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TRIANGLE SUBGROUPS OF HYPERBOLIC TETRAHEDRAL GROUPS

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It is known that there are nine compact tetrahedra in three-dimensional hyperbolic space for which the group of isometries generated by reflections in the faces is discrete. The subgroup of orientation-perserving isometries in such a group will be called a tetrahedral Kleinian group. Exactly eight such Γ are arithmetic and also a complete list of the finitely many arithmetic Fuchsian triangle groups G is known. In this paper, we determine for which pairs of groups (G, Γ) as above, with one possible exception, one can embed G into Γ. We find that there are many such pairs, contrasting with the single pair (G, Γ) which is known to arise when, instead of arithmeticity, the condition that G be realised as the subgroup of elements of Γ which centralise a reflection in one of the faces of the associated tetrahedron, is imposed.

1. Introduction.

There are nine compact tetrahedra in hyperbolic 3-space whose dihedral angles are submultiples of π so that the group generated by reflections in the faces of the tetrahedron is a discrete subgroup of the isometry group, $\mathrm{Isom}(\mathbf{H}^3)$. The centraliser of a reflection in one of the faces of the tetrahedron is then a discrete subgroup of the isometry group of the 2-dimensional hyperbolic plane on which that face lies. Thus if we restrict to orientation-preserving subgroups in both the ambient group and the subgroup, these tetrahedral Kleinian groups will contain Fuchsian subgroups. For only one of the tetrahedra and one of the faces of that tetrahedron, is the Fuchsian subgroup a triangle group [**1**]. Eight of the tetrahedral groups are arithmetic [**20**], and in these cases, these tetrahedral groups contain infinitely many commensurablilty classes of Fuchsian subgroups [**12**]. Here, using arithmetic techniques, it will be shown that these tetrahedral groups contain Fuchsian triangle groups in addition to the one "visible" on a face described above. For all except one of these arithmetic triangle groups complete information is obtained.

These tetrahedral groups arise in connection with various extremal problems in hyperbolic 3-manifolds and orbifolds. For example, one of them is the arithmetic orbifold of minimal volume [**3**] and conjecturally the 3-dimensional orbifold of minimal volume [**5**]. Others arise in the investigation of hyperbolic orbifolds with elliptic elements whose axes are as close as possible [**6**, **7**]. Indeed it was in investigating one such case, that the question addressed in this paper came to light and I am grateful to Gaven Martin for discussions on this.

2. Tetrahedral Groups.

The group of orientation-preserving isometries of hyperbolic 3-space $\mathbf{H}^3$, is isomorphic to $\mathrm{PSL}(2,\mathbf{C})$ via the Poincaré extension. Then 3-dimensional hyperbolic orbifolds (respectively manifolds) arise as the quotient of $\mathbf{H}^3$ by Kleinian groups which are discrete (discrete and torsion-free) subgroups Γ of $\mathrm{PSL}(2,\mathbf{C})$. Here we will only be concerned with the cases where Γ is cocompact i.e. the quotient space $\mathbf{H}^3/\Gamma$ is compact.

Let T denote a compact tetrahedron in $\mathbf{H}^3$ with vertices A, B, C, D such that the dihedral angles are submultiples of π. If the dihedral angles along the edges AB, BC, CA, CD, DA, BD are $\pi/l_1, \pi/l_2, \pi/l_3, \pi/m_1, \pi/m_2, \pi/m_3$ respectively, we denote the tetrahedron $[l_1, l_2, l_3; m_1, m_2, m_3]$. There are nine such tetrahedra [**8**]. If T_i denotes the tetrahedron let Γ_i denote the subgroup of index two which consists of orientation-preserving isometries in the group generated by reflections in the faces of T_i. These are the tetrahedral groups and with the notation just described are listed below.

$T_1[2,2,3;3,5,2]$	$T_2[2,2,3;2,5,3]$	$T_3[2,2,4;2,3,5]$
$T_4[2,2,5;2,3,5]$	$T_5[2,3,3;2,3,4]$	$T_6[2,3,4;2,3,4]$
$T_7[2,3,3;2,3,5]$	$T_8[2,3,4;2,3,5]$	$T_9[2,3,5;2,3,5]$

Table 1

Recall that Γ_i then has presentation

$$\Gamma_i = \{a, b, c \mid a^{l_1} = b^{l_2} = c^{l_3} = (bc)^{m_1} = (ca)^{m_2} = (ab)^{m_3} = 1\}.$$

Each face of the tetrahedron is a triangle on a hyperbolic plane in $\mathbf{H}^3$. If ρ denotes the reflection in that plane, let $C_{\Gamma_i}(\rho)$ denote the centraliser of ρ in Γ_i so that $C_{\Gamma_i}(\rho)$ will be a group of isometries of that hyperbolic plane. The subgroup $C^+_{\Gamma_i}(\rho)$ of elements which preserve the orientation of this plane is thus a Fuchsian group. A fundamental region for $C^+_{\Gamma_i}(\rho)$ on that hyperbolic plane can be obtained. This applies more generally to any discrete subgroup generated by reflections in the faces of a polyhedron and the structure of the Fuchsian groups so obtained in the cases of the nine tetrahedral groups has been determined (for all this see [**1**]).

For T_3, let ρ be the reflection in the face ABC. Then $C_{\Gamma_3}(\rho)$ has, as fundamental region, the triangular face ABC whose face angles are $\pi/4, \pi/5, \pi/2$ respectively. Thus Γ_3 contains the $(2,4,5)$ triangle group. Now the $(2,4,5)$ triangle group is well known to contain the triangle subgroups $(2,5,5), (4,4,5)$ as subgroups of indices 2 and 6 respectively. Thus Γ_3 also contains these. However, this is the only tetrahedron T_i such that a subgroup of the form $C^+_{\Gamma_i}(\rho)$ is a triangle group [**1**].

It should be noted that one can construct triangular prisms in $\mathbf{H}^3$ whose dihedral angles are all submultiples of π, with one triangular face meeting its neighbours orthogonally and having face angles $\pi/2, \pi/3, \pi/q$ for any $q \geq 7$ [**4**, **13**]. It follows that these Kleinian pentahedral groups contain the Fuchsian triangle groups $(2,3,q)$.

3. Arithmetic Groups.

Let k be a number field with one complex place and let A be a quaternion algebra over k which is ramified at the real places. If $\sigma : A \to M(2, \mathbf{C})$ is a representation of A and $\mathcal{O}$ an order in A, then $\mathrm{P}\,\sigma(\mathcal{O}^1)$ is a discrete subgroup of $\mathrm{PSL}(2,\mathbf{C})$ of finite covolume, where $\mathcal{O}^1$ is the group of elements in $\mathcal{O}$ of norm 1. An *arithmetic Kleinian group* is any Kleinian group commensurable with some such $P\sigma(\mathcal{O}^1)$. In fact if Γ is an arithmetic Kleinian group, then $\Gamma^{(2)} \subset \mathrm{P}\,\sigma(\mathcal{O}^1)$ for some order $\mathcal{O}$ where

$$\Gamma^{(2)} = \langle \gamma^2 \mid \gamma \in \Gamma \rangle.$$

Arithmetic Fuchsian groups are defined in a similar way with in this case k being a totally real number field and A ramified at all real places except one. For information on arithmetic Fuchsian and Kleinian groups see [**19**, **2**].

For any finite volume Kleinian group Γ then

$$\mathrm{k}\,\Gamma = \mathbf{Q}(\mathrm{tr}\,\delta \mid \delta \in \Gamma^{(2)})$$

is a finite extension of the rationals and an invariant of the commensurability class of Γ [**14**], called the invariant field of Γ. Furthermore

$$A\Gamma = \left\{ \sum a_i \delta_i \mid a_i \in k\Gamma, \delta_i \in \Gamma^{(2)} \right\}$$

is a quaternion algebra over $\mathrm{k}\,\Gamma$ [**11**]. In the cases where Γ is arithmetic, $\mathrm{k}\,\Gamma, A\Gamma$ define the arithmetic structure [**9**]. This last statement is also true for Fuchsian groups [**16**].

For a circle or straight line $\mathcal{C}$ in $\mathbf{C}$, and a Kleinian group Γ, define

$$\mathrm{Stab}(\mathcal{C}, \Gamma) = \{\gamma \in \Gamma \mid \gamma(\mathcal{C}) = \mathcal{C} \text{ and } \gamma \text{ preserves the components of } \mathbf{C} \setminus \mathcal{C}\}$$

Then $\mathrm{Stab}(\mathcal{C}, \Gamma)$ will be a maximal non-elementary Fuchsian subgroup if its limit set on $\mathcal{C}$ consists of more than two points.

In the cases where Γ is arithmetic, these groups $\mathrm{Stab}(\mathcal{C}, \Gamma)$ will be arithmetic Fuchsian groups F [**9**]. Furthermore, if Γ is determined by the quaternion algebra A over the number field k, then F will be determined by a quaternion algebra B over the totally real number field $l = k \cap \mathbf{R}$ where $[l : k] = 2$ and $A \cong B \otimes_l k$ [**9**]. This then forces the primes that ramify in A to have a certain form and also implies a relationship between these primes and the primes of l which are ramified in B [**12**]. For the applications, this information is gathered in the following theorem.

Theorem 3.1. *Let Γ be an arithmetic Kleinian group whose quaternion algebra A is defined over the number field k. Let F be a maximal non-elementary Fuchsian subgroup of Γ, which will then be arithmetic with quaternion algebra B defined over some totally real number field l. In addition*

(1) $k \cap \mathbf{R} = l$ *and* $[k : l] = 2$.

(2) $B \otimes_l k \cong A$.

(3) *A has finite ramification at a (possibly empty) set of pairs of prime ideals $\mathcal{Q}_i, \mathcal{Q}'_i (i = 1, 2, \cdots, n)$ of k such that $\mathcal{Q}_i \cap R_l = \mathcal{Q}'_i \cap R_l = \mathcal{P}_i$.*

(4) *B has finite ramification at the ideals $\mathcal{P}_i (i = 1, 2 \cdots, n)$ together with a (possibly empty) set of prime ideals $\mathcal{P}$ of l such that $\mathcal{P}$ is either inert or ramified in the extension $k \mid l$.*

The above theorem gives us information on which triangle groups are candidates to be subgroups of a given tetrahedral group. We now give results aimed at showing the existence of triangle subgroups of tetrahedral groups.

In the above circumstances, $B \otimes_l k \cong A$ and so there is an embedding $j : B \to A$. Now if $\mathcal{L}$ is an order in B, then $\mathcal{L} \otimes_{R_l} R_k$ is an order in $B \otimes_l k$. If $\mathcal{L}$ is maximal in B, $\mathcal{L} \otimes_{R_l} R_k$ is not in general maximal in $B \otimes_l k$. Indeed, if $d(\mathcal{L})$ is the discriminant of $\mathcal{L}$ then $d(\mathcal{L} \otimes_{R_l} R_k) = d(\mathcal{L}) R_k$ [**15**]. In all cases, we obtain an embedding $j : \mathcal{L} \to \mathcal{O}$ where $\mathcal{O}$ is a maximal order in A. Thus via j we obtain embeddings of Fuchsian groups into arithmetic Kleinian groups.

Theorem 3.2. *If $B \otimes_l k \cong A$, thus yielding $j : B \to A$ as in the above theorem, and $\mathcal{L}$ is a maximal order in B, then $j(\mathcal{L}^1) \subset \mathcal{O}^1$ for some maximal order in A.*

The type number of A is the number of conjugacy classes of maximal orders in A and can be determined from the arithmetic data of A [**19**]. For a maximal order $\mathcal{O}$, define

$$N\mathcal{O} = \{x \in A^* \mid x^{-1}\mathcal{O}x = \mathcal{O}\}.$$

Then if σ is a represenation $\sigma : A \to M(2, \mathbf{C})$, $P\sigma(\mathcal{O}^1) \subset P\sigma(N\mathcal{O})$ and $P\sigma(N\mathcal{O})$ is a maximal Kleinian group [**2**]. The same applies in the case of B and Fuchsian groups.

4. Arithmetic tetrahedral and triangle groups.

For all T_i, $i \neq 8$, the groups Γ_i are arithmetic [**20**], and the defining fields k_i and quaternion algebras A_i have been determined [**10**]. This information is recorded in Table 2 below together with some additional information readily deduced from the details in [**10**]. Here Δ_k denotes the discriminant of the field k and Ram_f, the set of finite primes at which the algebra A is ramified.

Group	Δ_k	Ram_f	$[P\sigma(N\mathcal{O}) : P\sigma(\mathcal{O}^1)]$	$P\sigma(\mathcal{O}^1)$	Type Number
Γ_1	-400	ϕ	2	Γ_1	1
Γ_2	-275	ϕ	2	Γ_2	1
Γ_3	-400	ϕ	2	$\Gamma_3^{(2)}$	1
Γ_4	-475	ϕ	2	Γ_4	1
Γ_5	-448	ϕ	2	Γ_5	1
Γ_6	-7	$\mathcal{Q}_2, \mathcal{Q}_2'$	8	$\Gamma_6^{(2)}$	1
Γ_7	-775	ϕ	2	Γ_7	1
Γ_9	-1375	ϕ	4	Γ_9	2

Table 2

Note that since only elements of orders 2,3,4,5 can occur in these tetrahedral groups, we need only consider the triangle groups in the table below. These are gathered together in commensurability classes and adjoined at the end of each class is the maximal triangle group in which these lie.

a) $(2,4,5), (2,5,5), (4,4,5); (2,4,5)$

b) $(3,3,4), (4,4,4); (2,3,8)$

c) $(3,3,5)(5,5,5) : (2,3,10)$

d) $(3,4,4); (2,4,6)$

e) $(3,5,5); (2,5,6)$

f) $(4,5,5); (2,5,8)$

g) $(3,4,5); (3,4,5)$

Table 3

Precisely which triangle groups are arithmetic has been determined [**17**]. Recalling that arithmetic Fuchsian groups are commensurable if and only if their quaternion algebras are isomorphic, the commensurability classes of these triangle groups were obtained by classifying their quaternion algebras [**18**]. From that information we note that the group $(3,4,5)$ is not arithmetic and also that the defining field of the group $(4,5,5)$ is $Q(\sqrt{5},\sqrt{2})$. Thus by Theorem 3.1, neither of these can be subgroups of any of the arithmetic tetrahedral groups. For the remainder the detailed information is given below. In this Table, we let F_α ,$\alpha = a,b,c,d,e$ denote the maximal triangle group in the class and label the corresponding field k_α and defining algebra B_α.

	F_α	$F_\alpha^{(2)}$	k_α	Finite Ramification
a	$(2,4,5)$	$(2,5,5)$	$Q(\sqrt{5})$	$\mathcal{P}_2$
b	$(2,3,8)$	$(3,3,4)$	$Q(\sqrt{2})$	$\mathcal{P}_2$
c	$(2,3,10)$	$(3,3,5)$	$Q(\sqrt{5})$	$\mathcal{P}_5$
d	$(2,4,6)$	$(2,2,3,3)$	Q	$\mathcal{P}_2,\mathcal{P}_3$
e	$(2,5,6)$	$(3,5,5)$	$Q(\sqrt{5})$	$\mathcal{P}_3$

Table 4

It should be noted in each of these cases that $F_\alpha^{(2)} = P\mu(\mathcal{L}^1)$ where μ is a representation of the quaternion algebra B_α and $\mathcal{L}$ is a maximal order of B_α.

5. Triangle Subgroups of Tetrahedral Groups.

In this section, the arithmetic tetrahedral groups Γ_i $i \neq 8$ are examined in turn to decide which triangle groups they contain.

Consider first the information obtained in Section 2, noting that $\Gamma_1 = \Gamma_3^{(2)}$ is a subgroup of index 2 in Γ_3. For the tetrahedral groups, this is the only pair that are commensurable [**10**]. From Section 2, it follows that $F_a \subset \Gamma_3$ and so $F_a^{(2)} \subset \Gamma_3^{(2)}$. Thus $(2,5,5)$ is a subgroup of Γ_1 but, since Γ_1 has no elements of order 4, the other two triangle groups in the commensurability class are not.

From Table 2, $A_1 (= A_3)$ has type number 1 so that for a maximal order $\mathcal{O}$ in A_1 we can assume, up to conjugacy, and again using Table 2, that $\Gamma_3^{(2)} = P\sigma(\mathcal{O}^1)$. So Γ_3 and $P\sigma(N\mathcal{O})$ lie in the normaliser of $P\sigma(\mathcal{O}^1)$ which is discrete of finite covolume. Since $P\sigma(N\mathcal{O})$ is a maximal Kleinian group, it follows that $\Gamma_3 = P\sigma(N\mathcal{O})$. Now $k_1 \cap \mathbf{R} = \mathbf{Q}(\sqrt{5})$ so we consider F_c, F_e by Theorem 3.1. Using Kummer's theorem applied to the extension $k_1 \mid \mathbf{Q}(\sqrt{5})$, both $\mathcal{P}_3, \mathcal{P}_5$ are inert in this extension. Then by the comments following Table 4, and the fact the A_1 has type number one,

$$F_c^{(2)} = P\sigma(j\mathcal{L}^1) \subset P\sigma(\mathcal{O}^1) = \Gamma_1$$

by Theorem 3.2. In exactly the same way $F_e^{(2)} \subset \Gamma_1$.

The cases Γ_i, $i = 2, 4, 5, 7$ follow in a very similar way using the data from Tables 2 and 4. The results are given in Theorem 5.1 below.

The group Γ_6 will now be considered. Note that A_6 is ramified at $\mathcal{Q}_2, \mathcal{Q}_2'$ and that $\mathcal{P}_3$ is inert in $\mathbf{Q}(\sqrt{-7}) \mid \mathbf{Q}$. Thus as before $F_d^{(2)}$ embeds in $\Gamma_6^{(2)}$. Additional information on A_6 is required. Recall from [**2**] that, in this case, the norm mapping defines an epimorphism $n : N\mathcal{O} \to R_f^*$ which induces an isomorphism

$$\frac{P\sigma(N\mathcal{O})}{P\sigma(\mathcal{O}^1)} \cong \frac{R_f^*}{R_f^{*2}} \tag{1}$$

where R_f^* is the group of units in the ring consisting of those elements of k_6 which are integral at all finite places not in the ramification set of A_6. The quotient group Q in (1) is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and is generated by the images of $x_1 = (1 + \sqrt{-7})/2$, $x_2 = (1 - \sqrt{-7})/2$, $x_3 = -1$. Now Γ_6 is a subgroup of $P\sigma(N\mathcal{O})$ of index 4, containing $P\sigma(\mathcal{O}^1)$ and so corresponds to one of the subgroups of order 2 in Q. Now Γ_6 contains elements of order 4. Since $\sqrt{2} \notin k_6$, there are no elements of order 4 in $P\sigma(\mathcal{O}^1)$. Let $\alpha \in N(\mathcal{O})$ be such that $P\sigma(\alpha)$ has order 4. Let n, t denote the norm and trace of α. Then $n = xy^2$, where x is a product of the $x_i's$ and $y \in R_f^*$. Since $P\sigma(\alpha)$ has order 4, $t^2 = 2n$. Thus $2x \in k_6^{*2}$ and so $x = x_1x_2 = 2$. Thus the subgroup of order 2 in Q generated by the image of x_1x_2 gives the unique subgroup of $P\sigma(N\mathcal{O})$ containing $P\sigma(\mathcal{O}^1)$ which contains elements of order 4. This unique subgroup must be Γ_6. Furthermore, if we choose in A_6 where

$$A_6 = \left(\frac{-1, -1}{\mathbf{Q}(\sqrt{-7})}\right)$$

the maximal order $\mathcal{O} = R_{k_6}[1, i, j, 1/2(1 + i + j + ij)]$ then $\alpha = 1 + i$ lies in $N\mathcal{O}$ and $n(\alpha) = 2$. Thus $\Gamma_6 = \langle P\sigma(\mathcal{O}^1), P\sigma(\alpha)\rangle$. Furthermore, if we choose $\beta = i + j + ij$ so that $n(\beta) = 3$ then $\Omega = \mathcal{O} \cap \beta\mathcal{O}\beta^{-1}$ is an Eichler order of level $\mathcal{Q}_3$ in A_6. Up to conjugacy there is only one class of Eichler orders of this level in A_6 [**19**] and one can check directly that $\alpha \in N\Omega$.

Now consider the group F_d. Recall the $F_d^{(2)} = P\sigma(j(\mathcal{L}^1))$ has signature $(2, 2, 3, 3)$. Also $F_d = P\sigma(j(N\mathcal{L}))$ and the group $(3, 4, 4)$ is its unique subgroup of index 2 which contains elements of order 4. The order $\mathcal{L} \otimes_{\mathbf{Z}} R_{k_6}$ has discriminant $\mathcal{Q}_2\mathcal{Q}_2'\mathcal{Q}_3$, which since it is square-free, is an Eichler order of level $\mathcal{Q}_3$. Thus we can assume that $\mathcal{L} \otimes_{\mathbf{Z}} R_{k_6} = \Omega$ and hence the image of one of the elements of order 4 in the $(3, 4, 4)$ group will coincide, after conjugacy, with $P\sigma(\alpha)$. Thus $(3, 4, 4)$ is a subgroup of Γ_6.

Finally, consider the group Γ_9. From Table 2, note that the type number of A_9 is 2 and that $\Gamma_9 = P\sigma(\mathcal{O}_1^1)$ for some maximal order $\mathcal{O}_1$ in A_9. But in this case there is a maximal order $\mathcal{O}_2$ in A_9 which is not conjugate to $\mathcal{O}_1$ in

A_9. Now $P\sigma(\mathcal{O}_2^1)$ cannot be isomorphic to Γ_9, otherwise by Mostow rigidity, $P\sigma(\mathcal{O}_2^1)$ would be conjugate to $P\sigma(\mathcal{O}_1^1)$ in Isom($\mathbf{H}^3$). But that implies that $\mathcal{O}_1^1$ and $\mathcal{O}_2^1$ are conjugate in A_9^*. Let $t\mathcal{O}_1^1 t^{-1} = \mathcal{O}_2^1$ and set $\mathcal{M} = t\mathcal{O}_1 t^{-1}$, so that $\mathcal{M}$ is a maximal order. Let $\mathcal{E} = \mathcal{M} \cap \mathcal{O}_2$ so that $\mathcal{E}^1 = \mathcal{O}_2^1$. If $\mathcal{M} \neq \mathcal{O}_2$, then $\mathcal{E}$ is an Eichler order, and for some prime $\mathcal{P} \in k_9$, $[((\mathcal{O}_2)_{\mathcal{P}})^1 : (\mathcal{E}_{\mathcal{P}})^1] > 1$. Since the index $[\mathcal{O}_2^1 : \mathcal{E}^1]$ is a product of local indices, this contradiction shows that $\mathcal{M} = \mathcal{O}_2$.

Let $P\sigma(\mathcal{O}_2^1) = \Gamma_9'$. Now $k_9 = \mathbf{Q}(\sqrt{5})(\theta)$ where θ satisfies $x^2 - x + (9 + 4\sqrt{5}) = 0$ and is such that $\{1, \theta\}$ is a relative integral basis of $k_9 \mid \mathbf{Q}(\sqrt{5})$ [**10**]. Now investigating the candidates F_a, F_c, F_e we find that $\mathcal{P}_2$ splits in the extension, $\mathcal{P}_5$ ramifies and $\mathcal{P}_3$ is inert. Thus we obtain that $F_c^{(2)}, F_e^{(2)}$ are contained in either Γ_9 or Γ_9'. In fact, by choosing a basis, one can prove directly that if $\mathcal{L}$ is a maximal order in B_c then the order $\mathcal{L} \otimes_{R_{\mathbf{Q}(\sqrt{5})}} R_{k_9}$, whose discriminant is $\mathcal{Q}_5^2$, lies inside an Ω whose discriminant is $\mathcal{Q}_5$. But then Ω is an Eichler order of level $\mathcal{Q}_5$ and there is only one class of Eichler orders of level $\mathcal{Q}_5$ in A_9, since $\mathcal{Q}_5$ is generated by the element $1/2(\sqrt{5} - \sqrt{(5 - 4\sqrt{5})})$ which is positive at one real ramified place and negative at the other [**19**]. It follows that the groups $(3,3,5), (5,5,5)$ lie in both Γ_9, Γ_9'. A similar argument does not work for $\mathcal{Q}_3$ as it is generated by 3 and thus we have a less precise statement.

Theorem 5.1. *The tetrahedral groups Γ_i, $i = 1, 2 \cdots, 7$ contain precisely the triangle groups given in Table 5 below. The group Γ_9 contains the groups $(3,3,5), (5,5,5)$ and Γ_9 or Γ_9' contains $(3,5,5)$.*

Tetrahedral Group	Triangle Subgroups
Γ_1	$(2,5,5), (3,3,5), (5,5,5), (3,5,5)$
Γ_2	$(2,5,5), (3,3,5), (5,5,5)$
Γ_3	$(2,4,5), (2,5,5), (4,4,5), (3,3,5), (5,5,5), (3,5,5)$
Γ_4	$(2,5,5), (3,5,5)$
Γ_5	$(3,3,4), (4,4,4)$
Γ_6	$(3,4,4)$
Γ_7	$(3,3,5), (5,5,5), (3,5,5)$

Table 5.

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