

$$\epsilon_i(j) = \begin{cases} -1 & \text{if } j = i - 1, \\ 1 & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\sum_{i=1}^{n+1} \epsilon_i = 0$.

Set $\alpha_i = \epsilon_i - \epsilon_{i+1}$. If λ, $\mu \in \mathbf{Z}^I$, we write $\lambda \geq \mu$ if $\lambda - \mu = \sum_{i=1}^{n} r_i \alpha_i$ for some non-negative integers r_i.

The elements $\lambda_i = \sum_{j=1}^{i} \epsilon_j$, $1 \leq i \leq n$, are called fundamental weights and the corresponding irreducible representations $V(\lambda_i)$ the fundamental representations of $U_q(sl_{n+1})$.

The representation $V(\lambda_1)$ is called the natural representation of $U_q(sl_{n+1})$; it will be denoted by V from now on. It has a basis $\{v_1, \ldots, v_{n+1}\}$ on which the action is given by:

$$\begin{aligned} x_i^+.v_r &= \delta_{r,i+1}v_{r-1}, \\ x_i^-.v_r &= \delta_{r,i}v_{r+1}, \\ k_i.v_r &= q^{\epsilon_r(i)}v_r \end{aligned}$$

(we set $v_{-1} = v_{n+2} = 0$).

Let $x_\theta^\pm$ be the operators on V defined by

$$x_\theta^+.v_r = \delta_{r,n+1}v_1, \quad x_\theta^-.v_r = \delta_{r,1}v_{n+1},$$

and let $k_\theta = k_1k_2\dots k_n$. It is easy to see that V can be made into a $U_q(\widehat{sl}_{n+1})$-module $V(a)$, for all $a \in \mathbb{C}^\times$, by letting k_0 act as k_θ^{-1} and $x_0^\pm$ as $a^{\pm 1}x_\theta^\mp$.

2.5.

Definition. If $\ell \le n$, a finite-dimensional $U_q(sl_{n+1})$-module W is said to be of level ℓ if every irreducible component of W is isomorphic to an irreducible component of $V^{\otimes \ell}$.

Note that every level ℓ representation of $U_q(sl_{n+1})$ is of type 1.

The next result follows immediately from Proposition 2.3 and the corresponding classical result (which is well-known and easy to prove).

Proposition. *Assume that $\ell \le n$. Then, the finite-dimensional $U_q(sl_{n+1})$-module $V(\lambda)$ is of level $\ell \le n$ iff $\sum_{i=1}^n i\lambda(i) = \ell$.*

Remark. This proposition shows that the concept of level is well-defined. The assumption that $\ell \le n$ is necessary, for if ℓ_1 or ℓ_2 is greater than n, it is possible for $V^{\otimes \ell_1}$ and $V^{\otimes \ell_2}$ to have an irreducible component in common even if $\ell_1 \ne \ell_2$.

2.6.

It is easy to check that $c = k_0k_1\dots k_n$ is central in $U_q(\widehat{sl}_{n+1})$.

Proposition. *The central element c of $U_q(\widehat{sl}_{n+1})$ acts as 1 on every finite-dimensional $U_q(\widehat{sl}_{n+1})$-module W of type 1.*

Proof. This was proved in [**2**] when $n = 1$ and W is irreducible. Essentially the same proof works for all n and the extension to arbitrary finite-dimensional W follows by an easy argument using Jordan–Hölder series. □

3. Hecke algebras and affine Hecke algebras.

In this section, we collect some well-known definitions and results concerning (affine) Hecke algebras (cf. [**9**], [**12**]). We continue to assume that $q \in \mathbb{C}^\times$ is not a root of unity.

3.1.

Definition. Fix $\ell \geq 1$. The affine Hecke algebra $\widehat{H}_\ell(q^2)$ is the unital associative algebra over $\mathbb{C}$ with generators $\sigma_i^{\pm 1}$, $i \in \{1, \dots, \ell-1\}$, $y_j^{\pm 1}$, $j \in \{1, \dots, \ell\}$, and the following defining relations:

$$\begin{aligned}
\sigma_i\sigma_i^{-1} &= \sigma_i^{-1}\sigma_i = 1,\\
\sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1},\\
\sigma_i\sigma_j &= \sigma_j\sigma_i \quad \text{if } |i-j| > 1,\\
(\sigma_i + 1)(\sigma_i - q^2) &= 0,\\
y_jy_j^{-1} &= y_j^{-1}y_j = 1,\\
y_jy_k &= y_ky_j,\\
y_j\sigma_i &= \sigma_iy_j \quad \text{if } j \neq i \text{ or } i+1,\\
\sigma_iy_i\sigma_i &= q^2y_{i+1}.
\end{aligned}$$

The unital associative algebra with generators $\sigma_i^{\pm 1}$, $i \in \{1, \dots, \ell-1\}$, defined by the first four sets of relations above is called the Hecke algebra $H_\ell(q^2)$.

There is an obvious homomorphism of $H_\ell(q^2)$ onto the subalgebra of $\widehat{H}_\ell(q^2)$ generated by the σ_i.

Lemma. *The multiplication map $C[y_1^{\pm 1}, \dots, y_\ell^{\pm 1}] \otimes H_\ell(q^2) \to \widehat{H}_\ell(q^2)$ is an isomorphism of vector spaces.*

3.2.

The following well-known result provides an analogue for affine Hecke and Hecke algebras of the canonical homomorphism $S_{\ell_1} \times S_{\ell_2} \to S_{\ell_1+\ell_2}$.

Proposition. *There exists a unique homomorphism of algebras*

$$\widehat{\iota}_{\ell_1,\ell_2} : \widehat{H}_{\ell_1}(q^2) \otimes \widehat{H}_{\ell_2}(q^2) \to \widehat{H}_{\ell_1+\ell_2}(q^2)$$

such that

$$\widehat{\iota}_{\ell_1,\ell_2}(\sigma_i\otimes 1) = \sigma_i,\ \ \widehat{\iota}_{\ell_1,\ell_2}(y_j\otimes 1) = y_j,\ \ i = 1,\dots,\ell_1-1,\ j = 1,\dots,\ell_1,$$
$$\widehat{\iota}_{\ell_1,\ell_2}(1\otimes\sigma_i) = \sigma_{i+\ell_1},\ \ \widehat{\iota}_{\ell_1,\ell_2}(1\otimes y_j) = y_{j+\ell_1},\ \ i = 1,\dots,\ell_2-1,\ j = 1,\dots,\ell_2.$$

Clearly the restriction of $\widehat{\iota}_{\ell_1,\ell_2}$ to $H_{\ell_1}(q^2)\otimes H_{\ell_2}(q^2)$ induces a homomorphism $\iota_{\ell_1,\ell_2} : H_{\ell_1}(q^2)\otimes H_{\ell_2}(q^2) \to H_{\ell_1+\ell_2}(q^2)$.

Let M_i be a right $H_{\ell_i}(q^2)$-module for $i = 1,2$, and let $M_1\otimes M_2$ be their outer tensor product (an $H_{\ell_1}(q^2)\otimes H_{\ell_2}(q^2)$-module). Then, the $H_{\ell_1+\ell_2}(q^2)$-module $M_1\odot M_2$, sometimes called the Zelevinsky tensor product of M_1 and M_2, is defined by

$$\begin{aligned} M_1\odot M_2 &= \operatorname{ind}_{H_{\ell_1}(q^2)\otimes H_{\ell_2}(q^2)}^{H_{\ell_1+\ell_2}(q^2)}(M_1\otimes M_2)\\ &= (M_1\otimes M_2)\bigotimes_{H_{\ell_1}(q^2)\otimes H_{\ell_2}(q^2)} H_{\ell_1+\ell_2}(q^2). \end{aligned}$$

The Zelevinsky tensor product $\widehat{\odot}$ for affine Hecke algebra modules is defined similarly. Standard properties of induced modules show that the Zelevinsky tensor products are associative up to isomorphism.

3.3.

Proposition. *Let M_i be a finite-dimensional $\widehat{H}_{\ell_i}(q^2)$-module, $i = 1,2$. Then, there is a canonical isomorphism of $H_{\ell_1+\ell_2}(q^2)$-modules*

$$(M_1\widehat{\odot}M_2)|_{H_{\ell_1+\ell_2}(q^2)} \cong M_1|_{H_{\ell_1}(q^2)}\odot M_2|_{H_{\ell_2}(q^2)},$$

where $M_i|_{H_{\ell_i}(q^2)}$ means M_i regarded as an $H_{\ell_i}(q^2)$-module by restriction, etc.

Proof. It is easy to see that the canonical map

$$M_1|_{H_{\ell_1}(q^2)}\odot M_2|_{H_{\ell_2}(q^2)} \to (M_1\widehat{\odot}M_2)|_{H_{\ell_1+\ell_2}(q^2)}$$

given by

$$(m_1\otimes m_2)\otimes h \mapsto (m_1\otimes m_2)\otimes h \quad (m_i\in M_i,\ h\in H_{\ell_1+\ell_2}(q^2))$$

is a well-defined surjective homomorphism of $H_{\ell_1+\ell_2}(q^2)$-modules. But, by Lemma 3.1, the rank of $\widehat{H}_{\ell_1+\ell_2}(q^2)$ as an $\widehat{H}_{\ell_1}(q^2)\otimes\widehat{H}_{\ell_2}(q^2)$-module is the same as that of $H_{\ell_1+\ell_2}(q^2)$ as an $H_{\ell_1}(q^2)\otimes H_{\ell_2}(q^2)$-module. It follows that

$$\dim_{\mathbb{C}}(M_1\widehat{\odot}M_2) = \dim_{\mathbb{C}}(M_1\odot M_2).$$

$\square$

3.4.

Affine Hecke algebras have a family of universal modules, defined as follows. Let $\mathbf{a} = (a_1, a_2, \dots, a_\ell) \in (\mathbb{C}^\times)^\ell$ and set

$$M_{\mathbf{a}} = \widehat{H}_\ell(q^2)/H_{\mathbf{a}},$$

the quotient of $\widehat{H}_\ell(q^2)$ by the right ideal $H_{\mathbf{a}}$ generated by $y_j - a_j$, $j = 1, \dots, \ell$.

Proposition ([**12**]).
(a) *Every finite-dimensional irreducible $\widehat{H}_\ell(q^2)$-module is isomorphic to a quotient of some $M_{\mathbf{a}}$.*
(b) *For all $\mathbf{a} \in (\mathbb{C}^\times)^\ell$, $M_{\mathbf{a}}$ is isomorphic as an $H_\ell(q^2)$-module to the right regular representation.*
(c) *$M_{\mathbf{a}}$ is reducible as an $\widehat{H}_\ell(q^2)$-module iff $a_j = q^2 a_k$ for some j, k.*

4. Duality between $U_q(\widehat{sl}_{n+1})$ and $\widehat{H}_\ell(q^2)$.

We begin by recalling the duality, established by Jimbo [**7**], between representations of $U_q(sl_{n+1})$ and $H_\ell(q^2)$.

4.1.

Let V be the natural $(n+1)$–dimensional representation of $U_q(sl_{n+1})$ defined in 2.4, and let $\check{R} : V \otimes V \to V \otimes V$ be the linear map given by

$$\check{R}(v_r \otimes v_s) = \begin{cases} q^2 v_r \otimes v_s & \text{if } r = s, \\ q v_s \otimes v_r & \text{if } s > r, \\ q v_s \otimes v_r + (q^2 - 1) v_r \otimes v_s & \text{if } r > s. \end{cases} \tag{1}$$

Fix $\ell > 1$ and let $\check{R}_i \in \operatorname{End}_{\mathbb{C}}(V^{\otimes \ell})$ be the map which acts as $\check{R}$ on the i^{th} and $(i+1)^{th}$ factors of the tensor product, and as the identity on the other factors.

Proposition ([**7**]). *Fix $\ell, n \geq 1$. There is a unique left $H_\ell(q^2)$-module structure on $V^{\otimes \ell}$ such that σ_i acts as $\check{R}_i$ for $i = 1, \dots, \ell - 1$. Moreover, the action of $H_\ell(q^2)$ commutes with the natural action of $U_q(sl_{n+1})$ on $V^{\otimes \ell}$.*

If M is a right $H_\ell(q^2)$-module, define

$$\mathcal{J}(M) = M \otimes_{H_\ell(q^2)} V^{\otimes \ell},$$

equipped with the natural left $U_q(sl_{n+1})$-module structure induced by that on $V^{\otimes \ell}$. Then, if $\ell \leq n$, the functor $M \to \mathcal{J}(M)$ is an equivalence from *the category of finite-dimensional $H_\ell(q^2)$-modules to the category of finite-dimensional $U_q(sl_{n+1})$-modules of level ℓ.*

4.2.

We now state the main result of this section, which is an analogue of Proposition 4.1 for quantum affine algebras. Recall the operators k_θ, $x_\theta^\pm \in \mathrm{End}_{\mathbb{C}}(V)$ defined in Section 2.4.

Theorem. *Fix $\ell, n \geq 1$. There is a functor $\mathcal{F}$ from the category of finite-dimensional right $\widehat{H}_\ell(q^2)$-modules to the category of finite-dimensional left $U_q(\widehat{sl}_{n+1})$-modules of type 1 which are of level ℓ as $U_q(sl_{n+1})$-modules, defined as follows. If M is an $\widehat{H}_\ell(q^2)$-module, then $\mathcal{F}(M) = \mathcal{J}(M)$ as a $U_q(sl_{n+1})$-module and the action of the remaining generators of $U_q(\widehat{sl}_{n+1})$ is given by*

$$x_0^\pm.(m \otimes \mathbf{v}) = \sum_{j=1}^{\ell} m.y_j^{\pm 1} \otimes Y_j^\pm.\mathbf{v}, \tag{2}$$

$$k_0.(m \otimes \mathbf{v}) = m \otimes (k_\theta^{-1})^{\otimes \ell}.\mathbf{v}, \tag{3}$$

where $m \in M$, $\mathbf{v} \in V^{\otimes \ell}$ and the operators $Y_j^\pm \in \mathrm{End}_{\mathbb{C}}(V^{\otimes \ell})$, $j = 1, \dots, \ell$, are defined by

$$Y_j^+ = 1^{\otimes j-1} \otimes x_\theta^- \otimes (k_\theta^{-1})^{\otimes \ell - j},$$
$$Y_j^- = k_\theta^{\otimes j-1} \otimes x_\theta^+ \otimes 1^{\otimes \ell - j}.$$

The functor $\mathcal{F}$ is an equivalence of categories if $\ell \leq n$.

Proof. We first show that the formulas (2) and (3) are well-defined. We do this for the action of x_0^+, leaving the verification for x_0^- and k_0 to the reader. Thus, we must prove that

$$x_0^+.(m.\sigma_i \otimes \mathbf{v}) = x_0^+.(m \otimes \sigma_i.\mathbf{v})$$

for $i = 1, \dots, \ell$, $\mathbf{v} \in V^{\otimes \ell}$. This is equivalent to proving that, as operators on $\mathcal{J}(M) = M \otimes_{H_\ell(q^2)} V^{\otimes \ell}$,

$$\sum_{j=1}^{\ell} \sigma_i y_j \otimes Y_j^+ = \sum_{j=1}^{\ell} y_j \otimes Y_j^+ \sigma_i. \tag{4}$$

If $j \neq i, i+1$, the j^{th} terms on the left and right-hand sides of (4) are equal, since $\sigma_i y_j = y_j \sigma_i$ and $\sigma_i Y_j^+ = Y_j^+ \sigma_i$. Hence we must show that

$$\sigma_i y_i \otimes Y_i^+ + \sigma_i y_{i+1} \otimes Y_{i+1}^+ = y_i \otimes Y_i^+ \sigma_i + y_{i+1} \otimes Y_{i+1}^+ \sigma_i.$$

Using the relation $\sigma_i - (q^2 - 1) = q^2\sigma_i^{-1}$, this reduces to

$$q^2 y_{i+1}\otimes\left(\sigma_i^{-1}Y_i^+ - Y_{i+1}^+\sigma_i^{-1}\right) + y_i\otimes\left(\sigma_i Y_{i+1}^+ - Y_i^+\sigma_i\right) = 0.$$

Thus, it suffices to prove that

$$\sigma_i Y_{i+1}^+ = Y_i^+\sigma_i,$$

i.e. that

$$\check{R}(1\otimes x_\theta^-) = (x_\theta^-\otimes k_\theta^{-1})\check{R} \tag{5}$$

as operators on $V\otimes V$. But this is easily checked by using the formula for $\check{R}$ in (1) and that for x_θ^- in 2.4.

In proving that the formulas (2) and (3) define a representation of $U_q(\widehat{sl}_{n+1})$, we shall assume that $n > 1$. The proof for the sl_2 case is similar (the difference arises because the Dynkin diagram of $\widehat{sl}_2$ has a double bond).

The only relations to be checked are those involving x_0^+, x_0^- and k_0. This is straightforward except for the quantized Serre relations:

$$[x_i^\pm, [x_0^\pm, x_i^\pm]_{q^{1/2}}]_{q^{1/2}} = 0, \tag{6}$$

$$[x_0^\pm, [x_i^\pm, x_0^\pm]_{q^{1/2}}]_{q^{1/2}} = 0, \tag{7}$$

for $i = 1, n$. We verify (7) for x_1^+, leaving the other cases to the reader.

Applying the left–hand side of (7) to $\mathcal{J}(M)$ and considering the terms involving $y_j y_k$, one sees that it suffices to prove that

$$\left[Y_j^+, \left[\Delta^{(\ell)}(x_1^+), Y_k^+\right]_{q^{1/2}}\right]_{q^{1/2}} + (j \leftrightarrow k) = 0, \tag{8}$$

where $(j \leftrightarrow k)$ means the result of interchanging j and k in the first term and $\Delta^{(\ell)}$ is the ℓ^{th} iterated comultiplication (so that $\Delta^{(2)} = \Delta$). Equation (8) will be proved by induction on ℓ, and we accordingly denote Y_k^+ by $Y_k^{+(\ell)}$. If $\ell = 1$, then (8) becomes

$$[x_\theta^-, [x_1^-, x_\theta^-]_{q^{1/2}}]_{q^{1/2}} = 0,$$

which holds by the remarks at the end of 2.4.

For the inductive step we distinguish three cases:

(i) $j, k < \ell$,

(ii) $j < \ell$, $k = \ell$ or $j = \ell$, $k < \ell$,

(iii) $j = k = \ell$.

For the first case, notice that the left–hand side of (8) is

$$\begin{aligned}
&\left[Y_j^{+(\ell-1)}\otimes k_\theta^{-1}, \left[\Delta^{(\ell-1)}(x_1^+)\otimes k_1 + 1\otimes x_1^+, Y_k^{+(\ell-1)}\otimes k_\theta^{-1}\right]_{q^{1/2}}\right]_{q^{1/2}} + (j \leftrightarrow k)\\
&= \left[Y_j^{+(\ell-1)}, \left[\Delta^{(\ell-1)}(x_1^+) + 1\otimes x_1^+, Y_k^{+(\ell-1)}\right]_{q^{1/2}}\right]_{q^{1/2}} \otimes k_1 k_\theta^{-2} + (j \leftrightarrow k)\\
&\quad + \left[Y_j^{+(\ell-1)}\otimes k_\theta^{-1}, Y_k^{+(\ell-1)}\otimes [x_1^+, k_\theta^{-1}]_{q^{1/2}}\right]_{q^{1/2}} + (j \leftrightarrow k).
\end{aligned}$$

The sum of the first two terms on the right–hand side vanishes by the induction hypothesis, and the sum of the last two terms is a multiple of

$$\begin{aligned}
&\left[Y_j^{+(\ell-1)}\otimes k_\theta^{-1}, Y_k^{+(\ell-1)}\otimes k_\theta^{-1} x_1^+\right]_{q^{1/2}} + (j \leftrightarrow k)\\
&\qquad\qquad = q^{1/2}\left[Y_j^{+(\ell-1)}, Y_k^{+(\ell-1)}\right]\otimes k_\theta^{-2} x_1^+ + (j \leftrightarrow k).
\end{aligned}$$

But the expression on the right–hand side is zero since $\left[Y_j^{+(l-1)}, Y_k^{+(l-1)}\right] = 0$, so the induction step is established in this case. The other two cases are similar; we omit the details.

We have thus proved that formulas (2) and (3) define a representation of $U_q(\widehat{sl}_{n+1})$. If $f : M \to M'$ is a homomorphism of $\widehat{H}_\ell(q^2)$–modules, we define $\mathcal{F}(f) : \mathcal{F}(M) \to \mathcal{F}(M')$ by

$$\mathcal{F}(f)(m\otimes\mathbf{v}) = f(m)\otimes\mathbf{v}.$$

The proof that $\mathcal{F}(f)$ is a well–defined homomorphism of $U_q(\widehat{sl}_{n+1})$–modules is completely straightforward. It is now obvious that $\mathcal{F}$ is a functor between the appropriate categories of representations. □

4.3.

Assume for the remainder of the proof that $\ell \le n$. To prove that $\mathcal{F}$ is an equivalence, we must prove that
(a) every finite–dimensional $U_q(\widehat{sl}_{n+1})$–module W of type 1 which is of level ℓ as a $U_q(sl_{n+1})$–module is isomorphic to $\mathcal{F}(M)$ for some $\widehat{H}_\ell(q^2)$–module M;
(b) $\mathcal{F}$ is bijective on sets of morphisms.
(See [**11**], p. 91.)

To prove (a), note that by Proposition 4.1, we may assume that $W = \mathcal{J}(M)$ for some $H_\ell(q^2)$–module M. We shall reconstruct the action of the $y_j^{\pm 1}$ on M from the known action of $x_0^\pm$ and k_0 on W.

We need the following lemma.

Lemma (a). *Let M be a finite–dimensional $H_\ell(q^2)$–module, and let $\mathbf{v} \in V^{\otimes \ell}$. The linear map $M \to \mathcal{J}(M)$ given by $m \to m\otimes\mathbf{v}$ is injective if $\mathbf{v}$ has non–zero component in each isotypical component of $\mathcal{J}(M)$.*

(b) *If* $\{v_1, \ldots, v_{n+1}\}$ *is the standard basis of* V, $i_1, \ldots, i_\ell \in \{1, \ldots, n+1\}$ *are distinct, and* $\mathbf{v} = v_{i_1} \otimes \cdots \otimes v_{i_\ell}$, *then* $V^{\otimes \ell} = U_q(sl_{n+1}).\mathbf{v}$. *In particular,* $\mathbf{v}$ *satisfies the condition in part* (a).

Proof. Part (a) follows easily from Proposition 4.1, and part (b) is elementary. □

4.4.

For $1 \leq j \leq n$, let

$$\mathbf{v}^{(j)} = v_2 \otimes \cdots \otimes v_j \otimes v_{n+1} \otimes v_{j+1} \otimes \cdots \otimes v_\ell,$$
$$\mathbf{w}^{(j)} = v_2 \otimes \cdots \otimes v_j \otimes v_1 \otimes v_{j+1} \otimes \cdots \otimes v_\ell.$$

Let $\mathbf{w}_\tau^{(j)}$ be the result of permuting the factors of $\mathbf{w}^{(j)}$ by $\tau \in S_\ell$. Since $\{\mathbf{w}_\tau^{(j)}\}_{\tau \in S_\ell}$ clearly spans the subspace of $V^{\otimes \ell}$ of weight λ_ℓ, we get, for any $m \in M$,

$$x_0^- . \left(m \otimes \mathbf{v}^{(j)}\right) = \sum_{\tau \in S_\ell} m_\tau \otimes \mathbf{w}_\tau^{(j)}$$

for some $m_\tau \in M$. By (1), $\mathbf{w}_\tau^{(j)}$ is a (non–zero) scalar multiple of $\sigma.\mathbf{w}^{(j)}$ for some $\sigma \in H_\ell(q^2)$ (depending on τ). It follows that

$$x_0^- . \left(m \otimes \mathbf{v}^{(j)}\right) = m' \otimes \mathbf{w}^{(j)}$$

for some $m' \in M$. By Lemma 4.3, there exists $\alpha_j^- \in \mathrm{End}_{\mathbb{C}}(M)$ such that $m' = \alpha_j^-(m)$ for all $m \in M$. By a similar argument, there exists $\alpha_j^+ \in \mathrm{End}_{\mathbb{C}}(M)$ such that

$$\begin{aligned} &x_0^+ . (m \otimes v_{n-\ell+2} \otimes \cdots \otimes v_{n-\ell+j} \otimes v_1 \otimes v_{n-\ell+j+1} \otimes \cdots \otimes v_n) \\ &= \alpha_j^+(m) \otimes v_{n-\ell+2} \otimes \cdots \otimes v_{n-\ell+j} \otimes v_{n+1} \otimes v_{n-\ell+j+1} \otimes \cdots \otimes v_n \end{aligned}$$

for all $m \in M$.

4.5.

We need to prove the following lemma. The proof of the theorem itself continues in Section 4.6.

Lemma. *For all* $m \in M$, $\mathbf{v} \in V^{\otimes \ell}$, *we have*

$$x_0^\pm . (m \otimes \mathbf{v}) = \sum_{j=1}^{\ell} \alpha_j^\pm(m) \otimes Y_j^\pm . \mathbf{v}.$$

Proof. Let $\mathbf{v} = v_{i_1}\otimes\cdots\otimes v_{i_\ell}$. If $\{i_1, \ldots, i_\ell\} \subset \{1, \ldots, n\}$, it is clear that $x_0^-.(m\otimes\mathbf{v}) = 0$, since $\epsilon_{i_1} + \ldots + \epsilon_{i_\ell} + \epsilon_1 + \ldots + \epsilon_n$ cannot be a weight of $V^{\otimes\ell}$.

Let $r \geq 0$, $s \geq 1$, $1 \leq j_1 < j_2 < \ldots < j_r \leq \ell$, $1 \leq j_1' < j_2' < \ldots < j_s' \leq \ell$, and assume that $\{j_1, \ldots, j_\ell\} \cap \{j_1', \ldots, j_s'\} = \emptyset$. Write $\mathbf{j} = (j_1, \ldots, j_r)$, $\mathbf{j}' = (j_1', \ldots, j_s')$, and let $V^{(\mathbf{j},\mathbf{j}')}$ be the subspace of $V^{\otimes\ell}$ spanned by vectors which have v_1 in positions $j_1, \ldots, j_r$, v_{n+1} in positions $j_1', \ldots, j_s'$, and vectors from $\{v_2, \ldots, v_n\}$ in the remaining positions. We shall prove the lemma when $\mathbf{v} \in V^{(\mathbf{j},\mathbf{j}')}$ for all such $\mathbf{j}, \mathbf{j}'$ in two steps:

(i) For $s = 1$, by induction on r;

(ii) for all r, by induction on s.

Observe that, by Lemma 4.3 (b) applied to the subalgebra of $U_q(sl_{n+1})$ generated by the $x_i^{\pm}$, $k_i^{\pm1}$ for $i \in \{2, \ldots, n\}$, to prove Lemma 4.5 for all $\mathbf{v} \in V^{(\mathbf{j},\mathbf{j}')}$, it suffices to prove it for one $\mathbf{v} \in V^{(\mathbf{j},\mathbf{j}')}$ with the property that no vector from the set $\{v_2, \ldots, v_n\}$ is repeated. (Note that such vectors $\mathbf{v}$ exist since $\ell + 1 - r - s \leq \ell \leq n$.)

Proof of Step (i). If $r = 0$ (and $s = 1$), there is nothing to prove, for we can take $\mathbf{v} = v_2\otimes\cdots\otimes v_{j_1'}\otimes v_{n+1}\otimes v_{j_1'+1}\otimes\cdots\otimes v_\ell$ and use the definition of $\alpha_{j_1'}^-$. Assume that the result holds for $r-1$, and let $\tilde{\mathbf{j}} = (j_1, \ldots, j_{r-1})$. Let $\mathbf{v}' \in V^{(\tilde{\mathbf{j}},j')}$ have v_2 in the j_r^{th} position, and distinct vectors from $\{v_3, \ldots, v_n\}$ in the remaining positions. Then,

$$\mathbf{v} = x_1^+.\mathbf{v}'.$$

Let $\mathbf{v}''$ (resp. $\mathbf{v}'''$) be the element obtained from $\mathbf{v}'$ by replacing v_{n+1} by v_1 (resp. v_2 by v_1). We then get, for all $m \in M$,

$$\begin{aligned}
x_0^-.(m\otimes\mathbf{v}) &= x_1^+x_0^-.(m\otimes\mathbf{v}') \\
&= q^{|\{t<r | j_t < j_1'\}|}\alpha_{j_1'}^-(m)\otimes(1^{\otimes j_r - 1}\otimes x_1^+\otimes k_1^{\ell - j_r}).\mathbf{v}'' \\
&= q^{|\{t<r | j_t < j_1'\}|}q^{\delta_{j_r < j_1'}}\alpha_{j_1'}^-(m)\otimes\mathbf{v}''' \\
&= q^{|\{t\leq r | j_t < j_1'\}|}\alpha_{j_1'}^-(m)\otimes\mathbf{v}''' \\
&= \alpha_{j_1'}^-(m)\otimes Y_{j_1'}^-.\mathbf{v}.
\end{aligned}$$

Proof of Step (ii). Assume that the result holds for all $\mathbf{v} \in V^{(\mathbf{j},\mathbf{j}')}$ with fewer than s v_{n+1}s. It suffices, as in step 1, to prove the result for one element $\mathbf{v} \in V^{(\mathbf{j},\mathbf{j}')}$ which has distinct entries from $\{v_3, \ldots, v_n\}$ in the remaining positions. Fix such a $\mathbf{v}$ and let $\mathbf{v}'$ be the element obtained from $\mathbf{v}$ by replacing v_{n+1} in positions j_1' and j_2' by v_n. Then,

$$\mathbf{v} = \frac{(x_n^-)^2}{q+q^{-1}}.\mathbf{v}'.$$

Using a quantized Serre relation we get

$$x_0^-.(m\otimes\mathbf{v}) = x_n^- x_0^- x_n^-.(m\otimes\mathbf{v}') - \frac{(x_n^-)^2 x_0^-}{q+q^{-1}}.(m\otimes\mathbf{v}').$$

Since x_n^- operates in the $j_1^{'th}$ and $j_2^{'th}$ positions in $\mathbf{v}'$, we obtain, using the induction hypothesis,

$$\frac{(x_n^-)^2 x_0^-}{q+q^{-1}}.(m\otimes\mathbf{v}') = q^2 \sum_{k=3}^{s} \alpha_{j'_k}^-(m)\otimes Y_{j'_k}^-.\mathbf{v}.$$

On the other hand,

$$x_n^-.(m\otimes\mathbf{v}') = m\otimes\mathbf{v}'' + q^{-1} m\otimes\mathbf{v}''',$$

where $\mathbf{v}''$ (resp. $\mathbf{v}'''$) is obtained from $\mathbf{v}'$ by replacing the v_n in its $j_1^{'th}$ position (resp. $j_2^{'th}$ position) by v_{n+1}. Using the induction hypothesis, we get

$$x_0^- x_n^-.(m\otimes\mathbf{v}') = \sum_{k\neq 2} \alpha_{j'_k}^-(m)\otimes Y_{j'_k}^-.\mathbf{v}'' + q^{-1} \sum_{k\neq 1} \alpha_{j'_k}^-(m)\otimes Y_{j'_k}^-.\mathbf{v}'''.$$

Noting that $\mathbf{v}'''$ has v_n only in the $j_2^{'th}$ position, we find that

$$x_n^-.\sum_{k\neq 2} \alpha_{j'_k}^-(m)\otimes Y_{j'_k}^-.\mathbf{v}'' = \alpha_{j'_1}^-(m)\otimes Y_{j'_1}^-.\mathbf{v}' + q^2 \sum_{k>2} \alpha_{j'_k}^-(m)\otimes Y_{j'_k}^-.\mathbf{v}'.$$

Similarly,

$$x_n^-.\sum_{k\neq 1} \alpha_{j'_k}^-(m)\otimes Y_{j'_k}^-.\mathbf{v}''' = q \sum_{k\neq 1} \alpha_{j'_k}^-(m)\otimes Y_{j'_k}^-.\mathbf{v}'.$$

Combining these computations we obtain finally,

$$\begin{aligned} x_0^-.(m\otimes\mathbf{v}) &= -q^2 \sum_{k>2} \alpha_{j'_k}^-(m)\otimes Y_{j'_k}^-.\mathbf{v} \\ &\quad + q^2 \sum_{k>2} \alpha_{j'_k}^-(m)\otimes Y_{j'_k}^-.\mathbf{v} + \alpha_{j'_1}^-(m)\otimes Y_{j'_1}^-.\mathbf{v} \\ &\quad + \sum_{k\neq 1} \alpha_{j'_k}^-(m)\otimes Y_{j'_k}^-.\mathbf{v} \\ &= \sum_{k=1}^{s} \alpha_{j'_k}^-(m)\otimes Y_{j'_k}^-.\mathbf{v}, \end{aligned}$$

as required.

This proves Lemma 4.5 for x_0^-. The proof for x_0^+ is similar. □

4.6.

We can now complete the proof of the theorem. We show that setting

$$m.y_j^{\pm 1} = \alpha_j^{\pm}(m)$$

defines a right $\widehat{H}_\ell(q^2)$–module structure on M, extending its $H_\ell(q^2)$–module structure. We have to check the following relations:

(i) $y_j y_j^{-1} = y_j^{-1} y_j = 1$,

(ii) $y_j y_k = y_k y_j$,

(iii) $q^2 y_{j+1} = \sigma_j y_j \sigma_j$.

Relations (i) and (ii) are proved by computing both sides of the equation

$$[x_0^+, x_0^-].(m \otimes \mathbf{v}) = \left(\frac{k_0 - k_0^{-1}}{q - q^{-1}}\right).(m \otimes \mathbf{v}),$$

where in the first case we take $\mathbf{v}$ to be a vector with v_{n+1} in the j^{th} place and $v_{n-\ell+2}, \ldots, v_n$ in the remaining places (in any order), and in the second case we take $\mathbf{v}$ to be a vector with v_1 in the j^{th} place, v_{n+1} in the k^{th} place and distinct vectors from $\{v_2, \ldots, v_n\}$ in the other places. Notice that since the central element $c \in U_q(\widehat{sl}_{n+1})$ acts as 1 on W we have $k_0.(m \otimes \mathbf{v}) = m \otimes (k_\theta^{-1})^{\otimes \ell}.\mathbf{v}$.

To prove (iii), let $\mathbf{v} = v_{i_1} \otimes \cdots \otimes v_{i_\ell} \in V^{\otimes \ell}$, where $i_j = 2$, $i_{j+1} = 1$, and the remaining i_k are distinct elements from $\{3, \ldots, n\}$ (this is possible since $\ell \leq n$). Let $\mathbf{v}'$ be the result of replacing v_1 in the i_{j+1}^{th} position in $\mathbf{v}$ by v_{n+1}. Since

$$\check{R}(v_2 \otimes v_{n+1}) = q v_{n+1} \otimes v_2, \quad \check{R}(v_1 \otimes v_2) = q v_2 \otimes v_1,$$

we have, for all $m \in M$,

$$m.\sigma_j y_j \sigma_j \otimes \mathbf{v}' = q m.\sigma_j y_j \otimes \mathbf{v}'',$$

where $\mathbf{v}''$ is obtained from $\mathbf{v}'$ by interchanging its j^{th} and $(j+1)^{th}$ factors, which

$$= q x_0^+.(m.\sigma_j \otimes \mathbf{v}'''),$$

where $\mathbf{v}'''$ is obtained from $\mathbf{v}$ by interchanging its j^{th} and $(j+1)^{th}$ factors, which

$$= q^2 x_0^+.(m \otimes \mathbf{v}) = q^2 m.y_{j+1} \otimes \mathbf{v}'.$$

Since $\mathbf{v}'$ has distinct components, Lemma 4.3 implies that

$$q^2 m.y_{j+1} = m.\sigma_j y_j \sigma_j,$$

for all $m \in M$.

The proof that $W \cong \mathcal{F}(M)$ as $U_q(\widehat{sl}_{n+1})$–modules is now complete. To show that $\mathcal{F}$ is an equivalence, we must prove that it is bijective on sets of morphisms. Injectivity of $\mathcal{F}$ follows from that of $\mathcal{J}$. For surjectivity, let $F : \mathcal{F}(M) \to \mathcal{F}(M')$ be a homomorphism of $U_q(\widehat{sl}_{n+1})$–modules. By Proposition 4.1 again, $F = \mathcal{J}(f)$ for some homomorphism $f : M \to M'$ of $H_\ell(q^2)$–modules. The fact that F commutes with the action of x_0^+ gives

$$\sum_{j=1}^{\ell} f(m.y_j)\otimes Y_j^+.\mathbf{v} = \sum_{j=1}^{\ell} f(m).y_j\otimes Y_j^+.\mathbf{v}$$

for all $m \in M$, $\mathbf{v} \in V^{\otimes \ell}$. By choosing $\mathbf{v}$ suitably, as in the preceding part of the proof, it is easy to see that this implies

$$f(m.y_j) = f(m).y_j$$

for all $j = 1, \ldots, \ell$. □

4.7.

The functor $\mathcal{F}$ is clearly one of $\mathbb{C}$–linear categories. The following result shows that it also captures part of the tensor structure of the category of $U_q(\widehat{sl}_{n+1})$–modules.

Proposition. *Let M_i be a finite-dimensional $\widehat{H}_{\ell_i}(q^2)$–module, $i = 1, 2$. Then, there is a canonical isomorphsm of $U_q(\widehat{sl}_{n+1})$–modules*

$$\mathcal{F}\left(M_1 \hat{\odot} M_2\right) \cong \mathcal{F}(M_1)\otimes\mathcal{F}(M_2).$$

Proof. We recall the following elementary fact: If $\iota : B \to A$ is a homomorphism of unital associative algebras over a field, M is a right B–module, W a left A–module, and $W|_B$ is W regarded as a left B–module via ι, there is a canonical isomorphism of vector spaces

$$\mathrm{ind}_B^A(M)\otimes W \cong M \bigotimes_B W|_B.$$

In fact, the isomorphism is given by

$$(m\otimes a)\otimes w \to m\otimes aw \qquad (m \in M, a \in A, w \in W).$$

Taking $A = H_{\ell_1+\ell_2}(q^2)$, $B = H_{\ell_1}(q^2)\otimes H_{\ell_2}(q^2)$, $\iota = \iota_{\ell_1,\ell_2}$, $M = M_1\otimes M_2$ and $W = V^{\otimes \ell_1+\ell_2}$, and noting that $W \cong (V^{\otimes \ell_1})\otimes(V^{\otimes \ell_2})$ as an $H_{\ell_1}(q^2)\otimes H_{\ell_2}(q^2)$-module, we get a canonical isomorphism of vector spaces

$$\mathcal{F}\left(M_1\widehat{\odot}M_2\right) \to (M_1\otimes M_2) \bigotimes_{H_{\ell_1}(q^2)\otimes H_{\ell_2}(q^2)} \left(V^{\otimes \ell_1}\otimes V^{\otimes \ell_2}\right).$$

The right–hand side is obviously isomorphic to $\mathcal{F}(M_1)\otimes\mathcal{F}(M_2)$ as a vector space. To complete the proof, one must check that the resulting isomorphism of vector spaces

$$\mathcal{F}\left(M_1\widehat{\odot}M_2\right) \to \mathcal{F}(M_1)\otimes\mathcal{F}(M_2)$$

commutes with the action of $U_q(\widehat{sl}_{n+1})$. This is completely straightforward. □

4.8.

We analyze the functor $\mathcal{F}$ in more detail in Section 7, when the parametrizations of the finite–dimensional irreducible representations of $\widehat{H}_\ell(q^2)$ and $U_q(\widehat{sl}_{n+1})$ have been described. The following result is, however, easy to prove now. Recall the universal $\widehat{H}_\ell(q^2)$–modules $M_{\mathbf{a}}$ and the $U_q(\widehat{sl}_{n+1})$–modules $V(a)$ defined in Sections 2.4 and 3.4 respectively.

Proposition. *Let* $\mathbf{a} = (a_1, \dots , a_\ell) \in (\mathbb{C}^\times)^\ell$, $\ell, n \geq 1$. *There is a canonical isomorphism of* $U_q(\widehat{sl}_{n+1})$*–modules*

$$\mathcal{F}(M_{\mathbf{a}}) \cong V(a_1)\otimes\cdots\otimes V(a_\ell).$$

Proof. As an $H_\ell(q^2)$–module, $M_{\mathbf{a}}$ is the right regular representation. It follows that the map

$$V^{\otimes \ell} \to \mathcal{J}(M_{\mathbf{a}}) \tag{9}$$

given by $\mathbf{v} \to 1\otimes\mathbf{v}$ is an isomorphism of $U_q(sl_{n+1})$–modules. Now,

$$x_0^+.(1\otimes\mathbf{v}) = \sum_{j=1}^{\ell} 1.y_j\otimes Y_j^+.\mathbf{v} = \left(\sum_{j=1}^{\ell} a_j Y_j^+\right).\mathbf{v}.$$

On the other hand,

$$\Delta^{(\ell)}(x_0^+) = \sum_{j=1}^{\ell} 1^{\otimes j-1}\otimes x_0^+\otimes k_0^{\otimes \ell-j}$$

acts on $V(a_1)\otimes\cdots\otimes V(a_\ell)$ as

$$\sum_{j=1}^{\ell} 1^{\otimes j-1}\otimes a_j x_\theta^-\otimes(k_\theta^{-1})^{\otimes \ell-j} = \sum_{j=1}^{\ell} a_j Y_j^+.$$

One checks in the same way that the map in (9) commutes with the action of x_0^- and k_0. $\square$

Corollary. *Let $1 \le \ell \le n$.*
(a) *Every finite-dimensional $U_q(\widehat{sl}_{n+1})$-module of type 1 and level ℓ as a $U_q(sl_{n+1})$-module is isomorphic to a quotient of $V(a_1)\otimes\cdots\otimes V(a_\ell)$, for some $a_1,\dots,a_\ell \in \mathbb{C}^\times$.*
(b) *If $a_1,\dots,a_\ell \in \mathbb{C}^\times$, then $V(a_1)\otimes\cdots\otimes V(a_\ell)$ is reducible as a $U_q(\widehat{sl}_{n+1})$-module iff $a_j = q^2 a_k$ for some j,k.*

Proof. This follows immediately from Proposition 3.4 and the fact that $\mathcal{F}$ is an equivalence of categories. $\square$

4.9.

Theorem 4.2 has a classical analogue, in which $U_q(\widehat{sl}_{n+1})$ is replaced by (the universal enveloping algebra of) the affine Lie algebra $\widehat{sl}_{n+1}$, and $\widehat{H}_\ell(q^2)$ by (the group algebra of) the affine Weyl group of $GL_\ell(\mathbb{C})$, i.e. the semi–direct product $S_\ell\tilde{\times}\mathbf{Z}^\ell$, where S_ℓ acts on the additive group $\mathbf{Z}^\ell$ by permuting the coordinates. We recall that $\widehat{sl}_{n+1}$ is the universal central extension (with one–dimensional centre) of the Lie algebra $L(sl_{n+1})$ of Laurent polynomial maps $\mathbb{C}^\times \to sl_{n+1}$. We identify sl_{n+1} with the subalgebra of $L(sl_{n+1})$ consisting of the constant maps.

Theorem. *There is a functor $\mathcal{F}_0$ from the category of finite-dimensional $S_\ell\tilde{\times}\mathbf{Z}^\ell$-modules to the category of finite-dimensional $L(sl_{n+1})$-modules which are of level ℓ as sl_{n+1}-modules, defined as follows. One takes*

$$\mathcal{F}_0(M) = M \bigotimes_{S_\ell} V_0^{\otimes \ell}$$

with the action of $f \in L(sl_{n+1})$ given by

$$f.(m\otimes\mathbf{v}) = \sum_{j=1}^{\ell} m.z_j\otimes\left(1^{\otimes j-1}\otimes f(1)\otimes 1^{\otimes \ell-j}\right).\mathbf{v},$$

where $z_j = (0,\dots,0,1,0,\dots,0) \in \mathbf{Z}^\ell \subset S_\ell\tilde{\times}\mathbf{Z}^\ell$ (with 1 in the j^{th} position). If $\ell \le n$, $\mathcal{F}_0$ is an equivalence.

Proof. The proof of this theorem is analogous to (but simpler than) that of Theorem 4.2. $\square$

Remark. The finite–dimensional irreducible representations of $L(sl_{n+1})$ were classified in [**1**]. For any $a \in \mathbb{C}^\times$, there is a homomorphism of Lie algebras

$$ev_a^0 : L(sl_{n+1}) \to sl_{n+1}$$

given by $ev_a^0(f) = f(a)$. If W is an irreducible sl_{n+1}–module, pulling back by ev_a^0 gives an irreducible $L(sl_{n+1})$–module $W(a)$. It is not difficult to prove that every finite–dimensional irreducible representation of $L(sl_{n+1})$ is isomorphic to a tensor product of $W(a)$s.

It is easy to identify the corresponding representations of $S_\ell \tilde{\times} \mathbf{Z}^\ell$. There is a homomorphism

$$\tilde{ev}_a^0 : S_\ell \tilde{\times} \mathbf{Z}^\ell \to S_\ell$$

which is the identity on S_ℓ and for which $\tilde{ev}_a^0(z_j) = a$ for all j. If M is an irreducible S_ℓ–module, pulling M back by $\tilde{ev}_a^0$ gives an irreducible $S_\ell \tilde{\times} \mathbf{Z}^\ell$–module $M(a)$. It is clear that

$$\mathcal{F}_0(M(a)) \cong \mathcal{F}S(M)(a).$$

By Theorem 4.9, every finite–dimensional irreducible $S_\ell \tilde{\times} \mathbf{Z}^\ell$–module is isomorphic to a Zelevinsky tensor product of $M(a)$s.

5. Evaluation Representations.

In this section, we construct analogues for $U_q(\widehat{sl}_{n+1})$ and $\widehat{H}_\ell(q^2)$ of the representations of sl_{n+1} and $S_\ell \tilde{\times} \mathbf{Z}^\ell$ described in Remark 4.9, and show how these representations are related by the functor $\mathcal{F}$.

5.1.

The following result was observed by Cherednik [**4**]. The proof is straightforward.

Proposition. *For every $a \in \mathbb{C}^\times$, there exists a homomorphism $\tilde{ev}_a$: $\widehat{H}_\ell(q^2) \to H_\ell(q^2)$ such that*

$$\tilde{ev}_a(\sigma_i) = \sigma_i,$$
$$\tilde{ev}_a(y_j) = aq^{-2(j-1)}\sigma_{j-1}\sigma_{j-2}\dots\sigma_2\sigma_1^2\sigma_2\dots\sigma_{j-1},$$

for $i = 1, \dots \ell - 1$, $j = 1, \dots, \ell$.

Note that $\tilde{ev}_a$ can be characterized as the unique homomorphism $\widehat{H}_\ell(q^2) \to H_\ell(q^2)$ which is the identity on $H_\ell(q^2) \subset \widehat{H}_\ell(q^2)$ and which maps y_1 to a.

If M is any $H_\ell(q^2)$–module, pulling back M by $\tilde{ev}_a$ gives an $\widehat{H}_\ell(q^2)$–module $M(a)$ which is isomorphic to M as an $H_\ell(q^2)$–module.

5.2.

In [7], Jimbo defined a quantum analogue of the homomorphism $ev_a^0 : \widehat{sl}_{n+1} \to sl_{n+1}$. To describe it, we need the following

Definition. $U_q(gl_{n+1})$ is the associative algebra over $\mathbb{C}$ with generators $x_i^{\pm}$, $i = 1, \ldots, n$, $t_r^{\pm 1}$, $r = 1, \ldots, n+1$, and the following defining relations:

$$\begin{aligned}
t_r t_r^{-1} &= 1 \ = t_r^{-1} t_r, \\
t_r t_s &= t_s t_r \ , \\
t_r x_i^{\pm} t_r^{-1} &= q^{\pm(\delta_{r,i} - \delta_{r,i+1})} x_i^{\pm} \ , \\
[x_i^{\pm}, [x_j^{\pm}, x_i^{\pm}]_{q^{1/2}}]_{q^{1/2}} &= 0 \ \text{ if } |i-j| = 1, \\
[x_i^{\pm}, x_j^{\pm}] &= 0 \ \text{ if } |i-j| > 1, \\
[x_i^{+}, x_j^{-}] &= \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}} \ ,
\end{aligned}$$

where $k_i = t_i t_{i+1}^{-1}$.

The algebra $U_q(gl_{n+1})$ has a Hopf algebra structure, but we shall not make any use of it.

Note that there is an obvious homomorphism $U_q(sl_{n+1}) \to U_q(gl_{n+1})$.

5.3.

Fix an $(n+1)^{th}$ root $q^{1/(n+1)}$ of q. We shall say that a finite–dimensional $U_q(gl_{n+1})$–module W is of type 1 if
(a) W is of type 1 regarded as a $U_q(sl_{n+1})$–module,
(b) the t_r act semisimply on W with eigenvalues which are integer powers of $q^{1/(n+1)}$,
(c) $t_1 t_2 \ldots t_{n+1}$ acts as 1 on W.

It is easy to see that restriction to $U_q(sl_{n+1})$ is an equivalence from the category of finite–dimensional $U_q(gl_{n+1})$–modules of type 1 to the category of finite–dimensional $U_q(sl_{n+1})$–modules of type 1. In particular the functor $\mathcal{J}$ of Proposition 4.1 may be viewed as taking values in the category of finite–dimensional $U_q(gl_{n+1})$–modules of type 1.

5.4. We can now state

Proposition ([7]). *For any $a \in \mathbb{C}^{\times}$, there exists a homomorphism ev_a :*

$U_q(\widehat{sl}_{n+1}) \to U_q(gl_{n+1})$ *such that*

$$\begin{aligned} ev_a(x_i^{\pm}) &= x_i^{\pm}, \quad ev_a(k_i) = k_i, \quad i = 1, \dots, n, \\ ev_a(k_0) &= (k_1 k_2 \dots k_n)^{-1}, \\ ev_a(x_0^{\pm}) &= (\pm 1)^{(n-1)} q^{\mp(n+1)/2} a^{\pm 1} (t_1 t_{n+1})^{\pm 1} \\ &\qquad \cdot [x_n^{\mp}, [x_{n-1}^{\mp}, \dots, [x_2^{\mp}, x_1^{\mp}]_{q^{1/2}} \dots]_{q^{1/2}}]_{q^{1/2}}. \end{aligned}$$

If W is a $U_q(sl_{n+1})$–module of type 1, we may regard W as a $U_q(gl_{n+1})$–module by (5.3). The pull-back of W by the homomorphism ev_a is a $U_q(\widehat{sl}_{n+1})$–module which we denote by $W(a)$.

5.5.

The main result of this section is

Theorem. *Let $1 \le \ell \le n$, and let M be a finite-dimensional right $H_\ell(q^2)$–module. Then there is a canonical isomorphism of $U_q(\widehat{sl}_{n+1})$–modules,*

$$\mathcal{F}\left(M\left(q^{-2\ell/(n+1)}a\right)\right) \cong \mathcal{J}(M)(a),$$

for all $a \in \mathbb{C}^\times$.

Proof. By Theorem 4.2 we know that $\mathcal{J}(M)(a) \cong \mathcal{F}(N)$, for some $\widehat{H}_\ell(q^2)$–module N which is isomorphic to M as an $H_\ell(q^2)$–module. It suffices to prove that y_1 acts as the scalar a on N. To prove this, we compute the action of x_0^+ on $m \otimes v_1 \otimes v_{n-\ell+2} \otimes v_{n-\ell+3} \otimes \cdots \otimes v_n \in \mathcal{F}(N)$ in two different ways, for all $m \in M$.

First, by the definition of $\mathcal{F}$, we have

$$\begin{aligned} (10) \quad & x_0^+.(m \otimes v_1 \otimes v_{n-\ell+2} \otimes v_{n-\ell+3} \otimes \cdots \otimes v_n) \\ &\qquad = m.y_1 \otimes v_{n+1} \otimes v_{n-\ell+2} \otimes v_{n-\ell+3} \otimes \cdots \otimes v_n. \end{aligned}$$

On the other hand, let $f_n = [x_n^-, [x_{n-1}^-, \dots, [x_2^-, x_1^-]_{q^{1/2}} \dots]_{q^{1/2}}]_{q^{1/2}}$. Then,

$$\begin{aligned} (11) \qquad & x_0^+.(m \otimes v_1 \otimes v_{n-\ell+2} \otimes \cdots \otimes v_n) \\ &= m \otimes ev_a(x_0^+).(v_1 \otimes v_{n-\ell+2} \otimes \cdots \otimes v_n) \\ &= a q^{-(n-1)/2 - 2\ell/(n+1)} m \otimes f_n.(v_1 \otimes v_{n-\ell+2} \otimes \cdots \otimes v_n). \end{aligned}$$

We prove by induction on n that

$$f_n.(v_1 \otimes v_{n-\ell+2} \otimes \cdots \otimes v_n) = q^{(n-1)/2} v_{n+1} \otimes v_{n-\ell+2} \otimes \cdots \otimes v_n.$$

The result is obvious if $n = 1$. Assuming it for $n-1$, note that $f_n = [x_n^-, f_{n-1}]_{q^{1/2}}$, so by the induction hypothesis,

$$\begin{aligned} f_n.(v_1 \otimes v_{n-\ell+2} \otimes \cdots \otimes v_n) &= q^{(n-1)/2} x_n^-.(v_n \otimes v_{n-\ell+2} \otimes \cdots \otimes v_n) \\ &\quad - q^{-1/2} f_{n-1}.(v_1 \otimes v_{n-\ell+2} \otimes \ldots \otimes v_{n-1} \otimes v_{n+1}). \end{aligned}$$

Since $x_i^-.v_{n+1} = 0$ for $1 \le i \le n-1$, we see that

$$\begin{aligned} & f_{n-1}.((v_1 \otimes v_{n-\ell+2} \otimes \cdots \otimes v_{n-1}) \otimes v_{n+1}) \\ &= (f_{n-1}.(v_1 \otimes v_{n-\ell+2} \otimes \cdots \otimes v_{n-1})) \otimes v_{n+1} \\ &= q^{(n-2)/2} v_n \otimes v_{n-\ell+2} \otimes \cdots \otimes v_{n-1} \otimes v_{n+1}, \end{aligned}$$

by the induction hypothesis again. Hence,

$$\begin{aligned} & f_n.(v_1 \otimes v_{n-\ell+2} \otimes \cdots \otimes v_n) \\ &= q^{(n-1)/2} (v_{n+1} \otimes v_{n-\ell+2} \otimes \cdots \otimes v_n + q^{-1} v_n \otimes v_{n-\ell+2} \otimes \cdots \otimes v_{n-1} \otimes v_{n+1}) \\ &\quad - q^{(n-3)/2} v_n \otimes v_{n-\ell+2} \otimes \cdots \otimes v_{n-1} \otimes v_{n+1} \\ &= q^{(n-1)/2} v_{n+1} \otimes v_{n-\ell+2} \otimes \cdots \otimes v_n, \end{aligned}$$

as required.

Hence, from (11), we obtain

$$x_0^+.(m \otimes v_1 \otimes v_{n-\ell+2} \otimes \cdots \otimes v_n) = a q^{-2\ell/(n+1)} m \otimes v_{n+1} \otimes v_{n-\ell+2} \otimes \cdots \otimes v_n.$$

Comparing with (10), and using Lemma 4.3, we obtain

$$m.y_1 = a q^{-2\ell/(n+1)} m$$

for all $m \in M$. □

6. Classification of finite–dimensional $U_q(\widehat{sl}_{n+1})$–modules.

6.1.

The finite–dimensional irreducible $U_q(\widehat{sl}_{n+1})$–modules of type 1 were classified in [**2**], [**3**]. To describe this result, we need an alternative presentation of $U_q(\widehat{sl}_{n+1})$ given in [**6**]. By Proposition 2.6, we need only consider the quantum loop algebra $L_q(sl_{n+1})$, the quotient of $U_q(\widehat{sl}_{n+1})$ by the two sided ideal generated by $c-1$.

Proposition. *$L_q(sl_{n+1})$ is isomorphic as an algebra to the algebra $\mathcal{A}$ with generators $X_{i,r}^{\pm}$ $(i \in \{1,\dots,n\}, r \in \mathbf{Z})$, $H_{i,r}$ $(i \in \{1,\dots,n\},\ r \in \mathbf{Z}\backslash\{0\})$, and $K_i^{\pm 1}$, $(i \in \{1,\dots,n\})$, and the following defining relations:*

$$\begin{aligned}
K_iK_i^{-1} &= 1 \ = K_i^{-1}K_i, \\
K_iH_{j,r} &= H_{j,r}K_i\ , \\
[H_{i,r}, H_{j,s}] &= 0\ , \\
K_iX_j^{\pm}K_i^{-1} &= q^{\pm a_{ij}}X_j^{\pm}\ , \\
[H_{i,r}, X_{j,s}^{\pm}] &= \ \pm\frac{1}{r}[ra_{ij}]_qX_{j,r+s}^{\pm}\ , \\
X_{i,r+1}^{\pm}X_{j,s}^{\pm} - q^{\pm a_{ij}}X_{j,s}^{\pm}X_{i,r+1}^{\pm} &= q^{\pm a_{ij}}X_{i,r}^{\pm}X_{j,s+1}^{\pm} - X_{j,s+1}^{\pm}X_{i,r}^{\pm}, \\
[X_{i,r}^{+}, X_{j,s}^{-}] &= \delta_{ij}\frac{\Phi_{i,r+s}^{+} - \Phi_{i,r+s}^{-}}{q-q^{-1}}\ ,
\end{aligned}$$

$$\sum_{\pi\in S_p}\sum_{k=0}^{p}(-1)^k\begin{bmatrix} p\\ k\end{bmatrix}_q X_{i,r_{\pi(1)}}^{\pm}\dots X_{i,r_{\pi(k)}}^{\pm}X_{j,s}^{\pm}X_{i,r_{\pi(k+1)}}^{\pm}\dots X_{i,r_{\pi(p)}}^{\pm} = 0\ ,\ i\neq j,$$

for all sequences $(r_1,\dots,r_p)\in\mathbf{Z}^p$, where $p = 1-a_{ij}$ and the elements $\Phi_{i,r}^{\pm}$ are determined by equating coefficients of powers of u in the formal power series

$$\sum_{r=0}^{\infty}\Phi_{i,\pm r}^{\pm}u^{\pm r} = K_i^{\pm 1}\exp\left(\pm(q-q^{-1})\sum_{s=1}^{\infty}H_{i,\pm s}u^{\pm s}\right).$$

The isomorphism $f : L_q(sl_{n+1}) \to \mathcal{A}$ is given by

$$f(x_i^{\pm}) = X_{i,0}^{\pm},\ \ f(k_i^{\pm 1}) = K_i^{\pm 1},$$

for $i\in\{1,\dots,n\}$, and

$$\begin{aligned}
f(k_0^{\pm 1}) &= (K_1K_2\dots K_n)^{\mp 1}, \\
f(x_0^{+}) &= (-1)^{m-1}q^{-(n-3)/2}[X_{n,0}^{-},[X_{n-1,0}^{-},\dots, \\
&\qquad [X_{m+1,0}^{-},[X_{1,0}^{-},\dots,[X_{m-1,0}^{-},X_{m,1}^{-}]_{q^{1/2}}\dots]]]]_{q^{1/2}}f(k_0), \\
f(x_0^{-}) &= \mu f(k_0^{-1})[X_{n,0}^{+},[X_{n-1,0}^{+},\dots,[X_{m+1,0}^{+}, \\
&\qquad [X_{1,0}^{+},\dots,[X_{m-1,0}^{+},X_{m,-1}^{+}]_{q^{1/2}}\dots]]]]_{q^{1/2}},
\end{aligned}$$

where $\mu\in\mathbb{C}^{\times}$ is determined by

$$[f(x_0^{+}),f(x_0^{-})] = \frac{f(k_0)-f(k_0^{-1})}{q-q^{-1}}.$$

Remark. Using the relations in $\mathcal{A}$, it is not difficult to see that the isomorphism f is independent of the choice of $m\in\{1,2,\dots,n\}$.

6.2.

The following result is proved in [**2**], [**3**].

Proposition. *Let W be a finite-dimensional irreducible $L_q(sl_{n+1})$-module of type* 1. *Then,*
(a) *W is generated by a vector w_0 satisfying*

$$X_{i,r}^{+}.w_0 = 0, \quad \Phi_{i,r}^{\pm}.w_0 = \phi_{i,r}^{\pm} w_0$$

for all $i \in \{1, \dots, n\}$, $r \in \mathbf{Z}$, and some $\phi_{i,r}^{\pm} \in \mathbb{C}$.
(b) *There exist unique monic polynomials $P_1(u), \dots, P_n(u)$ (depending on W) such that the $\phi_{i,r}^{\pm}$ satisfy*

$$\sum_{r=0}^{\infty} \phi_{i,r}^{+} u^r = q^{\deg P_i} \frac{P_i(q^{-2}u)}{P_i(u)} = \sum_{r=0}^{\infty} \phi_{i,r}^{-} u^{-r},$$

in the sense that the left and right-hand sides are the Laurent expansions of the middle term about 0 *and ∞ respectively. Assigning to W the corresponding n-tuple of polynomials defines a one to one correspondence between the isomorphism classes of finite-dimensional irreducible $L_q(sl_{n+1})$-modules of type 1 and the set of n-tuples of monic polynomials in one variable u with non-zero constant term.*

A consequence of this proposition is:

Corollary. *Let W be a finite-dimensional irreducible representation of $U_q(\widehat{sl}_{n+1})$ with associated polynomials P_i. Set $\lambda = (\deg\ P_1, \dots, \deg\ P_n)$. Then W contains the irreducible $U_q(sl_{n+1})$-module $V(\lambda)$ with multiplicity one. Further, if $V(\mu)$ is any other $U_q(sl_{n+1})$-module occurring in W, then $\lambda \geq \mu$.*

6.3.

The next proposition can be proved by studying the action of the comultiplication Δ of $U_q(\widehat{sl}_{n+1})$ on the generators $X_{i,r}^{+}$ etc., as in [**2**].

Proposition. *Let W and W' be two finite-dimensional irreducible $U_q(\widehat{sl}_{n+1})$-modules with associated monic polynomials P_i and P_i', $i = 1, \dots, n$. Let w_0 and w_0' be the generating vectors of W and W' as in Proposition* 6.2. *Then, in $W \otimes W'$ we have*

$$X_{i,r}^{+}.(w_0 \otimes w_0') = 0$$

for all $i \in \{1, \dots, n\}$, $r \in \mathbf{Z}$. *Further,* $w_0 \otimes w_0'$ *is a common eigenvector of the* $\Phi_{i,r}^{\pm}$ *with eigenvalues given as in Proposition* 6.2 (b) *by the polynomials* $P_i P_i'$.

This result suggests the following

Definition. If $i \in \{1, \dots, n\}$, $a \in \mathbb{C}^{\times}$, the irreducible finite-dimensional representation of $U_q(\widehat{sl}_{n+1})$ with associated polynomials

$$P_j(u) = \begin{cases} u - a & \text{if } j = i, \\ 1 & \text{otherwise,} \end{cases}$$

is called the i^{th} fundamental representation of $U_q(\widehat{sl}_{n+1})$ with parameter a, and is denoted by $V(\lambda_i, a)$.

Remark. Note that it follows from Corollary 6.2 that $V(\lambda_i, a) \cong V(\lambda_i)$ as $U_q(sl_{n+1})$–modules.

6.4.

We shall need the following result in Section 7.

Lemma. *Let* v_{λ_m} *be the* $U_q(sl_{n+1})$*–highest weight vector in* $V(\lambda_m, a)$, *where* $m \in \{1, \dots, n\}$, $a \in \mathbb{C}^{\times}$. *Then,*

$$x_0^+ . v_{\lambda_m} = (-1)^{m-1} a^{-1} x_n^- x_{n-1}^- \dots x_{m+1}^- x_1^- \dots x_m^- . v_{\lambda_m}.$$

Proof. By Proposition 2.3 and the preceding remark, we know that the weight spaces of $V(\lambda_m, a)$ as a $U_q(sl_{n+1})$–module are all one–dimensional and that the weights are precisely $\epsilon_{i_1} + \epsilon_{i_2} + \dots + \epsilon_{i_m}$, $1 \le i_1 < i_2 < \dots < i_m \le n+1$. It follows that

$$X_{m,1}^- . v_{\lambda_m} = b x_m^- . v_{\lambda_m}$$

for some $b \in \mathbb{C}$. Using Proposition 6.1 we get

$$\Phi_{m,1}^+ . v_{\lambda_m} = b(q - q^{-1}) v_{\lambda_m}.$$

Hence, from Proposition 6.2 (b), we get

$$q(q^{-2}u - a) = (u - a)(q + b(q - q^{-1})u + O(u^2)),$$

so that $b = a^{-1}$. Finally, from Proposition 6.1 again, we find that

$$x_0^+ . v_{\lambda_m} = (-1)^{m-1} a^{-1} x_n^- x_{n-1}^- \dots x_{m+1}^- x_1^- \dots x_m^- . v_{\lambda_m}.$$

□

7. Comparison with results of Zelevinsky and Rogawski.

In this section, we describe a parametrization, due to Zelevinsky [**13**] and Rogawski [**12**], of the finite–dimensional irreducible $\widehat{H}_\ell(q^2)$–modules. We then relate this, via the functor $\mathcal{F}$ defined in Theorem 4.2, to the parametrization of the finite–dimensional irreducible $U_q(\widehat{sl}_{n+1})$–modules given in Section 6.

7.1.

Since q is not a root of unity, $H_\ell(q^2) \cong \mathbb{C}[S_\ell]$ as an algebra. It follows that the finite–dimensional $H_\ell(q^2)$–modules are completely reducible and that the irreducibles are in one to one correspondence with the partitions of ℓ. We now describe this correspondence.

The defining relations of $H_\ell(q^2)$ imply that, if $w \in S_\ell$ and if

$$w = \tau_{i_1}\tau_{i_2}\dots\tau_{i_k}$$

is any reduced expression for w in terms of the simple transpositions $\tau_i = (i, i+1)$, the element

$$\sigma_w = \sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_k} \in H_\ell(q^2)$$

depends only on w.

Let $\leq$ be the Bruhat order on S_ℓ, and for $w' \leq w$, let $P_{w',w}(q)$ be the Kazhdan–Lusztig polynomial (see [**9**]). Define elements $C_w \in H_\ell(q^2)$ by

$$C_w = q^{\ell(w)} \sum_{w' \leq w} (-1)^{\ell(w)-\ell(w')} q^{-2\ell(w')} P_{w,w'}(q^{-2})\sigma_w.$$

We write C_i for C_{τ_i}. Note that $C_i = q^{-1}\sigma_i - q$. It is known (see [**9**]) that $\{C_w\}_{w\in W}$ is a basis of $H_\ell(q^2)$, and that

$$C_w\sigma_i = -C_w \text{ if } w\tau_i < w. \tag{12}$$

Let $\ell = \ell_1 + \ell_2 + \cdots + \ell_p$ be a partition π of ℓ, with each $\ell_r > 0$, and let S_ℓ^π be the subgroup $S_{\ell_1} \times S_{\ell_2} \times \cdots \times S_{\ell_p}$ of S_ℓ which fixes π. Let w_r be the longest element of the subgroup S_{ℓ_r}, i.e. the permutation which reverses the order of $(\ell_1 + \ell_2 + \cdots + \ell_{r-1} + 1, \dots, \ell_1 + \cdots + \ell_r)$, and set $w_\pi = w_1 w_2 \dots w_p$. Let I_π be the right ideal in $H_\ell(q^2)$ generated by C_{w_π}.

Proposition ([**12**]). *For every partition π of ℓ, I_π has a unique irreducible quotient J_π in which C_{w_π} has non-zero image. Conversely, every finite-dimensional irreducible right $H_\ell(q^2)$-module is isomorphic to some J_π.*

7.2.

Using Jimbo's functor $\mathcal{J}$, we can compare this parametrization of the finite–dimensional irreducible representations of $H_\ell(q^2)$ with that of the representations of $U_q(sl_{n+1})$ given by their highest weights.

Proposition. *Let $1 \le \ell \le n$ and let $\ell_1 + \ell_2 + \cdots + \ell_p$ be a partition π of ℓ. Then,*

$$\mathcal{J}(J_\pi) \cong V(\lambda_{\ell_1} + \lambda_{\ell_2} + \cdots + \lambda_{\ell_p})$$

as $U_q(sl_{n+1})$–modules.

Proof. We need the following lemma, which follows from (1).

Lemma. *Let π be as in the preceding proposition, and let $1 \le i \le \ell$ be such that $i \ne \sum_{j=1}^{r} \ell_j$ for any $1 \le r < p$. Let $\mathbf{v} \in V^{\otimes \ell}$ have $v_r \otimes v_s$ in the i^{th} and $(i+1)^{th}$ positions, and let $\mathbf{v}'$ be the result of interchanging the vectors in these positions. Then, in $\mathcal{J}(J_\pi)$, we have*

$$C_{w_\pi} \otimes \mathbf{v}' = \begin{cases} -q^{-1} C_{w_\pi} \otimes \mathbf{v} & \text{if } r < s, \\ -q C_{w_\pi} \otimes \mathbf{v} & \text{if } r > s, \\ 0 & \text{if } r = s. \end{cases}$$

Returning to the proof of the proposition, note that the weight space of $V^{\otimes \ell}$ of weight $\lambda_{\ell_1} + \lambda_{\ell_2} + \cdots + \lambda_{\ell_p}$ is spanned by the permutations of the vector

$$\mathbf{v}_\pi = v_1 \otimes v_2 \cdots \otimes v_{\ell_1} \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_{\ell_2} \otimes v_1 \cdots \otimes v_{\ell_p}.$$

By Proposition 4.1, there exists a partition π' of ℓ, say $\ell = \ell'_1 + \ell'_2 + \cdots + \ell'_r$, such that

$$\mathcal{J}(J_{\pi'}) \cong V(\lambda_{\ell_1} + \cdots + \lambda_{\ell_r}). \tag{13}$$

By the lemma, if $v_{i_1} \otimes \cdots \otimes v_{i_\ell}$ is any permutation of $\mathbf{v}_\pi$,

$$C_{w_\pi} \otimes v_{i_1} \otimes \cdots \otimes v_{i_\ell} = 0$$

unless the first ℓ'_1 vectors in the sequence $v_{i_1}, \ldots, v_{i_\ell}$ are distinct, together with the next ℓ'_2, ..., and the last ℓ'_r. It follows that, if $\le$ is the usual lexicographic ordering on the set of partitions of ℓ, we have $\pi' \le \pi$. But the map $\pi \to \pi'$ defined by (13) is a bijection since $\mathcal{J}$ is an equivalence. Since $\le$ is a total ordering it follows that this bijection is the identity map, i.e. $\pi' = \pi$. ∎

7.3.

We now turn to the representations of affine Hecke algebras. Recall the universal modules $M_{\mathbf{a}}$ defined in Section 3.4. We begin with the following elementary result.

Lemma ([**12**]). *Let* $\mathbf{a} = (a_1, \dots, a_\ell) \in (\mathbb{C}^\times)^\ell$, $w \in S_\ell$, $j \in \{1, \dots, \ell\}$. *Then, in* $M_{\mathbf{a}}$, *we have*

$$C_w . y_j = a_{w^{-1}(j)} C_w + \sum_{w' < w} \alpha_{w'} C_{w'}$$

for some $\alpha_{w'} \in \mathbb{C}$.

7.4.

Following Rogawski [**12**] and Zelevinsky [**13**], we make the following definition.

Definition. The segment s with centre $a \in \mathbb{C}^\times$ and length $|s| = k$ is the ordered sequence $s = (aq^{-k+1}, aq^{-k+3}, \dots, aq^{k-1}) \in (\mathbb{C}^\times)^k$.

If $\mathbf{s} = \{s_1, s_2, \dots, s_p\}$ is any (unordered) collection of segments, and if $|s_r| = \ell_r$, then $\ell = \ell_1 + \ell_2 + \cdots + \ell_p$ is a partition $\pi(\mathbf{s})$ of ℓ.

Proposition ([**12**]). *Let* $\ell \geq 1$ *and let* $\mathbf{s} = \{s_1, \dots, s_p\}$ *be any collection of segments, the sum of whose lengths is* ℓ. *Let* $\mathbf{a} = (s_1, \dots, s_p) \in (\mathbb{C}^\times)^\ell$ *be the result of juxtaposing the segments in* $\mathbf{s}$. *Then,*
(a) $I_{\pi(\mathbf{s})}$ *is an* $\widehat{H}_\ell(q^2)$*-submodule of* $M_{\mathbf{a}}$ (*this statement makes sense in view of Proposition* 3.4 (b));
(b) *with the* $\widehat{H}_\ell(q^2)$*-module structure from* $M_{\mathbf{a}}$, $I_{\pi(\mathbf{s})}$ *has a unique irreducible subquotient* $V_{\mathbf{a}}$ *in which* $C_{w_{\pi(\mathbf{s})}}$ *has non-zero image.*

Moreover, every finite-dimensional irreducible right $\widehat{H}_\ell(q^2)$*-module is isomorphic to some* $V_{\mathbf{a}}$.

7.5.

To prove the main result of this section, we shall need another description of $I_{\pi(\mathbf{s})}$ (we continue to use the notation of Section 7.4). Let $\Sigma^{\pi(\mathbf{s})} \subset S_\ell$ be the set of transpositions $\tau_i = (i, i+1)$ for $i \in \{1, \dots, \ell\} \backslash \{\ell_1, \ell_1 + \ell_2, \dots, \ell_1 + \cdots + \ell_{p-1}\}$. For $\tau_i \in \Sigma^{\pi(\mathbf{s})}$, let $\mathbf{a}_{\tau_i}$ be the result of interchanging the i^{th} and $(i+1)^{th}$ components of $\mathbf{a}$, and let

$$A_{\mathbf{a},i} : M_{\mathbf{a}_{\tau_i}} \to M_{\mathbf{a}}$$

be the map given by left multiplication by C_i (we identify $M_{\mathbf{a}}$ and $M_{\mathbf{a}_{\tau_i}}$ with $\widehat{H}_\ell(q^2)$ in the usual way).

Proposition ([**12**]). *With the above notation:*
(a) $A_{\mathbf{a},i}$ *is a homomorphism of* $\widehat{H}_\ell(q^2)$*-modules;*
(b) *regarded as an* $\widehat{H}_\ell(q^2)$*-submodule of* $M_{\mathbf{a}}$*,*

$$I_{\pi(\mathbf{s})} = \bigcap_{\tau_i \in \Sigma^{\pi(\mathbf{s})}} (\text{image of } A_{\mathbf{a},i}).$$

7.6.

We can now state the main result of this section.

Theorem. *Let* $\mathbf{s} = \{s_1, \dots, s_p\}$ *be a collection of segments, the sum of whose lengths is* ℓ*, let* a_r *be the centre of* s_r *and* ℓ_r *its length, and let* $\mathbf{a} = (s_1, \dots, s_p) \in (\mathbb{C}^\times)^\ell$ *be the result of juxtaposing* $s_1, \dots, s_p$*, as in Proposition 7.4. Then, if* $\ell \le n$*,* $\mathcal{F}(V_{\mathbf{a}})$ *is the irreducible* $U_q(\widehat{sl}_{n+1})$*-module defined by the polynomials*

$$P_i(u) = \prod_{\{j | \ell_j = i\}} (u - a_j^{-1}), \quad i = 1, \dots, n.$$

Proof. We first prove the result in the special case $p = 1$, so that $\mathbf{a} = (aq^{-\ell+1}, aq^{-\ell+3}, \dots, aq^{\ell-1})$ (we drop the subscripts for simplicity). Note that $w_{\pi(\mathbf{s})} = w_0$, the longest element of S_ℓ, and that $I_{\pi(\mathbf{s})}$ $(= J_{\pi(\mathbf{s})} = V_{\mathbf{a}})$ is one-dimensional and spanned by C_{w_0}. By Proposition 7.2,

$$\mathcal{J}(I_{\pi(\mathbf{s})}) \cong V(\lambda_\ell),$$

the highest weight vector being

$$\mathbf{v}_{\lambda_\ell} = C_{w_0} \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_\ell.$$

As a $U_q(\widehat{sl}_{n+1})$-module, $\mathcal{F}(V_{\mathbf{a}})$ is therefore defined by the polynomials

$$P_i(u) = \begin{cases} u - a' & \text{if } i = \ell, \\ 1 & \text{otherwise,} \end{cases}$$

for some $a' \in \mathbb{C}^\times$. To compute a', note first that, by the definition of $\mathcal{F}$,

$$x_0^+ . \mathbf{v}_{\lambda_\ell} = C_{w_0} . y_1 \otimes v_{n+1} \otimes v_2 \otimes \cdots \otimes v_\ell.$$

Since $I_{\pi(\mathbf{s})}$ is one–dimensional, Lemma 7.3 implies that

$$x_0^+ . \mathbf{v}_{\lambda_\ell} = q^{\ell-1} a C_{w_0} \otimes v_{n+1} \otimes v_2 \otimes \cdots \otimes v_\ell. \tag{14}$$

On the other hand Lemma 6.4 gives

$$\begin{aligned} x_0^+.\mathbf{v}_{\lambda_\ell} &= (-1)^{\ell-1}(a')^{-1}x_n^-x_{n-1}^-\cdots x_{\ell+1}^-x_1^-x_2^-\cdots x_\ell^-.\mathbf{v}_{\lambda_\ell} \\ &= (-1)^{\ell-1}(a')^{-1}(C_{w_0}\otimes v_2\otimes\cdots\otimes v_\ell\otimes v_{n+1}). \end{aligned}$$

Now by (12),

$$C_{w_0}\sigma_i^{-1} = -C_{w_0},$$

and by (1),

$$v_r\otimes v_{n+1} = q\check{R}^{-1}(v_{n+2}\otimes v_r), \quad \text{if } r \le n.$$

Hence,

$$C_{w_0}\otimes v_2\otimes\cdots\otimes v_\ell\otimes v_{n+1} = (-1)^{\ell-1}q^{\ell-1}C_{w_0}\otimes v_{n+1}\otimes v_2\otimes\cdots\otimes v_\ell,$$

and so

$$x_0^+.\mathbf{v}_{\lambda_\ell} = q^{\ell-1}(a')^{-1}C_{w_0}\otimes v_{n+1}\otimes v_2\otimes\cdots\otimes v_\ell.$$

Comparing with (14) gives $a' = a^{-1}$. (It follows from the proof of Proposition 7.2 that $C_{w_0}\otimes v_{n+1}\otimes v_2\otimes\cdots\otimes v_\ell \neq 0$.)

Suppose now that r is arbitrary. From Proposition 7.5 (b),

$$\mathcal{F}(I_{\pi(\mathbf{s})}) = \bigcap_{\tau_i\in\Sigma^{\pi(\mathbf{s})}} (\text{image of } \mathcal{F}(A_{\mathbf{a},i})). \tag{15}$$

To compute $\mathcal{F}(A_{\mathbf{a},i})$, note that $\mathbf{v}\mapsto 1\otimes\mathbf{v}$ defines an isomorphism of $U_q(sl_{n+1})$–modules $V^{\otimes\ell}\to\mathcal{F}(M_{\mathbf{a}})$, and that

$$\mathcal{F}(A_{\mathbf{a},i})(1\otimes\mathbf{v}) = C_i\otimes\mathbf{v} = 1\otimes C_i.\mathbf{v}.$$

It follows that

$$\mathcal{F}(A_{\mathbf{a},i}) = q^{-1}\check{R}_i - q \in \mathrm{End}_{\mathbb{C}}(V^{\otimes\ell}).$$

From (15) and the $r=1$ case, it follows that

$$\mathcal{F}(I_{\pi(\mathbf{s})}) = V(\lambda_{\ell_1}, a_1^{-1})\otimes\cdots\otimes V(\lambda_{\ell_p}, a_p^{-1}).$$

By Propositions 7.2 and 7.4 (b), $\mathcal{F}(V_{\mathbf{a}})$ is the unique irreducible subquotient of $\mathcal{F}(I_{\pi(\mathbf{s})})$ in which the tensor product of the highest weight vectors in the $V(\lambda_{\ell_r}, a_r^{-1})$ has non–zero image. The theorem now follows from the multiplicativity of the polynomials in Proposition 6.3. □

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UNIVERSITY OF CALIFORNIA
RIVERSIDE, CA 92521
E-mail address: chari@math.ucr.edu

AND

KING'S COLLEGE, STRAND
LONDON WC2R 2LS, U.K.