PACIFIC JOURNAL OF MATHEMATICS
Vol. 175, No. 1, 1996

# A CONSTRUCTION OF LOMONOSOV FUNCTIONS AND APPLICATIONS TO THE INVARIANT SUBSPACE PROBLEM

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**In this paper we give a constructive proof of an abstract approximation theorem inspired by the celebrated result of V.I. Lomonosov. This theorem is applied to obtain an alternative proof of some recent characterizations of the invariant subspace problem. We also establish density of non–cyclic vectors for the dual of a set of compact quasinilpotent operators, and conclude with the open problem of obtaining a similar result for the original set, rather than its dual.**

## 1. Introduction.

V.I. Lomonosov in his paper [**8**] conjectured that the adjoint of a bounded operator on a Banach space has a non–trivial closed invariant subspace. In view of the known examples of operators without an invariant subspace [**6**, **11**], this is the strongest version of the invariant subspace problem that can possibly have an affirmative answer. In particular, if the Lomonosov conjecture is true, then every operator on a reflexive Banach space has a non–trivial invariant subspace.

Considering the strong influence of Lomonosov's results on the theory of invariant subspaces, it is not surprising that both the conjecture and the techniques developed in the interesting paper [**8**] received further attention. L. de Branges used this result to obtain a characterization of the invariant subspace problem in terms of density of certain functions. This stimulated another characterization of the invariant subspace problem given by Y.A. Abramovich, C.D. Aliprantis, and O. Burkinshaw in [**1**]. Section 4 below presents a more detailed account of this work.

In this paper we take a slightly different approach. First we give a constructive proof of the approximation theorem, inspired by the well–known Lomonosov construction used in [**7**, **10**]. The theorem is of some independent interest, and it may have other applications besides the ones given in the subsequent sections. This theorem is then applied to give an alternative proof of the main result in [**1**]. Our proof applies to both real and complex Banach spaces, while the original result was established for complex Banach spaces only. The alternative proof somehow explains the role of compact operators that appear in the characterizations of the invariant subspace problem [**1**].

One may notice that the weak*-compactness of the unit ball in dual Banach spaces plays an important role in [**1**, **3**, **4**, **8**], as well as in the applications given in this chapter. In other words, if the Lomonosov conjecture is true, then the compactness of the unit ball, with respect to the weak* topology, is likely to be an important ingredient of its proof.

In the last section we put this observation to the test. A straightforward application of the approximation theorem obtained in Section 3, together with the Schauder–Tychonoff Fixed Point Theorem, yields density of non-cyclic vectors for the dual of a convex set of compact quasinilpotent operators. We end with the open problem of obtaining a similar result for the original set, rather than its dual.

This work is more or less self-contained and the notation and terminology used in it is (supposed to be) standard. However, here are a few conventions that hold throughout this paper:

By an operator we always mean a bounded linear operator acting on a real or complex Banach space. If $\mathcal{A}$ is a set of operators and $K$ is a fixed operator then $\mathcal{A}K$ stands for the set $\{AK \mid A \in \mathcal{A}\}$. Saying that a set of operators $\mathcal{A}$, acting on a Banach space $X$, admits an invariant subspace, means that there exists a non-trivial closed subspace of $X$ that is invariant under all operators in $\mathcal{A}$. The algebra of all bounded linear operators on a Banach space $X$ is denoted by $\mathbf{B}(X)$, while $C(S, X)$ stands for the space of all continuous functions $f\colon S \longrightarrow X$. If $S$ is a subset of a Banach space $X$, then in saying that a linear operator $A$ is in $C(S, X)$, we actually refer to the restriction of the operator $A$ to the set $S$.

## 2. Reflexive Topological Spaces and Continuous Indicator Functions.

This section introduces some topological preliminaries that lead to a fairly general treatment of the approximation theory in the next section, where an important role is played by the partition of unity and the "continuous indicator functions" associated with a basis for the topology on a compact domain of certain functions. The existence of continuous indicator functions can be characterized by a purely topological property of the underlying space, which is defined as "reflexivity" of the topological space. In this section we introduce both concepts and establish the connection between them.

**Definition 2.1.** Let $S = (S, \tau)$ be a topological space and denote by $C(S, \mathbb{R})$ the space of all continuous real-valued functions on $S$. A topological space $S$ is called *reflexive* if the topology $\tau$ coincides with the weakest topology $\tau_w$ on $S$ for which all the functions in $C(S, \mathbb{R})$ are continuous.

**Remark 2.1.** The reflexivity of topological spaces is not to be confused with the corresponding concept of the reflexivity of Banach spaces. Indeed, we conclude this section by showing that every subset of a locally convex space is reflexive.

**Proposition 2.1.** *Reflexivity is a hereditary property; i.e. a subspace $S$ of a reflexive topological space $X$ is reflexive with the relative topology.*

*Proof.* Consider the restrictions of the functions in $C(X, \mathbb{R})$ to the subset $S$, and observe that they induce the relative topology on $S$, whenever $X$ is reflexive. □

**Definition 2.2.** Suppose $U$ is an open subset of a topological space $S$. A continuous function $\Gamma\colon S \longrightarrow [0, \infty)$ is called a *continuous indicator function* of $U$ in $S$ if

$$U = \{s \in S \mid \Gamma(s) > 0\}.$$

**Remark 2.2.** If $X$ is a metric space then every open ball

$$U = U(x_0, r) = \{x \in X \mid d(x, x_0) < r\},$$

admits a continuous indicator function $\Gamma_U\colon X \longrightarrow [0, \infty)$, defined by

$$\Gamma_U(x) = \max\{0,\ r - d(x, x_0)\}.$$

Furthermore, suppose $f \in C(S, X)$. Then the open set $V = f^{-1}(U) \subset S$ "inherits" an indicator function from $U$ by setting: $\Gamma_V(s) = \Gamma_U(f(s))$.

This yields the following characterization of reflexivity.

**Proposition 2.2.** *A topological space $S = (S, \tau)$ is reflexive if and only if there exists an open basis $\mathcal{B}$ for the topology $\tau$ such that each set $V \in \mathcal{B}$ admits a continuous indicator function $\Gamma_V\colon S \longrightarrow [0, \infty)$.*

*Proof.* By definition of reflexivity, the family

$$\mathcal{B}_0 = \{f^{-1}(U) \mid f \in C(S, \mathbb{R}) \text{ and } U = (a, b) \subset \mathbb{R}\}$$

is a sub–basis for the topology $\tau$ on a reflexive space $S$. Clearly,

$$\Gamma_U(t) = \max\left\{0,\ \tfrac{b-a}{2} - \left|\tfrac{a+b}{2} - t\right|\right\}$$

is a continuous indicator function of the open interval $U = (a,b)$ in $\mathbb{R}$. Consequently, $\Gamma_V(s) = \Gamma_U(f(s))$ is a continuous indicator function for the set $V = f^{-1}(U)$ in $S$. Let $V = V_1 \cap \ldots \cap V_n$ for $V_k \in \mathcal{B}_0$. A continuous indicator function of $V$ can be defined by

$$\Gamma_V(s) = \prod_{k=1}^{n} \Gamma_{V_k}(s).$$

Therefore, each set in a basis

$$\mathcal{B} = \{V_1 \cap \ldots \cap V_n \mid V_k \in \mathcal{B}_0;\ n < \infty\},$$

admits a continuous indicator function.

The other direction is trivial, because the continuous indicator functions form a subset of $C(S, \mathbb{R})$. □

**Remark 2.3.** The argument in the proof of Proposition 2.2 shows that the space $\mathbb{R}$ can be replaced by any metric vector space over $\mathbb{R}$ in the definition of reflexivity. In particular, considering the complex valued functions would not change the definition of reflexivity.

**Remark 2.4.** While an open set $U$ is uniquely determined by any of its continuous indicator functions, the converse is of course not true. However, Proposition 2.2 allows us to choose a basis $\mathcal{B}$, and a corresponding family

$$\Gamma_{\mathcal{B}} = \{\Gamma_U \colon S \longrightarrow [0,\infty) \mid U \in \mathcal{B}\}$$

of continuous indicator functions associated with the basis $\mathcal{B}$ for the topology on a reflexive topological space $S$. In that sense, the correspondence between the elements of $\mathcal{B}$ and an associated family of continuous indicator functions $\Gamma_{\mathcal{B}}$ can be established.

Although not all topological spaces are reflexive (consider for example the topology of finite complements on any infinite set) the next proposition shows that convex balanced neighborhoods in a locally convex space admit continuous indicator functions, and consequently, all locally convex spaces are reflexive.

**Proposition 2.3.** *Every locally convex space $X$ is reflexive* (*as a topological space*).

*Proof.* Suppose $\mathcal{B}$ is a base for the topology on $X$ consisting of open convex balanced sets. Then for each $U \in \mathcal{B}$:

$$U = \{x \in X \mid \mu_U(x) < 1\},$$

where $\mu_U$ is the Minkowski functional of $U$. The function

$$\Gamma_U(x) = \max\{0,\, 1 - \mu_U(x)\}$$

is a continuous indicator function for $U$. By Proposition 2.2, $X$ is reflexive. $\square$

## 3. A Construction of the Lomonosov Functions.

The proof of the celebrated result of V.I. Lomonosov [**7, 10**] was based on the ingenious idea of defining a continuous function with compact domain in a Banach space, assuming that certain local conditions are met. In this section we generalize this idea in the form of an approximation theorem. Since our construction was greatly inspired by the proof of Lomonosov's Lemma [**7, 10**], we suggest the following definition.

**Definition 3.1.** Let $\mathcal{A} \subset C(S, X)$ be a subset of the space of continuous functions from a topological space $S$ to a locally convex space $X$. The convex subset $\mathcal{L}(\mathcal{A}) \subset C(S, X)$, defined by

$$\mathcal{L}(\mathcal{A}) = \left\{ \sum_{k=1}^{n} \alpha_k A_k \mid A_k \in \mathcal{A}, \alpha_k \in C(S, [0,1]) \text{ and } \sum_{k=1}^{n} \alpha_k \equiv 1;\ n < \infty \right\}$$

is called the *Lomonosov space* associated with the set $\mathcal{A}$, and a function $\Lambda \in \mathcal{L}(\mathcal{A})$ is called a *Lomonosov function*.

Recall that the *uniform topology* on $C(S, X)$ is induced by the topology on a linear space $X$. If $\mathcal{B}$ is a local basis for the topology on $X$ then the sets

$$\widehat{U} = \{f \in C(S, X) \mid f(S) \subset U \in \mathcal{B}\}$$

define a local basis for the uniform topology on $C(S, X)$. If $X$ is a locally convex space then so is $C(S, X)$. In particular, if $X$ is a Banach space then $C(S, X)$ with the uniform topology is a Banach space, as well.

We are now ready to give a construction of the Lomonosov function that uniformly approximates a continuous function within a given neighborhood.

**Lemma 3.1.** *Let $\mathcal{A} \subset C(S, X)$ be a subset of continuous functions from a reflexive compact topological space $S$ to a locally convex space $X$. Fix* *an open convex neighborhood $U$ of $0$ in $X$. Suppose $f\colon S \longrightarrow X$ is a continuous function that at each point of $S$ can be approximated within $U$ by some element of $\mathcal{A}$; i.e. for every point $s \in S$ there exists a function $A_s \in \mathcal{A}$ such that $A_s(s) - f(s) \in U$. Then there exists a finite subset $\{A_1, \dots, A_n\}$ of $\mathcal{A}$, together with continuous nonnegative functions $\alpha_k\colon S \longrightarrow [0,1]$, such that $\sum_{k=1}^n \alpha_k \equiv 1$, and the Lomonosov function $\Lambda \in \mathcal{L}(\mathcal{A})$, defined by*

$$\Lambda(s) = \sum_{k=1}^{n} \alpha_k(s) A_k(s),$$

*lies in the prescribed neighborhood $\widehat{U}$ of $f$ in $C(S,X)$; i.e. $(\Lambda - f)(S) \subset U$.*

*Proof.* By the hypothesis, for every point $s \in S$ there exists a function $A_s \in \mathcal{A}$ such that $A_s(s) - f(s) \in U$. Continuity of the functions $f$ and $A_s$ implies the existence of a (basic) neighborhood $W_s$ of $s$ in $S$ such that $A_s(w) - f(w) \in U$ for every $w \in W_s$. In this way we obtain an open cover for $S$ with the sets $W_s$. Compactness of $S$ yields a finite subcover: $W_{s_1} \cup \ldots \cup W_{s_n} \supset S$.

Each set $W_s$ is associated with a continuous indicator function $\Gamma_{W_s} : S \longrightarrow [0, \infty)$. Every point in $S$ lies in at least one neighborhood $W_{s_k}$; therefore the sum $\sum_{j=1}^n \Gamma_{W_{s_j}}(s)$ is strictly positive for all elements $s \in S$. Consequently, the functions $\alpha_k\colon S \longrightarrow [0,1]$, defined by

$$\alpha_k(s) = \frac{\Gamma_{W_{s_k}}(s)}{\sum_{j=1}^n \Gamma_{W_{s_j}}(s)} \qquad (k = 1, \dots, n),$$

are well defined and continuous on $S$. Also, $\sum_{k=1}^n \alpha_k(s) = 1$ for every $s \in S$, and $\alpha_k(s) > 0$ if and only if $s \in W_{s_k}$. Therefore, $\alpha_k(s) > 0$ implies that $A_{s_k}(s) - f(s) \in U$.

Set $A_k = A_{s_k}$ $(k = 1, \dots, n)$. Continuity of the functions $\alpha_k\colon S \longrightarrow [0,1]$ and $A_k\colon S \longrightarrow X$ implies that the Lomonosov function $\Lambda \in \mathcal{L}(\mathcal{A})$, defined by

$$\Lambda(s) = \sum_{k=1}^{n} \alpha_k(s) A_k(s),$$

is continuous. Observe that

$$\Lambda(s) - f(s) = \sum_{k=1}^{n} \alpha_k(s)\big(A_k(s) - f(s)\big)$$

is a convex combination of the elements in $U$, because only those coefficients $\alpha_k(s)$ for which $A_k(s) - f(s) \in U$ are nonzero. Since $U$ is a convex set, it follows that the image of $\Lambda - f$ is contained in $U$. In other words, $\Lambda$ lies in the prescribed neighborhood $\widehat{U}$ of $f$ in $C(S,X)$. $\square$

**Remark 3.1.** The proof of Lomonosov's Lemma [**7**, **10**] introduces a special case of the above construction: $S$ is a compact set in a Banach space $X$, defined as the closure of the image of the unit ball around a fixed vector $x_0$, under a given nonzero compact operator $K$. Furthermore, the vector $x_0$ is chosen so that the set $S$ doesn't contain the zero vector; $\mathcal{A}$ is the restriction to $S$ of an algebra of bounded linear operators on $X$ that admits no invariant subspaces. Under the stated hypothesis a construction of the function $\Lambda\colon S \longrightarrow X$ is given such that $\Lambda \in \mathcal{L}(\mathcal{A}K)$ maps $S$ into the unit ball around $x_0$; or equivalently, the constant function $f \equiv x_0$ can be approximated on $S$ within 1 by the elements of $\mathcal{L}(\mathcal{A}K)$. It is clear from the original construction as well as from Theorem 3.2 that in that case the set $S$ can be mapped into an arbitrary small neighborhood of $x_0$; or equivalently, the function $f \equiv x_0$ is in the closure of $\mathcal{L}(\mathcal{A}K)$.

The following approximation theorem follows immediately from Lemma 3.1.

**Theorem 3.2.** *Let $\mathcal{A} \subset C(S, X)$ be a subset of continuous functions from a reflexive compact topological space $S$ to a locally convex space $X$. Suppose that $f\colon S \longrightarrow X$ is a continuous function that at each point of $S$ can be approximated by some element of $\mathcal{A}$; i.e. for every $s \in S$ and every neighborhood $U$ of $0$ in $X$ there exists a function $A_s \in \mathcal{A}$ such that $A_s(s) - f(s) \in U$. Then the function $f$ can be approximated uniformly on $S$ by the elements of the associated Lomonosov space $\mathcal{L}(\mathcal{A})$.*

In the next section we employ Theorem 3.2 to obtain an alternative proof of a characterization of the existence of invariant subspaces for algebras of bounded linear operators acting on a real or complex Banach space. The complex version of this theorem was first established in [**1**], using rather different techniques built on the result of L. de Branges [**4**].

## 4. A Characterization of the Invariant Subspace Problem.

We introduce some basic concepts and notation that is consistent with [**1**]. However, for more details and further references on the *invariant subspace problem*, the reader is advised to consult the nicely written and comprehensible original [**1**].

In this section $X$ stands for a real or complex Banach space of dimension greater than one and $X'$ for its norm dual. The algebra of all bounded linear operators on $X$ is denoted by $\mathbf{B}(X)$. If $\mathcal{A}$ is any subset of $\mathbf{B}(X)$, then the adjoint set $\mathcal{A}'$ of $\mathcal{A}$ is defined by $\mathcal{A}' = \{A' \mid A \in \mathcal{A}\}$, where $A'$ is the Banach adjoint of $A$.

The set $S = \{x \in X' \mid \|x\| \le 1\}$ denotes the unit ball in the dual space $X'$, equipped with its weak* topology.

**Definition 4.1.** The vector space of all continuous functions from $S$ to $X'$, where both spaces are equipped with the weak* topology, is denoted by $C(S, X')$. As usual, $C(S)$ denotes the commutative Banach algebra of all continuous complex valued functions on $S$ with the uniform norm.

Note that for each $T \in \mathbf{B}(X)$ the restriction of the adjoint operator $T' \colon S \longrightarrow X'$ is a member of $C(S, X')$. The vector space $C(S, X')$, equipped with the norm

$$\|f\| = \sup_{s \in S} \|f(s)\|,$$

is a Banach space.

The Banach space $C(S, X')$ played the central role in [**1**, **4**, **8**]. Lomonosov [**8**] based his proof of an interesting extension of Burnside's Theorem on the characterization of the extreme points of the unit ball in the norm dual of $C(S, X')$ using the argument of the celebrated de Branges' proof of the Stone–Weierstrass Theorem [**3**]. Louis de Branges [**4**] performed a deep analysis of the behaviour of these extreme points that yielded a vector generalization of the Weierstrass approximation theorem. This approach resulted in a characterization of the existence of a nontrivial invariant subspace for the algebra $\mathcal{A}'$ in terms of density of the linear span of the set

$$\{\alpha A' \mid \alpha \in C(S) \text{ and } A \in \mathcal{A}\},$$

in the space of restrictions of the adjoint operators to $S$, with respect to a topology in $C(S, X')$, introduced by L. de Branges.

Building upon this work, Y.A. Abramovich, C.D. Aliprantis, and O. Burkinshaw in [**1**], obtained the following characterizations of the existence of a non–trivial invariant subspace for an algebra $\mathcal{A}$ of bounded linear operators acting on a complex Banach space $X$:

**Theorem 4.1** [Y.A. Abramovich, C.D. Aliprantis, and O. Burkinshaw]. *There is a non–trivial closed $\mathcal{A}$–invariant subspace of $X$ if and only if there exists an operator $T \in \mathbf{B}(X)$ and a compact operator $K \in \mathbf{B}(X)$ such that $K'T'$ does not belong to the norm closure of the vector subspace of $C(S, X')$ generated by the collection*

$$\{\alpha K'A' \mid \alpha \in C(S) \text{ and } A \in \mathcal{A}\}.$$

**Theorem 4.2** [Y.A. Abramovich, C.D. Aliprantis, and O. Burkinshaw]. *There is a non–trivial closed $\mathcal{A}'$–invariant subspace of $X'$ if and only if there exists an operator $T \in \mathbf{B}(X)$ and a compact operator $K \in \mathbf{B}(X)$ such that*

*$T'K'$ does not belong to the norm closure of the vector subspace of $C(S, X')$ generated by the collection*

$$\{\alpha A'K' \mid \alpha \in C(S) \text{ and } A \in \mathcal{A}\}.$$

We will give a short proof of both theorems as an application of Theorem 3.2. Our proof applies to real or complex Banach spaces, where in the case of a real Banach space, $C(S)$ stands for the Banach algebra of all real–valued continuous functions on the set $S$.

Observe that the Lomonosov spaces $\mathcal{L}(K'\mathcal{A}')$ and $\mathcal{L}(\mathcal{A}'K')$, as defined in the previous section, are subsets of the linear manifolds introduced in Theorems 4.1 and 4.2.

**Definition 4.2.** The vector $x$ in a Banach space $X$ is *cyclic* for the set of operators $\mathcal{A} \subset \mathbf{B}(X)$ whenever the orbit

$$\mathcal{A}x = \{Ax \mid A \in \mathcal{A}\}$$

is a dense subset of $X$. If every nonzero vector is cyclic for $\mathcal{A}$, we say that $\mathcal{A}$ acts *transitively* on $X$. The terms *$\tau$–cyclic* and *$\tau$–transitive* are defined in the same way, by considering the space $X$ equipped with a topology $\tau$, instead of the norm.

The following well known characterizations of the existence of a non-trivial invariant subspace for an algebra $\mathcal{A} \subset \mathbf{B}(X)$ follow immediately from the definition.

**Proposition 4.3.** *Suppose $\mathcal{A} \subset \mathbf{B}(X)$ is a subalgebra of bounded linear operators on $X$. The following are equivalent:*

(1) *$\mathcal{A}$ admits no nontrivial closed invariant subspace.*

(2) *$\mathcal{A}$ acts weak–transitively on $X$.*

(3) *$\mathcal{A}$ acts transitively on $X$.*

(4) *$\mathcal{A}'$ admits no nontrivial weak*–closed invariant subspace.*

(5) *$\mathcal{A}'$ acts weak*–transitively on $X'$.*

As in [**1**] we introduce the subspace of completely continuous functions in $C(S, X')$.

**Definition 4.3.** A function $f \in C(S, X')$ is said to be *completely continuous* if it is continuous with respect to the weak* topology on $S$ and the norm topology on $X'$. The subspace of all completely continuous functions is denoted by $\mathcal{K}(S, X')$.

Note that $K'\colon S \longrightarrow X'$ is completely continuous whenever $K \in \mathbf{B}(X)$ is a compact operator on $X$.

We are now ready to give a short proof of Theorems 4.1 and 4.2.

*Proof of Theorems* 1 *and* 2. We start with Theorem 4.2, which is an almost straightforward consequence of Proposition 4.3 and Theorem 3.2, applied to the space $\mathcal{K}(S, X')$.

Suppose $\mathcal{A}'$ has a non–trivial closed invariant subspace. Then by Proposition 4.3, there exists a pair of nonzero vectors $x', y' \in S$ such that $\|A'x' - y'\| \geq \varepsilon > 0$ for all $A' \in \mathcal{A}'$. Choose any vector $x \in X$ such that $\langle x', x\rangle = 1$, and define the rank–one operators $K = x \otimes x'$ and $T = x \otimes y'$. Clearly $T'K'x' = y'$, and since $T'K'$ cannot be approximated by the operators $A'K'$ at the point $x'$, it follows that $T'K'$ is not in the norm closure of the linear space generated by $\{\alpha A'K' \mid \alpha \in C(S) \text{ and } A \in \mathcal{A}\}$.

Conversely, suppose $\mathcal{A}'$ admits no non–trivial closed invariant subspaces. Therefore, $\mathcal{A}'$ acts transitively on $X'$, and consequently, every operator $T'K'$ can be approximated by $A'K'$ at each point of $S$. Furthermore, since $K$ is a compact operator in $\mathbf{B}(X)$, it follows that $T'K' \in \mathcal{K}(S, X')$. Theorem 3.2 implies that $T'K'$ is in the norm closure of the Lomonosov space $\mathcal{L}(\mathcal{A}'K')$ and thus completes the proof.

The proof of Theorem 4.1 is just slightly more complicated.

Suppose the algebra $\mathcal{A}$ admits a nontrivial closed invariant subspace $\mathcal{M}$. Then $\mathcal{M}^{\perp}$ is an invariant subspace for $\mathcal{A}'$. Fix a nonzero vector $x \in \mathcal{M}$ and a nonzero functional $y' \in M^{\perp}$, and choose a vector $y \in X$ such that $\langle y', y\rangle = 1$ and a functional $x' \in X'$, with $\langle x', x\rangle = 1$. Define the rank–one operators $K = x \otimes y'$ and $T = y \otimes x'$. Then $K'T'y' = y' \neq 0$, while $K'A'y' = 0$ for every $A' \in \mathcal{A}'$. Consequently, the operator $K'T'$ is not in the norm closure of the linear span of the completely continuous functions $\{\alpha K'A' \mid \alpha \in C(S) \text{ and } A \in \mathcal{A}\}$.

Conversely, suppose that there exists a compact operator $K$ and an operator $T$ such that $K'T'$ is not in the closure of the linear subspace generated by the completely continuous functions $\{\alpha K'A' \mid \alpha \in C(S) \text{ and } A \in \mathcal{A}\}$. Theorem 3.2 implies that there exists a nonzero vector $x' \in S$ such that the orbit $\mathcal{M} = \{K'A'x' \mid A \in \mathcal{A}\}$ is not a norm–dense manifold in the closure of the subspace $\mathcal{N} = \{K'T'x' \mid T \in \mathbf{B}(X)\}$. By the Hahn–Banach Theorem there exists a functional $y'' \in X''$ such that $\langle y'', K'A'x'\rangle = 0$ for every $A' \in \mathcal{A}'$, and $\langle y'', K'T'x'\rangle = 1$ for some $T \in \mathbf{B}(X)$. Consequently, $K''y'' \neq 0$. Compactness of $K$ implies that $y = K''y'' \in X$, where $X$ is considered naturally embedded in its second dual $X''$ (Theorem 5.5 [**2**, p. 185] or Theorem 2 [**5**, p. 482]). From $\langle x', Ay\rangle = 0$ for all $A \in \mathcal{A}$, it follows that the algebra $\mathcal{A}$ admits a non–trivial closed invariant subspace. $\square$

It is possible to obtain similar characterizations that do not involve compact operators, by considering some other topology on $C(S, X')$. Theorem 3.1 in [**1**] and Theorem 6 in [**4**] are examples of results in that direction. We conclude this section by giving yet another characterization of transitivity for an algebra $\mathcal{A}$ in terms of the closure of the Lomonosov space $\mathcal{L}(\mathcal{A}')$ with respect to the *uniform* topology $\tau_{w^*}$, induced on $C(S, X')$ by the weak* topology on the dual Banach space $X'$.

**Theorem 4.4.** *Suppose $\mathcal{A} \subset \mathbf{B}(X)$ is a set of bounded linear operators on $X$. Then the dual set $\mathcal{A}' = \{A' \mid A \in \mathcal{A}\}$ acts weak*–transitively on $S$ if and only if the $\tau_{w^*}$–closure of the Lomonosov space $\mathcal{L}(\mathcal{A}')$ is equal to the subspace*

$$C_0(S, X') = \{f \in C(S, X') \mid f(0) = 0\}.$$

*Proof.* The proof is almost identical to those of Theorems 4.1 and 4.2 except that Theorem 3.2 is now applied to the space $C(S, X')$ equipped with the topology $\tau_{w^*}$, instead of $\mathcal{K}(S, X')$ with the norm topology.

If the set $\mathcal{A}'$ does not act weak*–transitively on $X'$ then there exists a nonzero vector $x' \in S$ together with a weak* neighborhood $W$ of $y'$ in $S$ such that $A'x' \notin W$ for all $A' \in \mathcal{A}'$. Choose a vector $x \in X$ such that $\langle x', x \rangle = 1$ and let $T = x \otimes y'$. Then $T'x' = y'$, and since $T' \in C_0(S, X')$ cannot be approximated by the operators in $\mathcal{A}'$ at the point $x'$, it follows that $T'$ is not in the $\tau_{w^*}$–closure of the associated Lomonosov space $\mathcal{L}(\mathcal{A}')$.

Conversely, if the set $\mathcal{A}'$ acts weak*–transitively on $S$ it follows from Theorem 3.2 that every function $f \in C_0(S, X')$ can be uniformly approximated by the elements of $\mathcal{L}(\mathcal{A}')$, and thus $f$ is in the $\tau_{w^*}$–closure of the Lomonosov space $\mathcal{L}(\mathcal{A}')$. $\square$

**Corollary 4.5.** *The algebra $\mathcal{A}$ admits no non–trivial closed invariant subspace if and only if the $\tau_{w^*}$–closure of the Lomonosov space $\mathcal{L}(\mathcal{A}')$ is equal to the subspace*

$$C_0(S, X') = \{f \in C(S, X') \mid f(0) = 0\}.$$

*Proof.* By Proposition 4.3, the fact that $\mathcal{A}$ admits no non–trivial invariant subspace is equivalent to $\mathcal{A}'$ acting weak*–transitively on $S$. The result now follows from Theorem 4.4. $\square$

Note that the $\tau_{w^*}$–closure of the Lomonosov space $\mathcal{L}(\mathbf{B}(X)')$ is always equal to $C_0(S, X')$. This observation yields a few alternative formulations of Corollary 4.5, which are left to the reader.

## 5. On Convex Sets of Compact Quasinilpotent Operators.

In this section we combine Lemma 3.1 with the Schauder–Tychonoff Fixed Point Theorem, to establish a density result for non–cyclic vectors for the dual of a convex set of compact quasinilpotent operators. We discuss in what sense this result generalizes the celebrated Lomonosov Lemma [**7**], and conclude with a problem of establishing a similar result for the original set, rather than its dual.

Recall that an operator is called *quasinilpotent* if 0 is the only point in its spectrum.

**Theorem 5.1.** *Suppose $\mathcal{A}$ is a convex set of compact quasinilpotent operators acting on a real or complex Banach space $X$, and let $\mathcal{A}' = \{A' \mid A \in \mathcal{A}\}$ be its dual in $\mathbf{B}(X')$. Then the set of non–cyclic vectors for $\mathcal{A}'$ is dense in $X'$.*

*Proof.* Suppose not; then there exists a functional $x_0 \in X'$ and a positive number $r > 0$ such that all vectors in the ball

$$S = \{x \in X' \mid \|x - x_0\| \leq r\}$$

are cyclic for $\mathcal{A}'$. In particular, for every functional $x \in S$ there exists an operator $A' \in \mathcal{A}'$ such that $\|A'x - x_0\| < r$. By Lemma 3.1 it follows that there exists a Lomonosov function $\Lambda \in \mathcal{L}(\mathcal{A}')$ such that $\|\Lambda(x) - x_0\| < r$ for all $x \in S$. Consequently, $\Lambda$ maps $S$ into itself (weak*–continuously).

The Schauder–Tychonoff Fixed Point Theorem [**5**, p. 456] implies that $\Lambda$ has a fixed point $x_1 = \Lambda(x_1)$ in $S$. By the definition of the Lomonosov space

$$\Lambda = \sum_{k=1}^{n} \alpha_k A'_k, \quad \text{where } A_k \in \mathcal{A},\ \alpha_k \in C(S, [0,1]) \text{ and } \sum_{k=1}^{n} \alpha_k \equiv 1;\ n < \infty.$$

Therefore $A' = \sum_{k=1}^{n} \alpha_k(x_1) A'_k$ is an operator in the convex set $\mathcal{A}'$. From $\Lambda(x_1) = x_1$, we conclude that $A'x_1 = x_1$. Since $x_1 \neq 0$, it follows that 1 is an eigenvalue for $A'$, contradicting the assumption that $A'$ is a quasinilpotent operator. □

**Remark 5.1.** Note that (unless $\mathcal{A}$ is assumed to be an algebra) it is not enough to require that the operators in $\mathcal{A}'$ have no common invariant subspace, in order to ensure that $\mathcal{A}'$ acts transitively on $X'$. It is indeed possible to give examples of manifolds of nilpotent operators without a non–trivial closed common invariant subspace. For such examples on finite–dimensional vector spaces see [**9**]. By Theorem 5.1 a manifold of such operators cannot act transitively on the underlying space.

Theorem 5.1 does not follow from the original work of V.I. Lomonosov [**7**]. On the other hand, Lomonosov's Lemma [**7**] easily follows from Theorem 5.1, in the case when the underlying Banach space is reflexive. In that sense Theorem 5.1 is a generalization of the Lomonosov Lemma.

This discussion suggests the following question, which we have not been able to resolve:

*Does there exist a convex set $\mathcal{A}$ of compact quasinilpotent operators acting on a real or complex Banach space $X$ such that the set of non-cyclic vectors for $\mathcal{A}$ is* not *dense in $X$?*

By Theorem 5.1 the underlying Banach space in such an example (if it exists) cannot be reflexive. Furthermore, Lomonosov's Lemma implies that the set $\mathcal{A}$ cannot be of the form $\mathcal{A}K$ or $K\mathcal{A}$, where $K$ is a fixed compact operator. In particular, the set $\mathcal{A}$ in such an example can never be an algebra.

Since, according to Theorems 4.1 and 4.2, compact operators are closely related to the existence of invariant subspaces for algebras of operators, the answer to the above question might be of some interest to the theory of invariant subspaces.

**Acknowledgements.** The author is thankful to Professor Heydar Radjavi, whose suggestions resulted in a significant improvement of the formulation of Theorem 3.2. He also thanks Professor Y.A. Abramovich for sending the preprints of some of his recent joint work with C.D. Aliprantis, and O. Burkinshaw, that stimulated this research.

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Received April 28, 1994 and revised December 16, 1994. This work was enabled by the award of the Izaak Walton Killam Memorial Scholarship, and also supported in part by the Research Council of Slovenia.

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