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L^P-BOUNDS FOR HYPERSINGULAR INTEGRAL OPERATORS ALONG CURVES

SHARAD CHANDARANA

It is known that the Hilbert transform along curves:

$$\mathcal{H}_\Gamma f(x) = pv \int_{-\infty}^{\infty} f(x - \Gamma(t)) \frac{dt}{t} \qquad (x \in \mathbf{R}^n)$$

is bounded on L^p, $1 < p < \infty$, where $\Gamma(t)$ is an appropriate curve in $\mathbf{R}^n$. In particular, $\|\mathcal{H}_\Gamma f\|_p \le C\|f\|_p$, $1 < p < \infty$, where $\Gamma(t) = (t, |t|^k \operatorname{sgn} t)$, $k \ge 2$, is a curve in $\mathbf{R}^2$.
It is easy to see that the *hypersingular integral operator*

$$\mathcal{T} f(x) = pv \int_{-1}^{1} f(x - \Gamma(t)) \frac{dt}{t|t|^\alpha} \qquad (\alpha > 0),$$

in which the singularity at the origin is worse than that in the Hilbert transform, is not bounded on $L^2(\mathbf{R}^2)$. To counterbalance this worsened singularity, we introduce an additional oscillation $e^{-2\pi i|t|^{-\beta}}$ and study the operator

$$\mathcal{T}_{\alpha,\beta} f(x,y) = pv \int_{-1}^{1} f(x - t, y - \gamma(t))\, e^{-2\pi i|t|^{-\beta}} \frac{dt}{t|t|^\alpha} \qquad (\alpha, \beta > 0)$$

along the curve $\Gamma(t) = (t, \gamma(t))$, where $\gamma(t) = |t|^k$ or $\gamma(t) = |t|^k \operatorname{sgn} t$, $k \ge 2$, in $\mathbf{R}^2$ and show that

(i) $\|\mathcal{T}_{\alpha,\beta} f\|_2 \le A_{\alpha,\beta}\|f\|_2$ **if and only if** $\beta \ge 3\alpha$;

(ii) $\|\mathcal{T}_{\alpha,\beta} f\|_p \le B_{\alpha,\beta}\|f\|_p$ **whenever** $\beta > 3\alpha$, **and**

$$1 + \frac{3\alpha(\beta+1)}{\beta(\beta+1) + (\beta - 3\alpha)} < p < \frac{\beta(\beta+1) + (\beta - 3\alpha)}{3\alpha(\beta+1)} + 1.$$

1. Introduction.

In recent years, several mathematicians have studied the Hilbert transform along curves:

$$\mathcal{H}_\Gamma f(x) = pv \int_{-\infty}^{\infty} f(x - \Gamma(t)) \frac{dt}{t} \qquad (x \in \mathbf{R}^n),$$

where $\Gamma(t)$ is an appropriate curve in $\mathbf{R}^n$. Fabes and Rivière were led to the study of $\mathcal{H}_\Gamma$ in their attempt to generalize the Method of Rotation of Calderon and Zygmund; for details see [**Fa, Ri**] and [**Wa2**].

Nagel, Rivière, Stein and Wainger, and several other mathematicians have studied the L^p-boundedness of $\mathcal{H}_\Gamma$ for a variety of curves Γ. A detailed survey of these results can be found in [**St, Wa**]; also see [**Wa1**]. Nagel, Rivière and Wainger proved in [**NRW1**] that $\mathcal{H}_\Gamma$ is a bounded operator on L^p, $1 < p < \infty$, when $\Gamma(t) = (|t|^{\alpha_1} \operatorname{sgn} t, \cdots, |t|^{\alpha_n} \operatorname{sgn} t)$, each $\alpha_k > 0$, is a curve in $\mathbf{R}^n$. In particular, $\|\mathcal{H}_\Gamma f\|_p \leq C\|f\|_p$, $1 < p < \infty$, where $\Gamma(t) = (t, |t|^k \operatorname{sgn} t)$, $k \geq 2$, is a curve in $\mathbf{R}^2$. For more general curves see [**Na, Wa**], [**NVWW**], and [**Wa3**].

The kernel, $K(x) = \frac{1}{\pi x}$, of the Hilbert transform,

$$\mathcal{H}f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x-y)}{y} \, dy \qquad (x \in \mathbf{R}),$$

owing to its order of magnitude, is not integrable either at 0 or ∞. It does, however, compensate for this deficiency by cancellation due to oscillation; this oscillatory property being reflected in the fact that its Fourier transform, $\hat{K}(x) = i \operatorname{sgn} x$, is bounded.

It is tempting to explore a situation where the order of magnitude of the singularity of K at the origin is greater than that of $|x|^{-1}$, say of the order of $|x|^{-1-\alpha}$, $\alpha > 0$. It is reasonable to expect that some additional oscillation is required to compensate for this worsened singularity. This translates to the requirement that the Fourier transform of K, in addition to being bounded, have some decay at infinity; that is, $|\hat{K}(x)| \leq C(1+|x|)^{-\beta}$ for some $\beta > 0$. For further discussion see Theorem 5 of [**St**].

Integral operators with strong singularities of the type described above, were studied by Hirschman in one dimension [**Hi**], Wainger in k-dimensions [**Wa**], Stein [**St**], Fefferman [**Fe**], and Fefferman and Stein [**Fe, St**].

It is not hard to see that the *hypersingular integral operator*

$$\mathcal{T}f(x) = pv \int_{-1}^{1} f(x - \Gamma(t)) \frac{dt}{t|t|^\alpha} \qquad (\alpha > 0)$$

along $\Gamma(t) = (t, \gamma(t))$, where $\gamma(t) = |t|^k$ or $\gamma(t) = |t|^k \operatorname{sgn} t$, $k \geq 2$, is not bounded on $L^2(\mathbf{R}^2)$. The L^2-boundedness of this operator is equivalent to the *uniform* boundedness, in $\mathbf{R}^2$, of the *multiplier*

$$m(x,y) = pv\int_{-1}^{1} e^{-2\pi i[xt+y\gamma(t)]}\,\frac{dt}{t|t|^\alpha} \qquad (\alpha > 0).$$

It is easy to see that $|m(\frac{1}{4},0)| = \infty$ for $\alpha \geq 1$; for $0 < \alpha < 1$ and $x > 0$,

$$|m(x,0)| = 2\left|\int_0^1 \sin(2\pi x t)\,\frac{dt}{t^{1+\alpha}}\right| = 2\,(2\pi x)^\alpha \left|\int_0^{2\pi x} \sin s\,\frac{ds}{s^{1+\alpha}}\right| \to \infty$$

as $x \to \infty$.

One can ask if the worsened singularity at the origin can be counterbalanced by an oscillation. This leads us to the operator

$$\mathcal{T}_{\alpha,\beta} f(x,y) = pv\int_{-1}^{1} f(x-t, y-\gamma(t))\, e^{-2\pi i|t|^{-\beta}}\,\frac{dt}{t|t|^\alpha} \qquad (\alpha,\, \beta > 0)$$

along the curve $\Gamma(t) = (t, \gamma(t))$, $\gamma(t) = |t|^k$ or $\gamma(t) = |t|^k \operatorname{sgn} t$, $k \geq 2$, in $\mathbf{R}^2$.

Zielinski, in his thesis [**Zi**], studied the L^2-boundedness of $\mathcal{T}_{\alpha,\beta}$ along the parabola $\gamma(t) = (t,t^2)$, and proved that $\|\mathcal{T}_{\alpha,\beta} f\|_2 \leq C\|f\|_2 \iff \beta \geq 3\alpha$.

1.1. Statement of the Main Result. We state the main result of this paper as:

Theorem 1. *Suppose that* $\gamma(t) = |t|^k$ *or* $\gamma(t) = |t|^k \operatorname{sgn} t$, $k \geq 2$, *and*

$$\mathcal{T}_{\alpha,\beta} f(x,y) = pv\int_{-1}^{1} f(x-t, y-\gamma(t)) e^{-2\pi i|t|^{-\beta}}\frac{dt}{t|t|^\alpha} \qquad (\alpha,\, \beta > 0).$$

Then

(i) $\|\mathcal{T}_{\alpha,\beta} f\|_2 \leq A_{\alpha,\beta}\|f\|_2$ *if and only if* $\beta \geq 3\alpha$;

(ii) $\|\mathcal{T}_{\alpha,\beta} f\|_p \leq B_{\alpha,\beta}\|f\|_p$ *whenever* $\beta > 3\alpha$, *and*

$$1 + \frac{3\alpha\,(\beta+1)}{\beta\,(\beta+1) + (\beta - 3\alpha)} < p < \frac{\beta\,(\beta+1) + (\beta-3\alpha)}{3\alpha\,(\beta+1)} + 1.$$

Here $A_{\alpha,\beta}$ also depends on k, and $B_{\alpha,\beta}$ also depends on p.

1.2. Outline of Proof. In Section 2, we define an appropriate *one parameter family of dilations* $\{\delta_t\}_{t>0}$, and a corresponding distance function ρ, whose *homogeneity* with respect to δ_t is essential in proving the L^2 and L^p-boundedness of $\mathcal{T}_{\alpha,\beta}$.

In Section 3, we prove that $\mathcal{T}_{\alpha,\beta}$ is a bounded operator on L^2 if and only if $\beta \geq 3\alpha$. This is achieved by applying van der Corput's Lemma and its corollary to judiciously subdivided intervals, and the *asymptotics of oscillatory integrals.*

The L^p-boundedness, as stated in the second assertion of Theorem 1, is proven in Section 4. This is accomplished by showing that a certain *analytic family*, $\{\mathcal{T}_z^\epsilon\}$, of truncated operators is bounded on L^2 for an appropriate $\Re z > 0$; and it is bounded on L^p, $1 < p < \infty$, for an appropriate $\Re z < 0$; and that the bound in each case grows at most as fast as a polynomial in $|z|$. The result then follows by *analytic interpolation*.

2. Dilations and Homogeneity.

We define a *one parameter group of dilations* $\{\delta_t\}_{t>0}$, $\delta_t : \mathbf{R}^2 \to \mathbf{R}^2$, by $\delta_t = \operatorname{diag}[t^{1+\beta}, t^{k+\beta}]$, with $A = \operatorname{diag}[1+\beta, k+\beta]$ and $a = \operatorname{trace} A = 2\beta+k+1$, and a corresponding *distance function* ρ defined by: $\rho = \rho(x,y) = t$ such that

$$\left\|\delta_\rho^{-1}(x,y)\right\|^2 = \left(\frac{x}{\rho^{1+\beta}}\right)^2 + \left(\frac{y}{\rho^{k+\beta}}\right)^2 = 1$$

if $(x,y) \neq (0,0)$, and $\rho(0,0) = 0$. Then ρ is *homogeneous* with respect to δ_t: $\rho(\delta_t x) = t\rho(x), t > 0, x \in \mathbf{R}^2$; $\rho(x)$ is continuous and is in $C^\infty(\mathbf{R}^2 - 0)$; $\rho(x+y) \leq C[\rho(x) + \rho(y)]$, for some $C > 0$; and $\mathbf{R}^2$ can be coordinatized by the *polar-like* coordinates $\rho = \rho(x)$ and $u = \delta_\rho^{-1} x$, with $dx = \rho^{a-1}\, d\rho\, (Au, u)\, d\varsigma = \rho^{a-1}\, d\rho\, d\varphi$, where $d\varsigma$ is the linear measure on $\mathbf{S}^1$. For proofs of these assertions and additional properties of δ_t and ρ see [**St**, **Wa**].

3. L^2-Boundedness.

The proof of sufficiency in the first assertion of Theorem 1 is accomplished as an easy consequence of Theorem 2, which we prove next. Our point of departure is the observation that

$$\widehat{(\mathcal{T}_{\alpha,\beta} f)}(x,y) = m_{\alpha,\beta}(x,y)\hat{f}(x,y) \qquad (f \in L^2),$$

where $\hat{}$ denotes the *Fourier transform*, and $m_{\alpha,\beta}(x,y)$ is the *multiplier* given by

$$m_{\alpha,\beta}(x,y) = pv \int_{-1}^{1} e^{-2\pi i[xt+y\gamma(t)+|t|^{-\beta}]} \frac{dt}{t|t|^{\alpha}} \qquad (\alpha, \beta > 0)\,.$$

Thus, the boundedness of $\mathcal{T}_{\alpha,\beta}$ on L^2 is, by the Plancherel Theorem, equivalent to the *uniform boundedness*, in x and y, of the multiplier $m_{\alpha,\beta}$. So we first prove:

Theorem 2. *The multiplier* $m_{\alpha,\beta}(x,y)$ *is uniformly bounded in* $\mathbf{R}^2$ *for* $\beta \geq 3\alpha$. *More precisely:*

$$|m_{\alpha,\beta}(x,y)| \leq \begin{cases} C & \text{if } 0 \leq \rho \leq 1 \\ C\,\rho^{-\frac{\beta-3\alpha}{3}} & \text{if } \rho > 1 \end{cases}, \quad \beta \geq 3\alpha,\ (x,y) \in \mathbf{R}^2.$$

The proof of Theorem 2 depends mainly on the following:

Lemma 3.1. *Suppose that*

(i) g *is real-valued and smooth for all* $t \in [a,b]$, $0 < a < b$ *;*

(ii) $|g^{(k)}(t)| \geq \rho > 0$ *for all* $t \in [a,b]$ *with* $k \geq 2$ *; in addition,* g' *is monotone on* $[a,b]$ *if* $k = 1$ *;*

(iii) $z = \sigma + i\tau$, $\sigma \geq 0$, $\tau \in \mathbf{R}$

(iv) $\alpha \geq 0$.

Then,

$$\left| \int_a^b e^{-2\pi i g(t)}\, \frac{dt}{t^{1+\alpha+z}} \right| \;\leq\; \frac{C(1+|z|)}{a^{1+\alpha+\sigma}}\; \rho^{-\frac{1}{k}}\,.$$

Proof. Let

$$G(t) \;=\; \int_a^t e^{-2\pi i g(s)}\, ds\,.$$

Then, by van der Corput's Lemma (see **[St3]**, Chapter VIII),

$$|G(t)| \;\leq\; C_k\, \rho^{-\frac{1}{k}}\,, \quad t \in [a,b].$$

Integrating by parts, we get

$$\left| \int_a^b e^{-2\pi i g(t)}\, \frac{dt}{t^{1+\alpha+z}} \right| \;=\; \left| \left[\frac{G(t)}{t^{1+\alpha+z}} \right]_{t=a}^{t=b} \;-\; (1+\alpha+z) \int_a^b \frac{G(t)}{t^{2+\alpha+z}}\, dt \right|$$

$$\begin{aligned}
&\leq C\,\rho^{-\frac{1}{k}}\left[\frac{1}{b^{1+\alpha+\sigma}}+\frac{1}{a^{1+\alpha+\sigma}}\right] \\
&\quad + C(1+\alpha+|z|)\,\rho^{-\frac{1}{k}}\int_a^b \frac{dt}{|t^{2+\alpha+z}|} \\
&\leq \frac{2C}{a^{1+\alpha+\sigma}}\,\rho^{-\frac{1}{k}} + \frac{C(1+\alpha+|z|)}{1+\alpha+\sigma}\left[\frac{1}{b^{1+\alpha+\sigma}}+\frac{1}{a^{1+\alpha+\sigma}}\right]\rho^{-\frac{1}{k}} \\
&\leq \frac{C(1+|z|)}{a^{1+\alpha+\sigma}}\,\rho^{-\frac{1}{k}}\ .
\end{aligned}$$

This completes the proof of Lemma 3.1. □

Proof of Theorem 2: We only need look at

$$m^{+}(x,y) = m^{+}_{\alpha,\beta}(x,y) = \int_0^1 e^{-2\pi i[xt+yt^k+t^{-\beta}]}\,\frac{dt}{t^{1+\alpha}}\ ,$$

since the other half can be dealt with similarly.

Since $\rho(0,0)=0$ and $m(0,0)=0$; for $(x,y)\neq(0,0)$ but $x^2+y^2\leq 1$, so that $0<\rho\leq 1$, if we let

$$g(s) = xs + ys^k + s^{-\beta},$$

then

$$g'(s) = x + yk\,s^{k-1} - \beta\,s^{-(\beta+1)},$$

and so there exists a $T>0$ *independent* of x and y such that $g'(s)\leq -\frac{\beta}{2}\,s^{-(\beta+1)}$ for $s\in(0,T]$. Then if we let

$$G(s) = \int_0^s e^{-2\pi i g(t)}\,dt\ ,$$

we get $|\,G(s)\,|\leq Cs^{\beta+1}$, by van der Corput's Lemma. Hence integrating by parts we get,

$$\begin{aligned}
\left|\int_0^T e^{-2\pi i g(s)}\,\frac{ds}{s^{1+\alpha}}\right| &\leq \left|\left[\frac{G(s)}{s^{1+\alpha}}\right]_{s=0}^{s=T}\right| + (1+\alpha)\int_0^T \frac{|G(s)|}{s^{\alpha+2}}\,ds \\
&\leq C\left[\frac{s^{\beta+1}}{s^{\alpha+1}}\right]_{s=0}^{s=T} + C\,(1+\alpha)\int_0^T \frac{s^{\beta+1}}{s^{\alpha+2}}\,ds
\end{aligned}$$

$$= C \left[s^{\beta-\alpha} \right]_{s=0}^{s=T} + C\,(1+\alpha) \int_0^T s^{(\beta-\alpha)-1}\, ds \, .$$

Both of these exist if $\beta > \alpha$. Thus,

$$\left| \int_0^T e^{-2\pi i g(s)} \, \frac{ds}{s^{1+\alpha}} \right| \leq C.$$

For $s \in [T, 1]$,

$$\left| \int_T^1 e^{-2\pi i g(s)} \, \frac{ds}{s^{1+\alpha}} \right| \leq \int_T^1 \frac{ds}{s^{1+\alpha}} = \frac{1}{\alpha} \left[\frac{1}{T^\alpha} - 1 \right] \leq C.$$

Thus $m^+(x,y)$ is uniformly bounbed when $0 \leq \rho \leq 1$. We now turn to the case when $\rho > 1$. With $\rho = \rho(x,y)$ as defined above, the change of variable $t = s\,\rho^{-1}$ leads us to

$$m^+\,(x,y) = \rho^\alpha \int_0^\rho e^{-2\pi i [\frac{x}{\rho} s + \frac{y}{\rho^k} s^k + \rho^\beta s^{-\beta}]} \, \frac{ds}{s^{1+\alpha}} \, .$$

Thus, to prove the theorem, we need only show that

$$\left| \int_0^\rho e^{-2\pi i [\frac{x}{\rho} s + \frac{y}{\rho^k} s^k + \rho^\beta s^{-\beta}]} \, \frac{ds}{s^{1+\alpha}} \right| \leq C\, \rho^{-\frac{\beta}{3}}$$

for all $(x,y) \in \mathbf{R}^2$. To this end, we show that the above integral is *uniformly bounded* in each of the four quadrants of $\mathbf{R}^2$.

Note: For notational convenience, we shall write x *(resp.* y*) if* x *(resp.* y*) is positive, and* $-x$ *(resp.* $-y$*) if* x *(resp.* y*) is negative.*

Case I: $x < 0,\ y < 0$.

Let

$$g(s) = -\,\frac{x}{\rho}\, s \;-\; \frac{y}{\rho^k}\, s^k \;+\; \rho^\beta s^{-\beta}.$$

Then,

$$g'(s) = -\,\frac{x}{\rho} \;-\; \frac{y}{\rho^k}\, k\, s^{k-1} \;-\; \rho^\beta\, \beta\, s^{-(\beta+1)},$$

and

$$g''(s) = -\frac{y}{\rho^k}\, k\,(k-1)\, s^{k-2} \;+\; \rho^\beta\, \beta\,(\beta+1)\; s^{-(\beta+2)}.$$

Let

$$G(s) \;=\; \int_0^s e^{-2\pi i g(t)}\, dt\;.$$

Since near 0 we have $g'(s) \le -\rho^\beta\ \beta\ s^{-(\beta+1)}$, van der Corput's Lemma gives $|G(s)| \;\le\; C\,\rho^{-\beta}\, s^{\beta+1}$. Hence, integrating by parts as before, we get

$$\left| \int_0^1 e^{-2\pi i g(s)}\, \frac{ds}{s^{1+\alpha}} \right| \;\le\; C\,\rho^{-\beta} \qquad \text{for } \beta > \alpha.$$

To tackle the integral from 1 to ρ, we need to consider the following two cases:

$$\text{(i)}\ \frac{x}{\rho} \;\ge\; \frac{\rho^\beta}{2}\,; \qquad \text{(ii)}\ \frac{x}{\rho} \;\le\; \frac{\rho^\beta}{2}\,.$$

(i) $\frac{x}{\rho} \ge \frac{\rho^\beta}{2}$

This implies that $-\frac{x}{\rho} \;\le\; -\frac{\rho^\beta}{2}$. Thus $g'(s) \;\le\; -\frac{x}{\rho} \;\le\; -\frac{\rho^\beta}{2}$ on $[1,\rho]$, together with Lemma 3.1, yields

$$\left| \int_1^\rho e^{-2\pi i g(s)}\, \frac{ds}{s^{1+\alpha}} \right| \;\le\; C\,\rho^{-\beta}\,.$$

(ii) $\frac{x}{\rho} \le \frac{\rho^\beta}{2}$

By the definition of ρ , this implies that $-\frac{y}{\rho^k} \;\le\; -\frac{\rho^\beta}{2}$. Then,

$$\begin{aligned} g'(s) &= -\frac{x}{\rho} \;-\; \frac{y}{\rho^k}\, k\, s^{k-1} \;-\; \rho^\beta\, \beta\, s^{-(\beta+1)} \\ &\le -\frac{x}{\rho} \;-\; k\,\frac{\rho^\beta}{2} \;-\; \rho^\beta\, \beta\, s^{-(\beta+1)} \\ &\le -\frac{k}{2}\,\rho^\beta \quad \text{ for } \quad s \in [1,\rho]. \end{aligned}$$

This, along with Lemma 3.1, gives

$$\left| \int_1^\rho e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \le C\,\rho^{-\beta}\,.$$

Hence, $|\,m^+(x,y)\,| \le C\,\rho^{-(\beta-\alpha)}$ whenever $x, y < 0$ and $\beta > \alpha$. This completes *Case* I.

Case II: $x \ge 0,\ y \ge 0$.

In this case,

$$\begin{aligned} g(s) &= +\frac{x}{\rho}\,s + \frac{y}{\rho^k}\,s^k + \rho^\beta s^{-\beta}\,, \\ g'(s) &= +\frac{x}{\rho} + \frac{y}{\rho^k}\,k\,s^{k-1} - \rho^\beta\,\beta\,s^{-(\beta+1)}\,, \\ g''(s) &= +\frac{y}{\rho^k}\,k\,(k-1)\,s^{k-2} + \rho^\beta\,\beta\,(\beta+1)\,s^{-(\beta+2)}\,. \end{aligned}$$

In the vicinity of 0, we have $g''(s) \ge C\rho^\beta s^{-(\beta+2)}$; and so

$$\left| \int_0^b e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \le C\rho^{-\frac{\beta}{2}} \quad \text{for } \beta > 2\alpha\,,$$

using van der Corput's lemma, where b can be chosen later.

Away from 0, we have the following two cases:

$$\text{(i)}\ \frac{y}{\rho^k} \le \frac{\rho^\beta}{2}\,; \qquad \text{(ii)}\ \frac{y}{\rho^k} \ge \frac{\rho^\beta}{2}\,.$$

(i) $\frac{y}{\rho^k} \le \frac{\rho^\beta}{2}$

This, and the definition of ρ imply that $\frac{x}{\rho} \ge \frac{\rho^\beta}{2}$.

Then,

$$\begin{aligned} g'(s) &\ge \frac{\rho^\beta}{2} - \beta\,\rho^\beta\,s^{-(\beta+1)} \\ &\ge \frac{\rho^\beta}{2} - \frac{\rho^\beta}{4} \\ &\ge \frac{\rho^\beta}{4} \quad \text{whenever } s \ge (4\beta)^{\frac{1}{\beta+1}}\,. \end{aligned}$$

Note that g' is increasing since $g'' > 0$. Choosing $b = (4\beta)^{\frac{1}{\beta+1}}$, and using Lemma 3.1, we get

$$\left| \int_b^\rho e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\beta} .$$

(ii) $\frac{y}{\rho^k} \geq \frac{\rho^\beta}{2}$

Here, $g''(s) \geq C \frac{\rho^\beta}{2}$. Choosing $b = 1$, and using Lemma 3.1, we get

$$\left| \int_1^\rho e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\frac{\beta}{2}} .$$

Thus, $| m^+(x, y) | \leq C \rho^{-(\frac{\beta}{2} - \alpha)}$ whenever $x, y \geq 0$ and $\beta > 2\alpha$.

This completes *Case* II.

Case III : $x < 0$, $y \geq 0$.

Here,

$$g(s) = - \frac{x}{\rho} s + \frac{y}{\rho^k} s^k + \rho^\beta s^{-\beta} ,$$
$$g'(s) = - \frac{x}{\rho} + \frac{y}{\rho^k} k s^{k-1} - \rho^\beta \beta s^{-(\beta+1)} ,$$
$$g''(s) = + \frac{y}{\rho^k} k (k-1) s^{k-2} + \rho^\beta \beta (\beta + 1) s^{-(\beta+2)}.$$

Close to 0 , $g''(s) \geq \beta (\beta + 1) \rho^\beta s^{-(\beta+2)}$; and so

$$\left| \int_0^b e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C\rho^{-\frac{\beta}{2}} \qquad \text{for } \beta > 2\alpha ,$$

using van der Corput's Lemma , where b can be chosen later.

Farther from 0 , the following two cases need to be considered:

$$\text{(i) } \frac{y}{\rho^k} \geq \frac{\rho^{\frac{\beta}{3}}}{8k} ; \qquad \text{(ii) } \frac{y}{\rho^k} \leq \frac{\rho^{\frac{\beta}{3}}}{8k} .$$

(i) $\frac{y}{\rho^k} \geq \frac{\rho^{\frac{\beta}{3}}}{8k}$

Here,

$$g''(s) \geq \frac{y}{\rho^k} k (k-1) s^{k-2}$$

$$\begin{aligned}
&\geq \frac{(k-1)}{8}\,\rho^{\frac{\beta}{3}}\,s^{k-2}\\
&\geq C\rho^{\frac{\beta}{3}}\,\rho^{\frac{\beta}{3}}\\
&= C\rho^{\frac{2\beta}{3}} \qquad \text{whenever } s\in I=\left[\,\rho^{\frac{\beta}{3(k-2)}},\ \rho\,\right].
\end{aligned}$$

Choosing $b=\rho^{\frac{\beta}{3(k-2)}}$ in the above, and using Lemma 3.1 we get

$$\left|\int_I e^{-2\pi i g(s)}\,\frac{ds}{s^{1+\alpha}}\right| \leq C\,\rho^{-\frac{\beta}{3(k-2)}}\,\rho^{-\frac{\beta}{3}} \leq C\,\rho^{-\frac{\beta}{3}}\ .$$

(ii) $\frac{y}{\rho^k} \leq \frac{\rho^{\frac{\beta}{3}}}{8k}$

This implies that $\frac{x}{\rho} \geq \frac{\rho^\beta}{2}$, and so

$$\begin{aligned}
g'(s) &\leq -\frac{x}{\rho} + \frac{y}{\rho^k}\,k\,s^{(k-1)}\\
&\leq -\frac{\rho^\beta}{2} + \frac{\rho^{\frac{\beta}{3}}}{8}\,\rho^{\frac{2\beta}{3}}\\
&\leq -\frac{\rho^\beta}{4} \qquad \text{whenever } s\in\left[1,\ \rho^{\frac{2\beta}{3(k-1)}}\,\right].
\end{aligned}$$

If $\beta \geq \frac{3(k-1)}{2}$, we are done using Lemma 3.1. If not, we need to subdivide further:

$$\text{(ii a)}\ \frac{y}{\rho^k} \leq \frac{\rho^{\beta-(k-1)}}{8k}\,; \qquad \text{(ii b)}\ \frac{\rho^{\beta-(k-1)}}{8k} \leq \frac{y}{\rho^k} \leq \frac{\rho^{\frac{\beta}{3}}}{8k}\,.$$

(ii a) $\frac{y}{\rho^k} \leq \frac{\rho^{\beta-(k-1)}}{8k}$; $\quad 0<\beta<\frac{3(k-1)}{2}$, $\quad s\in[\,1,\ \rho\,]$.

In this case,

$$\begin{aligned}
g'(s) &\leq -\frac{\rho^\beta}{2} + \frac{\rho^{\beta-(k-1)}}{8}\,s^{(k-1)}\\
&\leq -\frac{\rho^\beta}{2} + \frac{\rho^\beta}{8}\\
&\leq -\frac{\rho^\beta}{4}\,.
\end{aligned}$$

Thus, Lemma 3.1 gives

$$\left|\int_1^\rho e^{-2\pi i g(s)}\,\frac{ds}{s^{1+\alpha}}\right| \leq C\rho^{-\beta}\ .$$

(ii b) $\dfrac{\rho^{\beta-(k-1)}}{8k} \le \dfrac{y}{\rho^k} \le \dfrac{\rho^{\frac{\beta}{3}}}{8k}$; $\quad 0<\beta< \dfrac{3(k-1)}{2}$, $\quad s\in[1,\rho]$.

There is a real number $j>1$ such that $0<j-1\le \frac{2\beta}{3} \le j<k$. Then $\frac{\beta}{3} = \beta - \frac{2\beta}{3} \le \beta - (j-1) = \beta - [k - \{k - (j-1)\}]$.

Now, let

$$S_N = \sum_{n=0}^{N} \left(\frac{k-1}{k}\right)^n .$$

Then, $S_{m+1} = 1+\left(\frac{k-1}{k}\right) S_m$; $m\ge 0$. We choose N so that

$$S_N = \frac{1-\left(\frac{k-1}{k}\right)^{N+1}}{1-\left(\frac{k-1}{k}\right)} = k\left[1-\left(\frac{k-1}{k}\right)^{N+1}\right] \ge k - (j-1)\,;$$

i.e., $(j-1) \ge k\left(\frac{k-1}{k}\right)^{N+1}$. This can be done since $\frac{k-1}{k} < 1$.

We now look at:

$$\frac{\rho^{\beta-[k-S_m]}}{8k} \le \frac{y}{\rho^k} \le \frac{\rho^{\beta-[k-S_{m+1}]}}{8k}; \quad m = 0,1,2,\dots,N-1 .$$

For $s\in I = \left[1,\, \rho^{1-\frac{S_m}{k}}\right]$, we have

$$\begin{aligned} g'(s) &= -\frac{x}{\rho} + \frac{y}{\rho^k}\, k\, s^{k-1} - \rho^\beta\, \beta\, s^{-(\beta+1)} \\ &\le -\frac{\rho^\beta}{2} + \frac{1}{8}\,\rho^{\beta-[k-S_{m+1}]}\,\rho^{k-1-\left(\frac{k-1}{k}\right)S_m} \\ &= -\frac{\rho^\beta}{2} + \frac{\rho^\beta}{8} \\ &\le -\frac{\rho^\beta}{4} . \end{aligned}$$

Hence,

$$\left| \int_I e^{-2\pi i g(s)}\, \frac{ds}{s^{1+\alpha}} \right| \le C\,\rho^{-\beta} \qquad \text{using Lemma 3.1.}$$

Next, for $s\in I = \left[\rho^{1-\frac{S_m}{k}},\, \rho\right]$, we have

$$\begin{aligned} g''(s) &\ge k\,(k-1)\,\frac{y}{\rho^k}\, s^{k-2} \\ &\ge \frac{(k-1)}{8}\,\rho^{\beta-[k-S_m]}\, s^{k-2} \end{aligned}$$

$$\geq \frac{(k-1)}{8} \rho^{\beta-[k-S_m]} \rho^{k-2-\left(\frac{k-2}{k}\right) S_m}$$
$$= C\rho^{\beta-2+\frac{2}{k} S_m}.$$

This, along with Lemma 3.1, gives

$$\left| \int_I e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C\rho^{-\frac{\beta}{2}+1-\frac{S_m}{k}} \rho^{-1+\frac{S_m}{k}} = C\rho^{-\frac{\beta}{2}}.$$

Thus, $|m^+(x,y)|$ is uniformly bounded when $x, y < 0$ and $\beta \geq 3\alpha$.
This completes *Case* III.

Case IV: $x > 0$, $y < 0$.

Here,

$$g(s) = +\frac{x}{\rho} s - \frac{y}{\rho^k} s^k + \rho^\beta s^{-\beta},$$
$$g'(s) = +\frac{x}{\rho} - \frac{y}{\rho^k} k s^{k-1} - \rho^\beta \beta s^{-(\beta+1)},$$
$$g''(s) = -\frac{y}{\rho^k} k(k-1) s^{k-2} + \rho^\beta \beta(\beta+1) s^{-(\beta+2)},$$
$$g'''(s) = -\frac{y}{\rho^k} k(k-1)(k-2) s^{k-3} - \rho^\beta \beta(\beta+1)(\beta+2) s^{-(\beta+3)}.$$

We need to split as follows:

$$\text{(i) } \frac{y}{\rho^k} \geq C_1 \rho^\beta; \qquad \text{(ii) } \frac{y}{\rho^k} \leq C_1 \rho^\beta$$

where $0 < C_1 < 1$ is to be chosen appropriately at a later stage.

Note that, in the vicinage of 0,

$$g'(s) \leq \frac{x}{\rho} - \rho^\beta \beta s^{-(\beta+1)}$$
$$\leq \rho^\beta - \rho^\beta \beta s^{-(\beta+1)}$$
$$\leq -\frac{\beta}{2} \rho^\beta s^{-(\beta+1)}$$

whenever $s \in I = \left(0, \left(\frac{\beta}{2}\right)^{\frac{1}{\beta+1}}\right]$.

Therefore,

$$\left| \int_I e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\beta} \quad \text{for } \beta > \alpha$$

using van der Corput's Lemma.

(i) $\frac{y}{\rho^k} \geq C_1 \rho^\beta$

Note that, $|g'''(s)| \geq \frac{y}{\rho^k} k(k-1)(k-2) s^{k-3} \geq C \rho^\beta$ whenever $s \in I = \left[\left(\frac{\beta}{2}\right)^{\frac{1}{\beta+1}}, \rho \right]$. Thus,

$$\left| \int_I e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\frac{\beta}{3}}$$

using Lemma 3.1. This completes (i).

(ii) $\frac{y}{\rho^k} \leq C_1 \rho^\beta$

This needs to be split further:

$$\text{(ii a)}\ \frac{\rho^{\frac{2\beta}{3}}}{8k} \leq \frac{y}{\rho^k} \leq C_1 \rho^\beta ; \qquad \text{(ii b)}\ \frac{y}{\rho^k} \leq \frac{\rho^{\frac{2\beta}{3}}}{8k} .$$

(ii a) $\frac{\rho^{\frac{2\beta}{3}}}{8k} \leq \frac{y}{\rho^k} \leq C_1 \rho^\beta$

At $s_0 = \left[\frac{\beta(\beta+1)}{k(k-1)y}\right]^{\frac{1}{\beta+k}} \rho$, we have $g''(s_0) = 0$. Since $g''' < 0$, g' has a maximum at s_0.

Now,

$$\begin{aligned} g'(s_0) &= \frac{x}{\rho} - \frac{y}{\rho^k} k \left[\frac{\beta(\beta+1)}{k(k-1)y}\right]^{\frac{k-1}{\beta+k}} \rho^{k-1} - \rho^{-1} \beta \left[\frac{\beta(\beta+1)}{k(k-1)y}\right]^{-\frac{\beta+1}{\beta+k}} \\ &= \frac{x}{\rho} - C_{\beta,k} \frac{y^{\frac{\beta+1}{\beta+k}}}{\rho} , \end{aligned}$$

where $C_{\beta,k} = \left[\frac{\beta(\beta+k)}{(k-1)}\right] \left[\frac{k(k-1)}{\beta(\beta+1)}\right]^{\frac{\beta+1}{\beta+k}}$.

Now, choose C_1 so that $g'(s_0) \geq \frac{1}{2} \frac{x}{\rho}$. Next, choose $a < 1$ and $b > 1$ such that in the neighborhood $I_{a,b} = [\, a s_0 , b s_0 \,]$ of s_0, we have

$$g'(s) \geq \frac{x}{4\rho} \geq \frac{1}{4}(1 - C_1^2)^{\frac{1}{2}} \rho^\beta .$$

Then,

$$\left| \int_{I_{a,b}} e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\beta}$$

using van der Corput's Lemma on $[as_0, s_0]$ and $[s_0, bs_0]$.

Since $g'''(s) < 0$, $g''(s)$ is decreasing; and so on $I = \left[\left(\frac{\beta}{2}\right)^{\frac{1}{\beta+1}}, as_0 \right]$, we have

$$\begin{aligned} g''(s) \ \geq \ g''(as_0) &= -\frac{y}{\rho^k}\, k\,(k-1)\,(as_0)^{k-2} \ + \ \rho^\beta\, \beta\,(\beta+1)(as_0)^{-(\beta+2)} \\ &= C_{a,\beta,k}\, \frac{y^{\frac{\beta+2}{\beta+k}}}{\rho^2} \ \geq \ C'_{a,\beta,k}\, \rho^{\frac{2\beta}{3}}, \end{aligned}$$

as a simple calculation shows.

Thus on I,

$$\left| \int_I e^{-2\pi i g(s)}\, \frac{ds}{s^{1+\alpha}} \right| \ \leq \ C\, \rho^{-\frac{\beta}{3}}$$

by Lemma 3.1.

Now, on $I = [bs_0, \rho]$,

$$g''(s) \ \leq \ g''(bs_0) \ \leq \ -C'_{b,\beta,k}\, \rho^{\frac{2\beta}{3}},$$

as before. Hence, once again,

$$\left| \int_I e^{-2\pi i g(s)}\, \frac{ds}{s^{1+\alpha}} \right| \ \leq \ C\, \rho^{-\frac{\beta}{3}}$$

using Lemma 3.1. This completes (ii a).

(ii b) $\dfrac{y}{\rho^k} \leq \dfrac{\rho^{\frac{2\beta}{3}}}{8k}$

Have,

$$\begin{aligned} g'(s) &= +\,\frac{x}{\rho} \ - \ \frac{y}{\rho^k}\, k\, s^{k-1} \ - \ \rho^\beta\, \beta\, s^{-(\beta+1)} \\ &\geq \ \frac{\rho^\beta}{2} \ - \ \frac{\rho^{\frac{2\beta}{3}}}{8}\, s^{(k-1)} \ - \rho^\beta\, \beta\, s^{-(\beta+1)} \\ &\geq \ \frac{\rho^\beta}{4} \quad \text{whenever } \ s \in I \ = \ \left[(8\beta)^{\frac{1}{\beta+1}}, \ \rho^{\frac{\beta}{3(k-1)}} \right]. \end{aligned}$$

If $\beta \geq 3(k-1)$, we are done using Lemma 3.1. If not, we need to subdivide further:

$$\text{(ii b A)}\ \frac{y}{\rho^k} \ \leq \ \frac{\rho^{\beta-(k-1)}}{8k}\,; \qquad \text{(ii b B)}\ \frac{\rho^{\beta-(k-1)}}{8k} \ \leq \ \frac{y}{\rho^k} \ \leq \ \frac{\rho^{\frac{2\beta}{3}}}{8k}$$

with $0 < \beta < 3(k-1)$, and $s \in I = [1, \rho]$.

(ii b A) $\dfrac{y}{\rho^k} \le \dfrac{\rho^{\beta-(k-1)}}{8k}$; $0 < \beta < 3(k-1)$, $s \in I = [\,1,\ \rho\,]$
Have,

$$\begin{aligned} g'(s) &= + \frac{x}{\rho} - \frac{y}{\rho^k}\, k\, s^{k-1} - \rho^\beta\, \beta\, s^{-(\beta+1)} \\ &\ge \frac{\rho^\beta}{2} - \frac{\rho^{\beta-(k-1)}}{8}\, \rho^{k-1} - \frac{\rho^\beta}{8} \\ &\ge \frac{\rho^\beta}{4} \quad \text{whenever } s \in I = \left[(8\beta)^{\frac{1}{\beta+1}} ,\ \rho \right] . \end{aligned}$$

Using Lemma 3.1 once again, we are done.

(ii b B) $\dfrac{\rho^{\beta-(k-1)}}{8k} \le \dfrac{y}{\rho^k} \le \dfrac{\rho^{\frac{2\beta}{3}}}{8k}$; $0 < \beta < 3(k-1)$, $s \in I = [\,1,\ \rho\,]$
We proceed here as in *Case* III (ii b) :
There is a real number $j > 1$ such that $0 < j - 1 \le \frac{\beta}{3} \le j < k$.

Then $\frac{2\beta}{3} = \beta - \frac{\beta}{3} \le \beta - (j-1) = \beta - [\, k - \{k - (j-1)\}]$.

With N , S_N , and S_m as in *Case* III (ii b) , for

$$\frac{\rho^{\beta-[k-S_m]}}{8k} \le \frac{y}{\rho^k} \le \frac{\rho^{\beta-[k-S_{m+1}]}}{8k} , \quad m = 0, 1, 2, \ldots, N-1 ,$$

and $s \in I = \left[(8\beta)^{\frac{1}{\beta+1}} ,\ \rho^{1-\frac{S_m}{k}} \right]$, we note that

$$\begin{aligned} g'(s) &\ge \frac{\rho^\beta}{2} - \frac{1}{8}\, \rho^{\beta-[k-S_{m+1}]}\, \rho^{k-1-\left(\frac{k-1}{k}\right) S_m} - \frac{\rho^\beta}{8} \\ &= \frac{\rho^\beta}{2} - \frac{\rho^\beta}{8} - \frac{\rho^\beta}{8} \ge \frac{\rho^\beta}{4} . \end{aligned}$$

Hence Lemma 3.1 implies that

$$\left| \int_I e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \le C\, \rho^{-\beta} .$$

For $s \in I = \left[\rho^{1-\frac{S_m}{k}} ,\ \rho \right]$, we use the fact that

$$\begin{aligned} |\, g'''(s)\, | &\ge k\,(k-1)\,(k-2)\, \frac{y}{\rho^k}\, s^{k-3} \\ &\ge C_k\, \rho^{\beta-[k-S_m]}\, \rho^{k-3-\frac{k-3}{k} S_m} \\ &\ge C_k\, \rho^{\beta-3+\frac{3}{k} S_m} . \end{aligned}$$

Lemma 3.1 now yields,

$$\left| \int_I e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C\,\rho^{-\frac{\beta}{3}+1-\frac{S_m}{k}}\,\rho^{-1+\frac{S_m}{k}} = C\,\rho^{-\frac{\beta}{3}}\,.$$

On $\left[\left(\frac{\beta}{2}\right)^{\frac{1}{\beta+1}}, (8\beta)^{\frac{1}{\beta+1}}\right]$ we use the fact that $|g'''(s)| \geq C'\,\rho^{\beta}$, and Lemma 3.1. This completes *Case* IV, and shows that $|m(x,y)| \leq C\,\rho^{-\frac{\beta}{3}+\alpha}$; i.e., the *multiplier* $m(x,y)$ is uniformly bounded in $\mathbf{R}^2$ whenever $\beta \geq 3\alpha$. Thus the proof of Theorem 2 is complete. □

Plancherel's Theorem now shows that

$$\|\mathcal{T}_{\alpha,\beta} f\|_2 = \|\widehat{\mathcal{T}_{\alpha,\beta} f}\|_2 \leq A_{\alpha,\beta}\,\|\hat{f}\|_2 = A_{\alpha,\beta}\|f\|_2 \quad \text{for } \beta \geq 3\alpha.$$

Theorem 3. *Along the curve* $y = -C_{\beta,k}\,x^{\frac{\beta+k}{\beta+1}}$ $(x>0)$,

$$\left| m\left(x, -C_{\beta,k}\,x^{\frac{\beta+k}{\beta+1}}\right)\right| \sim C\,\rho^{-\left[\frac{\beta}{3}-\alpha\right]} \quad \textit{as } \rho \longrightarrow \infty.$$

Proof. As before, it suffices to prove the above estimate for

$$m^+(x,y) = \int_0^1 e^{-2\pi i\left[xs+ys^k+s^{-\beta}\right]}\,\frac{ds}{s^{1+\alpha}}\,.$$

For (x,y) on the above curve, write $x = C_{\beta,k}\,\tau^{\beta+1}$ and $y = -\tau^{\beta+k}$ $(\tau > 0)$. The change of variable $s \mapsto s\tau^{-1}$ yields

$$m^+\left(C_{\beta,k}\,\tau^{\beta+1}, -\tau^{\beta+k}\right) = \tau^{\alpha} \int_0^{\tau} e^{-2\pi i \tau^{\beta} g(s)}\,\frac{ds}{s^{1+\alpha}}\,,$$

with $g(s) = [C_{\beta,k}\,s - s^k + s^{-\beta}]$. We split the above integral as

$$\int_0^{\tau} = \int_0^a + \int_a^b + \int_b^{\tau}$$

where $[a,b]$ is a small fixed interval centered at $s_0 = \left[\frac{\beta(\beta+1)}{k(k-1)}\right]^{\frac{1}{\beta+k}}$. Then since $g'(s_0) = g''(s_0) = 0$, but $g'''(s_0) \neq 0$, we have

$$\tau^{\alpha}\left|\int_a^b e^{-2\pi i\tau^{\beta} g(s)}\,\frac{ds}{s^{1+\alpha}}\right| \leq C\,\tau^{\left(\alpha-\frac{\beta}{3}\right)} + O\left(\tau^{\left(\alpha-\frac{2\beta}{3}\right)}\right) \quad \text{as } \tau \longrightarrow \infty,$$

by a standard result on integral asymptotics; see **[St3]**, Chapter VIII. Next, on $I_1 = \left(0, \left[\frac{\beta}{2C_{\beta,k}}\right]^{\frac{1}{\beta+1}}\right]$, we have

$$g'(s) \leq -\frac{\beta}{2}\, s^{-(\beta+1)} \leq -C_{\beta,k}\,.$$

Hence, by van der Corput's Lemma we get

$$\tau^{\alpha}\left|\int_{I_1} e^{-2\pi i \tau^{\beta} g(s)}\,\frac{ds}{s^{1+\alpha}}\right| \leq C\,\tau^{-(\beta-\alpha)}\,.$$

Since $g''' < 0$, g'' is decreasing on $I_2 = \left[\left[\frac{\beta}{2C_{\beta,k}}\right]^{\frac{1}{\beta+1}}, a\right]$. Thus

$$g''(s) \geq g''(a) = \left[-k(k-1)\,a^{k-2} + \beta(\beta+1)\,a^{-(\beta+2)}\right] = C > 0,$$

since $0 < a < s_0$ and $g''(s_0) = 0$. Hence,

$$\tau^{\alpha}\left|\int_{I_2} e^{-2\pi i \tau^{\beta} g(s)}\,\frac{ds}{s^{1+\alpha}}\right| \leq C\,\tau^{-\left[\frac{\beta}{2}-\alpha\right]}\,.$$

Since $g''' < 0$, as seen before, g'' is decreasing on $I_3 = [b, \tau]$. Then

$$g''(s) \leq g''(b) = \left[-k(k-1)\,b^{k-2} + \beta(\beta+1)\,b^{-(\beta+2)}\right] = -C < 0,$$

since $b > s_0$ and $g''(s_0) = 0$. Hence, by van der Corput's Lemma,

$$\tau^{\alpha}\left|\int_{I_3} e^{-2\pi i \tau^{\beta} g(s)}\,\frac{ds}{s^{1+\alpha}}\right| \leq C\,\tau^{-\left[\frac{\beta}{2}-\alpha\right]}\,.$$

Thus on $(0, a] \cup [b, \tau]$, $m^+(x, y)$ decays *faster* than required. This shows that

$$\left|m\left(C_{\beta,k}\,\tau^{\beta+1}, -\tau^{\beta+k}\right)\right| \sim C\,\tau^{-\left[\frac{\beta}{3}-\alpha\right]} \quad \text{as } \tau \longrightarrow \infty;$$

that is,

$$\left|m\left(x, -C_{\beta,k}\,x^{\frac{\beta+k}{\beta+1}}\right)\right| \sim C\,\rho^{-\left[\frac{\beta}{3}-\alpha\right]} \quad \text{as } \rho \longrightarrow \infty.$$

This completes the proof of Theorem 3. □

This shows that on the curve $y = -C_{\beta,k}\,x^{\frac{\beta+k}{\beta+1}}$ $(x > 0)$, the multiplier $m(x, y)$ becomes unbounded if $\beta < 3\alpha$; hence the bound $\beta \geq 3\alpha$ on $m(x, y)$ is *sharp*, and the first assertion of Theorem 1 is proved.

4. L^p-Boundedness.

To prove the second assertion of Theorem 1, we introduce an *analytic family* of truncated operators defined by

$$\widehat{(\mathcal{T}_z^\epsilon f)}(x,y) = \rho^z(x,y)\, m_z^\epsilon(x,y)\, \hat{f}(x,y) \qquad (f \in \mathcal{S}),$$

where

$$m_z^\epsilon(x,y) = \int\limits_{\epsilon \le |t| \le 1} e^{-2\pi i[xt + y\gamma(t) + |t|^{-\beta}]}\, |t|^{-z} \frac{dt}{t|t|^\alpha}\,;\ \alpha > 0,\ \beta \ge 3\alpha, \text{ and } \epsilon > 0.$$

We note at the outset that $\mathcal{T}_0^0 = \mathcal{T}_{\alpha,\beta}$ is bounded on L^2. We need to prove that

$$\|\mathcal{T}_0^0 f\|_p \le C\|f\|_p \qquad (f \in L^p),$$

where p is as in the statement of Theorem 1.

Lemma 4.1. *Let* $z = \sigma + i\tau$; $0 \le \sigma \le \frac{1}{2}\left[\frac{\beta}{3} - \alpha\right]$, $\tau \in \mathbf{R}$. *Then for simple* f

$$\|\mathcal{T}_z^\epsilon f\|_2 \le C(1 + |z|)\, \|f\|_2 .$$

Proof. It suffices to show that for each z, $|m_z^\epsilon(x,y)|$ is uniformly bounded for $(x,y) \in \mathbf{R}^2$. The proof of this fact is very similar to that of Theorem 2 of Section 3, and shows that

$$|m_z^\epsilon(x,y)| \le \begin{cases} C(1+|z|) & \text{if } 0 \le \rho \le 1 \\ C(1+|z|)\, \rho^{-\frac{\beta}{3}+(\alpha+\sigma)} & \text{if } \rho > 1 \end{cases} \quad \text{for all } (x,y) \in \mathbf{R}^2.$$

Then for $\rho > 1$,

$$\begin{aligned} |\rho^z\, m_z^\epsilon(x,y)| &\le C(1+|z|)\, \rho^\sigma \rho^{-\frac{\beta}{3}+(\alpha+\sigma)} \\ &= C(1+|z|)\, \rho^{-(\frac{\beta}{3}-\alpha)+2\sigma} . \end{aligned}$$

For each z, this is uniformly bounded whenever $0 \le \sigma \le \frac{1}{2}\left[\frac{\beta}{3} - \alpha\right]$. The result now follows from the definition of $\mathcal{T}_z^\epsilon$ and the Plancherel theorem. This completes the proof of Lemma 4.1. $\square$

To prove the L^p-boundedness of T_z^ϵ, we need the following:

Lemma 4.2. *For* $-a < \Re z < 0$,

$$\rho^z(x,y) = \hat{h}_z(x,y)$$

where

(i) $h_z(x,y)$ *is a locally integrable function;*

(ii) $h_z \in C^\infty(\mathbf{R}^2 - 0)$*;*

(iii) $h_z(\delta_\lambda(x,y)) = \lambda^{-a-z} h_z(x,y)$, $\lambda > 0$, $(x,y) \neq (0,0)$ *;*

(iv) *each derivative of* $h_z(x,y)$ *is bounded by a polynomial in* $|z|$ *if* $\rho(x,y) \geq 1$.

Here $a = (2\beta + k + 1) =$ *trace* A, *and the Fourier transform is to be taken in the sense of distributions.*

Proof. See [**St**, **Wa**]. □

Remark 4.3. If the line joining x and $x - w$ avoids the origin, and $\frac{|w|}{|x|}$ is sufficiently small, then

$$\begin{aligned}
|h_z(x-w) - h_z(x)| &= \left| \int_0^1 \frac{d}{dt} h_z(x - tw)\, dt \right| \\
&= \left| - \int_0^1 \nabla h_z(x - tw) \cdot w\, dt \right| \\
&\leq |w| \int_0^1 |\nabla h_z(x - tw)|\, dt \\
&\leq C(z)\,|w| \qquad (4.3\text{-}1)
\end{aligned}$$

since the derivatives of h_z are bounded by $C(z)$, by Lemma 4.2. This observation, and the homogeneity of h_z with $\lambda = \rho(x)$ and $\|x\|$ sufficiently large, then imply that,

$$\begin{aligned}
&|h_z(x-w) - h_z(x)| \\
&= \left| h_z\left(\delta_{\rho(x)}\left(\delta^{-1}_{\rho(x)} x - \delta^{-1}_{\rho(x)} w\right)\right) - h_z\left(\delta_{\rho(x)}\left(\delta^{-1}_{\rho(x)} x\right)\right) \right| \\
&\leq C(z) \frac{\left|\delta^{-1}_{\rho(x)}\right| |w|}{\rho(x)^{(2\beta+k+1)+\sigma}} \qquad \text{by } (4.3-1) \\
&\leq C(z) \frac{|w|}{\rho(x)^{(2\beta+k+1)+\sigma+\beta+1}}\,. \qquad (4.3\text{-}2)
\end{aligned}$$

Lemma 4.4. *Suppose that*

(i) $\mathcal{T}_z^\epsilon f$ *is defined by*

$$\widehat{(\mathcal{T}_z^\epsilon f)}(x,y) = \rho^z(x,y)\, m_z^\epsilon(x,y)\, \hat{f}(x,y), \quad f \in \mathcal{S};$$

(ii) $z = \sigma + i\tau\,;\ -\alpha < \sigma \le -\alpha \left[\dfrac{\beta+1}{\beta+2}\right] < 0\,,\ \tau \in \mathbf{R}.$

Then

$$\|\mathcal{T}_z^\epsilon f\|_p \le C(z)\, \|f\|_p \qquad (1 < p < \infty),$$

where, for fixed α *and* β, $C(z)$ *grows at most as fast as a polynomial in* $|z|$.

Proof. By Lemma 4.2, for $f \in \mathcal{S}$, we see that

$$(\mathcal{T}_z^\epsilon f)\,(x) = (K_z * f)\,(x)\,, \tag{4.4-1}$$

where

$$K_z(x) = \int_{\epsilon \le |t| \le 1} h_z(x - \Gamma(t))\, |t|^{-z}\, e^{-2\pi i |t|^{-\beta}} \frac{dt}{t|t|^\alpha}$$

with $x \in \mathbf{R}^2$, and $\Gamma(t) = [t, \gamma(t)] \in \mathbf{R}^2$. It follows that (4.4-1) holds when f is *simple*. Our aim now is to show that, for $x, y \in \mathbf{R}^2$,

$$\int_{\rho(x) > C\rho(y)} |\, K_z(x-y) - K_z(x)\,|\, dx \le C_1(z)\,, \tag{4.4-2}$$

where $C_1(z)$ has at most polynomial growth in $|z|$. Now $U_\alpha = \{x : \rho(x) < \alpha\}$ is a regular Vitali family; and proving (4.4-2) will prove our lemma by virtue of Theorem 4.1 of [**Ri**].

There are two cases to consider: $0 < \rho(y) \le 1$, and $\rho(y) \ge 1$.

Case I: $0 < \rho(y) \le 1$.

Since

$$\int_{\epsilon \le |t| \le 1} h_z(x)\, e^{-2\pi i |t|^{-\beta}}\, |t|^{-z} \frac{dt}{t|t|^\alpha} = 0\,;$$

$$K_z(x) = \int_{\epsilon \le |t| \le 1} [h_z(x - \Gamma(t)) - h_z(x)]\, e^{-2\pi i |t|^{-\beta}}\, |t|^{-z} \frac{dt}{t|t|^\alpha}\,.$$

The change of variable $t = s\,\rho(y)^{\beta+1}$ gives $dt = \rho(y)^{\beta+1}\, ds$, and

$$K_z(x) = \int_{\epsilon\, \rho(y)^{-(\beta+1)} \le |s| \le 1} \left[h_z(x - \Gamma(s\rho(y)^{\beta+1})) - h_z(x)\right] e^{-2\pi i |s\, \rho(y)^{\beta+1}|^{-\beta}}$$

$$\cdot \left| s\,\rho(y)^{\beta+1} \right|^{-z} \rho(y)^{-(\beta+1)\alpha} \frac{ds}{s|s|^{\alpha}}$$

$$+ \int\limits_{1 \le |s| \le \rho(y)^{-(\beta+1)}} \cdots$$

$$= K_z^1 + K_z^2 .$$

Now,

$$\int\limits_{\rho(x) > C\rho(y)} |\, K_z^1(x) \,|\, dx$$

$$\le \int\limits_{\rho(x) > C\rho(y)} \int\limits_{\epsilon\,\rho(y)^{-(\beta+1)} \le |s| \le 1} \left| h_z \left(x - \Gamma\left(s\,\rho(y)^{\beta+1} \right) \right) - h_z(x) \right|$$

(4.4-3)

$$|s|^{-1-\alpha-\sigma} \rho(y)^{-(\beta+1)(\alpha+\sigma)}\, ds\, dx .$$

The change of variable $x = \delta_{\rho(y)}\, x'$ implies $dx = \rho(y)^{(2\beta+k+1)}\, dx'$; $\rho(x) = \rho(y)\,\rho(x')$; and that $\|x'\|$ is large. The right-hand side of (4.4-3) now becomes:

$$\int\limits_{\rho(x') > C} \int\limits_{\epsilon\,\rho(y)^{-(\beta+1)} \le |s| \le 1} \left| h_z \left(\delta_{\rho(y)} \left[x' - \delta^{-1}_{\rho(y)}\, \Gamma\left(s\,\rho(y)^{\beta+1} \right) \right] \right) h_z \left(\delta_{\rho(y)}\, x' \right) \right|$$

$$\cdot\, |s|^{-1-\alpha-\sigma} \rho(y)^{-(\beta+1)(\alpha+\sigma)} \rho(y)^{(2\beta+k+1)}\, ds\, dx'.$$

Now, using the homogeneity of h_z:

$$h_z\left(\delta_{\rho(y)}\, x \right) = \rho(y)^{-(2\beta+k+1)-z}\, h_z(x) ,$$

and writing $x = x'$, this

$$= \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int\limits_{\rho(x) > C} \int\limits_{\epsilon\,\rho(y)^{-(\beta+1)} \le |s| \le 1}$$

$$\left| h_z \left(x - \left[s,\, \gamma(s)\, \rho(y)^{\beta(k-1)} \right] \right) - h_z(x) \right| |s|^{-1-\alpha-\sigma}\, ds\, dx$$

$$= \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int\limits_{\rho(x) > C} \int\limits_{\epsilon\,\rho(y)^{-(\beta+1)} \le |s| \le 1}$$

$$\left| h_z \left(\delta_{\rho(x)} \left(\delta^{-1}_{\rho(x)}\, x - \delta^{-1}_{\rho(x)} \left[s,\, \gamma(s)\, \rho(y)^{\beta(k-1)} \right] \right) \right) - h_z \left(\delta_{\rho(x)} \left(\delta^{-1}_{\rho(x)}\, x \right) \right) \right|$$

$$\cdot\, |s|^{-1-\alpha-\sigma}\, ds\, dx.$$

Note that $\left\|\delta_{\rho(x)}^{-1}\, x\right\| = 1$; and since $\rho(x)$ is large, $\|w\| = \left\|\delta_{\rho(x)}^{-1}\left[s,\, \gamma(s)\, \rho(y)^{\beta(k-1)}\right]\right\|$ is small. Fubini's theorem and (4.3-2) then imply that the above is

$$\leq C(z)\, \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int\limits_{\epsilon\, \rho(y)^{-(\beta+1)} \leq |s| \leq 1} |s|^{-1-\alpha-\sigma}\left[s^2 + s^{2k}\rho(y)^{2\beta(k-1)}\right]^{\frac{1}{2}} ds$$
$$\cdot \int\limits_{\rho(x)>C} \frac{dx}{\rho(x)^{(2\beta+k+1)+\sigma+\beta+1}}\,.$$

Changing to *polar-like* coordinates with $dx = \rho(x)^{(2\beta+k+1)-1}\, d\rho(x)\, d\varphi$, the above is

$$\leq C(z)\, \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int\limits_{\epsilon\, \rho(y)^{-(\beta+1)} \leq |s| \leq 1} |s|^{-\alpha-\sigma}\left[1 + s^{2k-2}\rho(y)^{2\beta(k-1)}\right]^{\frac{1}{2}} ds$$
$$\cdot \int\limits_{\mathbf{S}^1} d\varphi \int\limits_{\rho(x)>C} \frac{d\rho(x)}{\rho(x)^{\beta+\sigma+2}}.$$

For $\rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma}$ to be bounded, we need $-(\beta+1)(\alpha+\sigma)-\sigma \geq 0$; that is, $\sigma \leq -\alpha\left[\dfrac{\beta+1}{\beta+2}\right] < 0$. With σ as in the preceding statement, $-\alpha-\sigma \geq -\dfrac{\alpha}{\beta+2} > -1$ since $\beta \geq 3\alpha$; and so $|s|^{-\alpha-\sigma}$ is integrable on $\epsilon\, \rho(y)^{-(\beta+1)} \leq |s| \leq 1$. For the ρ-integral to be bounded, we need $\beta + \sigma + 2 > 1$; that is, $\sigma > -(\beta+1)$. Thus, whenever $-(\beta+1) < -\alpha < \sigma \leq -\alpha\left[\dfrac{\beta+1}{\beta+2}\right]$, we have that $\displaystyle\int\limits_{\rho(x)>C\rho(y)} |K_z^1(x)|\, dx$ is bounded by $C(z)$. Similarly, $\displaystyle\int\limits_{\rho(x)>C\rho(y)} |K_z^1(x-y)|\, dx \leq C(z)$ using the fact that $\rho(x+y) \leq C[\rho(x)+\rho(y)]$.

Next,

$$\int\limits_{\rho(x)>C\rho(y)} \left|\, K_z^2(x-y) \;-\; K_z^2(x)\,\right| dx$$
$$\leq \int\limits_{\rho(x)>C\rho(y)} \;\int\limits_{1\leq|s|\leq\rho(y)^{-(\beta+1)}}$$
$$\left|h_z\left(x-y-\Gamma\left(s\,\rho(y)^{\beta+1}\right)\right) \;-\; h_z\left(x-\Gamma\left(s\,\rho(y)^{\beta+1}\right)\right)\right|$$

(4.4-4)
$$\cdot\, |s|^{-1-\alpha-\sigma}\, \rho(y)^{-(\beta+1)(\alpha+\sigma)}\, ds\, dx\,.$$

Again, with $x = \delta_{\rho(y)}\, x'$ so that $\|x'\|$ is large, $dx = \rho(y)^{(2\beta+k+1)}\, dx'$, $\rho(x) = \rho(y)\,\rho(x')$, and using the homogeneity of h_z with $\lambda = \rho(y)$, the right-hand side of (4.4-4) becomes

$$= \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int\limits_{\rho(x')>C} \int\limits_{1\le|s|\le\rho(y)^{-(\beta+1)}} \left|h_z\left(x' - \delta^{-1}_{\rho(y)}\, y - \left[s, \gamma(s)\,\rho(y)^{\beta(k-1)}\right]\right) - h_z\left(x' - \left[s, \gamma(s)\,\rho(y)^{\beta(k-1)}\right]\right)\right| \cdot |s|^{-1-\alpha-\sigma}\, ds\, dx'.$$

Writing $x = x'$ and $w = x - \left[s,\, \gamma(s)\,\rho(y)^{\beta(k-1)}\right]$, so that $dw = dx$ and $\|x\|$ is large, and using Fubini's theorem, this is

$$= \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int\limits_{1\le|s|\le\rho(y)^{-(\beta+1)}} |s|^{-1-\alpha-\sigma}\, ds \cdot \left[\int\limits_{\rho(w)>C_2} + \int\limits_{\rho(w)\le C_2} \left|h_z\left(w - \delta^{-1}_{\rho(y)}\, y\right) - h_z(w)\right| dw\right]$$
$$= \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int\limits_{1\le|s|\le\rho(y)^{-(\beta+1)}} |s|^{-1-\alpha-\sigma}\, ds\, [I + II]\,;$$

where C_2 is a large constant. Now, using the homogeneity of h_z, and (4.3-2) we see that,

$$I = \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int\limits_{1\le|s|\le\rho(y)^{-(\beta+1)}} |s|^{-1-\alpha-\sigma}\, ds \cdot \int\limits_{\rho(w)>C_2} \left|h_z\left(\delta_{\rho(w)}\left(\delta^{-1}_{\rho(w)} w - \delta^{-1}_{\rho(w)}\left(\delta^{-1}_{\rho(y)} y\right)\right)\right) - h_z\left(\delta_{\rho(w)}\left(\delta^{-1}_{\rho(w)} w\right)\right)\right| dw$$
$$\le C(z)\rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int\limits_{1\le|s|\le\rho(y)^{-(\beta+1)}} |s|^{-1-\alpha-\sigma}\, ds \cdot \int\limits_{\rho(w)>C_2} \frac{\left|\delta^{-1}_{\rho(y)}\, y\right|}{\rho(w)^{(2\beta+k+1)+\sigma+\beta+1}}\, dw$$
$$\le C(z)\rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int\limits_{1\le|s|\le\rho(y)^{-(\beta+1)}} |s|^{-1-\alpha-\sigma}\, ds \cdot \int\limits_{\mathbf{S}^1} d\varphi \int\limits_{\rho(w)>C_2} \frac{d\rho(w)}{\rho(w)^{\beta+\sigma+2}}\,.$$

For $-\alpha < \sigma \leq -\alpha \left[\frac{\beta+1}{\beta+2}\right] < 0$, we have $0 < \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \leq 1$, and $1+\alpha+\sigma > 1$; and so $|s|^{-1-\alpha-\sigma}$ is integrable on $|s| \geq 1$. The ρ-integral is bounded, since $\beta+\sigma+2 > 1$ whenever $\sigma > -\alpha > -(\beta+1)$. Hence, I is bounded by $C(z)$.

Next,

$$II = \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int_{1\leq|s|\leq\rho(y)^{-(\beta+1)}} |s|^{-1-\alpha-\sigma}\, ds \cdot \int_{\rho(w)\leq C_2} \left|h_z\left(w - \delta_{\rho(y)}^{-1} y\right) - h_z(w)\right| dw.$$

The inner w-integral is bounded, since h_z is locally integrable; the outer s-integral is bounded whenever $-\alpha < \sigma \leq -\alpha \left[\frac{\beta+1}{\beta+2}\right]$.

Case II: $\rho(y) > 1$.

Fubini's theorem, homogeneity of h_z, and (4.3-2) together imply that,

$$\begin{aligned}
&\int_{\rho(x)>C\rho(y)} |K_z(x)|\, dx \\
&\leq \int_{\epsilon\leq|t|\leq1} |t|^{-1-\alpha-\sigma}\, dt \int_{\rho(x)>C\rho(y)} |h_z(x-\Gamma(t)) - h_z(x)|\, dx \\
&= \int_{\epsilon\leq|t|\leq1} |t|^{-1-\alpha-\sigma}\, dt \\
&\quad \cdot \int_{\rho(x)>C} \left|h_z\left(\delta_{\rho(x)}\left(\delta_{\rho(x)}^{-1} x - \delta_{\rho(x)}^{-1}\Gamma(t)\right)\right) - h_z\left(\delta_{\rho(x)}\left(\delta_{\rho(x)}^{-1} x\right)\right)\right| dx \\
&\leq C(z) \int_{\epsilon\leq|t|\leq1} |t|^{-1-\alpha-\sigma}|\Gamma(t)|\, dt \int_{\rho(x)>C} \frac{dx}{\rho(x)^{(2\beta+k+1)+\sigma+\beta+1}} \\
&\leq C(z) \int_{\epsilon\leq|t|\leq1} |t|^{-\alpha-\sigma}\, dt \int_{\mathbf{S}^1} d\varphi \int_{\rho(x)>C} \frac{d\rho(x)}{\rho(x)^{\beta+\sigma+2}}
\end{aligned}$$

since $|\Gamma(t)| = (t^2+t^{2k})^{\frac{1}{2}} \leq \sqrt{2}\,|t|$ for $|t| \leq 1$. The last expression is bounded whenever $-\alpha < \sigma \leq -\alpha \left[\frac{\beta+1}{\beta+2}\right]$. Similarly, $\int_{\rho(x)>C\rho(y)} |K_z(x-y)|\, dx$ is bounded by $C(z)$. This completes the proof of Lemma 4.4. □

This brings us to the final step of the proof of Theorem 1:

4.1. Interpolation. Lemma 4.1 shows that, for f simple, $\|\mathcal{T}_z^\epsilon f\|_2 \leq C_1(z)\|f\|_2$ whenever $0 < \Re z \leq \frac{1}{2}\left[\frac{\beta}{3} - \alpha\right]$, $\beta > 3\alpha$; Lemma 4.4 shows that $\|\mathcal{T}_z^\epsilon f\|_p \leq C_2(z)\|f\|_p$, $1 < p < \infty$, whenever $-\alpha < \Re z \leq -\alpha\left[\frac{\beta+1}{\beta+2}\right] < 0$; each $C_i(z)$ $(i = 1,2)$ grows at most as fast as a polynomial in $|z|$. It follows that $\{\mathcal{T}_z^\epsilon\}$ is an *admissible analytic family* for the *Stein analytic interpolation theorem* (see [**St**, **We**], page 205), defined for z in the strip

$$S = \left\{ z \in \mathbf{C} : -\alpha\left[\frac{\beta+1}{\beta+2}\right] \leq \Re z \leq \frac{1}{2}\left[\frac{\beta}{3} - \alpha\right]\right\}.$$

Analytic interpolation and *duality* now imply that $\mathcal{T}_0^\epsilon = \mathcal{T}_{\alpha,\beta}^\epsilon$ is bounded on L^p whenever

$$1 + \frac{3\alpha\,(\beta+1)}{\beta\,(\beta+1) + (\beta - 3\alpha)} < p < \frac{\beta\,(\beta+1) + (\beta - 3\alpha)}{3\alpha\,(\beta+1)} + 1,$$

for all *simple* f on $\mathbf{R}^2$. An easy limiting argument shows that $\|\mathcal{T}_{\alpha,\beta}^\epsilon f\|_p \leq B_{\alpha,\beta}\|f\|_p$ for all $f \in \mathcal{S}$. The constant $B_{\alpha,\beta}$ is independent of ϵ. Letting $\epsilon \to 0$, Fatou's lemma gives $\|\mathcal{T}_{\alpha,\beta} f\|_p \leq B_{\alpha,\beta}\|f\|_p$ for all $f \in \mathcal{S}$. Now, another limiting argument shows that the last inequality holds for all $f \in L^p$. This completes the proof of Theorem 1. □

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University of Wisconsin-Madison
Madison, WI 53706
E-mail address: chandara@math.wisc.edu