PACIFIC JOURNAL OF MATHEMATICS
Vol. 176, No. 1, 1996

SOLVABILITY OF DIRICHLET PROBLEMS FOR SEMILINEAR ELLIPTIC EQUATIONS ON CERTAIN DOMAINS

ZHIREN JIN

We demonstrate a method to solve Dirichlet problems for semilinear elliptic equations on certain domains by a combination of change of variables, variational method and super-sub- solutions method. We show that Dirichlet problems for a semilinear elliptic equation have a least one solution as long as a relationship between the growth rate of the nonlinear term and the size of the domain is satisfied. The result can be applied to semilinear elliptic equations with super-critical growth.

1. Introduction and Results.

Let Ω be a bounded domain in R^n, $n > 2$. We consider the Dirichlet problem for a semilinear elliptic equation

$$(D_0) \qquad \begin{cases} -\Delta u = f(x,u) & \text{in} \quad \Omega; \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where Δ is the standard Laplace operator, $f(x,u)$ is a local Hölder continuous function defined on $\overline{\Omega} \times R$.

Throughout the paper, we assume that:

($\dagger$) There are positive constants M_1, M_2, $q \geq 1$, such that

$$|f(x,t)| \leq M_1 + M_2|t|^q \qquad \text{for all} \;\; x \in \overline{\Omega}, \; t \in R.$$

The main result of paper is

Theorem 1. *There is a constant $c(n,q)$ depending only on n and q, such that if we assume*

(1) ($\dagger$);

(2) $|\Omega| \leq c(n,q)\left(M_2 M_1^{q-1}\right)^{-\frac{n}{2q}}$,

then (D_0) has at least one solution.

When $q < \frac{n+2}{n-2}$, a result similar to Theorem 1 was shown in [**3**]. The method used in [**3**] is the variational method. When $q > \frac{n+2}{n-2}$, a direct variational approach does not work. We shall use a combination of changes of variables, super- sub- solutions method and variational method to show the result.

As in [**3**], since the result requires the volume of the domain Ω to be dominated by something related to the nonlinear term, we need to distinguish the result from the triviality of using an implicit function theorem to get a similar result. Here are a few points. First of all, an implicit function theorem tells us that (D_0) has at least one solution when the size of the domain Ω is small, but usually one will not be able to get an explicit upper bound for the size of the domain as we do here. Secondly, in the case that M_2 is small relative to M_1, the bound in Theorem 1 is not necessarily small at all. Lastly, the bound obtained in the result is invariant under the scaling of the domain (as explained in [**3**]).

When $f(x, 0) = 0$ on Ω, (D_0) has a trivial solution $u = 0$. And (1) and (2) in Theorem 1 are not enough to assure the existence of a nontrivial solution as indicated by the well known Pohozaev identity [**5**] for the case that $f(x,t) = |t|^{q-1}t$, $q > \frac{n+2}{n-2}$ and Ω is any ball (see [**6**] also). To get a non-trivial solution, additional conditions are needed. Let λ_1 be the first eigenvalue of $-\Delta$ on Ω with Dirichlet boundary conditions. Then we have

Theorem 2. *There is a constant $c(n,q)$ depending only on n and q, such that if*
(1) $(\dagger)$;
(2) $|\Omega| \leq c(n,q)\left(M_2 M_1^{q-1}\right)^{-\frac{n}{2q}}$;
(3) $\lim_{t \longrightarrow 0^+} \frac{f(x,t)}{t} > \lambda_1$ *uniformly for* $x \in \overline{\Omega}$,
then (D_0) has a positive solution.

Remark. Any function $f(x,t)$ will satisfy (3) in Theorem 2 if near $t = 0$, $t > 0$, $f(x,t)$ behaves like ct^β for some $c > 0$ and $\beta < 1$. Indeed, (3) assures that (D_0) has a family of very small positive subsolutions. And (3) can be replaced by any other conditions which assure the existence of small positive subsolutions for (D_0).

The ideas of the proofs: since there is no restriction on q, one can not use the variational method directly to solve (D_0). What we shall do is to combine a change of variable and the variational method to construct a pair of super- sub- solutions. For the purpose of illustration, we give a rough sketch of the proof of Theorem 1 here. Let $f^+(x,t) = \max\{f(x,t), 0\}$,

$f^-(x,t) = \min\{f(x,t), 0\}$. We look at a pair of quasilinear elliptic equations (α is a constant to be chosen).

$$(1) \qquad \begin{cases} -\Delta u_1 = f^+(x,u_1) + \dfrac{\alpha - 1}{u_1}|\nabla u_1|^2 & \text{in } \Omega; \\ u_1 > 0 & \text{in } \Omega; \\ u_1 = 0 & \text{on } \partial\Omega; \end{cases}$$

and

$$(2) \qquad \begin{cases} -\Delta u_2 = f^-(x,u_2) + \dfrac{\alpha - 1}{u_2}|\nabla u_2|^2 & \text{in } \Omega; \\ u_2 < 0 & \text{in } \Omega; \\ u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

If we can solve (1) and (2) for u_1 and u_2, then $u_2 \le u_1$, and we have a pair of super- sub- solutions. Thus (D_0) has a solution (for example, see Theorem 6.5 in [**4**]).

Usually it is not a good idea to solve a semilinear equation by looking at a quasilinear one. But here a change of variable will change the whole picture. For example if $q > \frac{n+2}{n-2}$, $\alpha > \frac{(q-1)(n-2)}{4}$, let $v = \frac{1}{\alpha}|u_1|^\alpha$ in (1), then v satisfies

$$\begin{cases} -\Delta v = f^+\left(x, (\alpha|v|)^{\frac{1}{\alpha}}\right)(\alpha|v|)^{\frac{(\alpha-1)}{\alpha}} & \text{in } \Omega; \\ v > 0 & \text{in } \Omega; \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus the change of variable has transformed the quasilinear equation into semilinear one with sub- critical growth! Now we can use the variational method and the method used in [**3**] to get a super- solution u_1. A sub- solution u_2 can be obtained similarly.

Acknowlegment: The author would like to thank the referee for valuable suggestions.

2. Proofs.

2.1. Proof of Theorem 1. We may assume that $f(x,0)$ is not identically zero, otherwise $u = 0$ is a trivial solution.

Step 1: Existence of a super- solution u_1.

We may assume $f^+(x_1, 0) > 0$ for some $x_1 \in \Omega$, otherwise $u_1 = 0$ is a super- solution.

Let $\alpha \geq \max\left\{\frac{(q-1)(n-2)}{4}, 1\right\}$. The exact value of α will be determined later. Consider

$$
(3) \qquad \begin{cases} -\Delta u_1 = f^+(x, u_1) + \dfrac{\alpha - 1}{u_1}|\nabla u_1|^2 & \text{in } \Omega; \\ u_1 > 0 & \text{in } \Omega; \\ u_1 = 0 & \text{on } \partial\Omega. \end{cases}
$$

Change variable $v = \frac{1}{\alpha}|u_1|^\alpha$, then v satisfies

$$
(4) \qquad \begin{cases} -\Delta v = f^+\left(x, (\alpha|v|)^{\frac{1}{\alpha}}\right)(\alpha|v|)^{\frac{(\alpha-1)}{\alpha}} & \text{in } \Omega; \\ v > 0 & \text{in } \Omega; \\ v = 0 & \text{on } \partial\Omega. \end{cases}
$$

It is clear that every solution of (4) corresponds to a solution of (3).

Set $f_1(x,v) = f^+\left(x, (\alpha|v|)^{\frac{1}{\alpha}}\right)|\alpha v|^{\frac{(\alpha-1)}{\alpha}}$. Then $f_1(x,v) \geq 0$ for all v and is Hölder continuous about v. (†) implies that for all v

$$
(5) \qquad 0 \leq f_1(x,v) \leq M_1|\alpha v|^{\frac{(\alpha-1)}{\alpha}} + M_2|\alpha v|^{\frac{(q+\alpha-1)}{\alpha}}.
$$

Here we observe that $\frac{(q+\alpha-1)}{\alpha} < \frac{n+2}{n-2}$ if $\alpha > \frac{(q-1)(n-2)}{4}$. Thus $f_1(x,v)$ has subcritical growth if $\alpha > \frac{(q-1)(n-2)}{4}$.

Consider the functional

$$
J_\alpha(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega F_1(x,v)dx, \quad v \in H_0^1(\Omega),
$$

where $F_1(x,v) = \int_0^v f_1(x,s)ds$.

We shall show that $J_\alpha(v)$ has a nontrivial critical point for suitable choice of α (and under the assumption of Theorem 1). Then the regularity theory (see [1]) and the maximum principle imply that the non-trivial critical point is a positive solution to (4).

For any $v \in H_0^1(\Omega)$, from (5) we have

$$
\int_\Omega F_1(x,v)dx \leq \frac{\alpha^{2-\frac{1}{\alpha}}}{2\alpha-1}M_1\int_\Omega |v|^{\frac{2\alpha-1}{\alpha}}dx + \alpha^{\frac{q+2\alpha-1}{\alpha}}\frac{1}{q+2\alpha-1}M_2\int_\Omega |v|^{\frac{q+2\alpha-1}{\alpha}}dx.
$$

Then

$$
\begin{aligned} J_\alpha(v) \geq \frac{1}{2}\int_\Omega |\nabla v|^2 dx &- \frac{\alpha^{2-\frac{1}{\alpha}}}{2\alpha-1}M_1\int_\Omega |v|^{\frac{2\alpha-1}{\alpha}}dx \\ &- \alpha^{\frac{q+2\alpha-1}{\alpha}}\frac{1}{q+2\alpha-1}M_2\int_\Omega |v|^{\frac{q+2\alpha-1}{\alpha}}dx. \end{aligned}
$$

Let q_1, q_2 be defined by $\frac{1}{q_1} = \frac{2}{n} + \frac{1}{\alpha}\frac{n-2}{2n}$ and $\frac{1}{q_2} = \frac{2}{n} - \frac{(q-1)}{\alpha}\frac{(n-2)}{2n}$. Using Hölder inequality and Sobolev embedding inequality (see [**8**])

$$\left(\int_\Omega |v|^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{2n}} \le S(n)\left(\int_\Omega |\nabla v|^2 dx\right)^{\frac{1}{2}} \qquad v \in H_0^1(\Omega),$$

we have

$$\begin{aligned} J_\alpha(v) \ge \frac{1}{2}\int_\Omega |\nabla v|^2 dx &- \frac{\alpha^{2-\frac{1}{\alpha}}}{2\alpha-1}S(n)^{\frac{2\alpha-1}{\alpha}}M_1|\Omega|^{\frac{1}{q_1}}\left(\int_\Omega |\nabla v|^2 dx\right)^{\frac{2\alpha-1}{2\alpha}} \\ &- \frac{1}{q+2\alpha-1}\alpha^{\frac{q+2\alpha-1}{\alpha}}S(n)^{\frac{q+2\alpha-1}{\alpha}}M_2|\Omega|^{\frac{1}{q_2}}\left(\int_\Omega |\nabla v|^2 dx\right)^{\frac{q+2\alpha-1}{2\alpha}}. \end{aligned}$$

Denote $\left(\int_\Omega |\nabla v|^2 dx\right)^{\frac{1}{2}} = \rho$, we get

$$\begin{aligned} J_\alpha(v) &\ge \frac{1}{2}\rho^2 - \frac{\alpha^{2-\frac{1}{\alpha}}}{2\alpha-1}S(n)^{\frac{2\alpha-1}{\alpha}}M_1|\Omega|^{\frac{1}{q_1}}\rho^{\frac{2\alpha-1}{\alpha}} \\ &\quad - \frac{1}{q+2\alpha-1}\alpha^{\frac{q+2\alpha-1}{\alpha}}S(n)^{\frac{q+2\alpha-1}{\alpha}}M_2|\Omega|^{\frac{1}{q_2}}\rho^{\frac{q+2\alpha-1}{\alpha}} \\ &= \left(\frac{1}{2} - \frac{1}{q+2\alpha-1}\alpha^{\frac{q+2\alpha-1}{\alpha}}S(n)^{\frac{q+2\alpha-1}{\alpha}}M_2|\Omega|^{\frac{1}{q_2}}\rho^{\frac{q-1}{\alpha}}\right)\rho^2 \\ &\quad - \frac{\alpha^{2-\frac{1}{\alpha}}}{2\alpha-1}S(n)^{\frac{2\alpha-1}{\alpha}}M_1|\Omega|^{\frac{1}{q_1}}\rho^{\frac{2\alpha-1}{\alpha}}. \end{aligned}$$

Let ρ be defined by

$$\rho = \left(\frac{4}{q+2\alpha-1}\alpha^{\frac{q+2\alpha-1}{\alpha}}S(n)^{\frac{q+2\alpha-1}{\alpha}}|\Omega|^{\frac{1}{q_2}}\right)^{-\frac{\alpha}{q-1}}M_2^{-\frac{\alpha}{q-1}}. \tag{6}$$

Then

$$J_\alpha(v) \ge \frac{1}{4}\rho^2 - \frac{\alpha^{2-\frac{1}{\alpha}}}{2\alpha-1}S(n)^{\frac{2\alpha-1}{\alpha}}M_1|\Omega|^{\frac{1}{q_1}}\rho^{\frac{2\alpha-1}{\alpha}}.$$

Thus if

$$\frac{1}{4}\rho^2 \ge \frac{\alpha^{2-\frac{1}{\alpha}}}{2\alpha-1}S(n)^{\frac{2\alpha-1}{\alpha}}M_1|\Omega|^{\frac{1}{q_1}}\rho^{\frac{2\alpha-1}{\alpha}}, \tag{7}$$

we shall have $J_\alpha(v) \ge 0$ on $\left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}} = \rho$ with ρ determined by (6). (7) is equivalent to

$$\frac{1}{4}\rho^{\frac{1}{\alpha}} \ge \frac{\alpha^{2-\frac{1}{\alpha}}}{2\alpha-1}S(n)^{\frac{2\alpha-1}{\alpha}}M_1|\Omega|^{\frac{1}{q_1}}.$$

Combining this with (6) and definitions of q_1, q_2, we have

$$|\Omega| \le c(n,q,\alpha)\left(M_2 M_1^{q-1}\right)^{-\frac{n}{2q}}, \tag{8}$$

for some constant $c(n,q,\alpha)$ depending only on n, q and α. And $c(n,q,\alpha)$ is continuous for $\alpha \ge 1$. Now we choose $\alpha = \frac{(q-1)(n-2)}{4} + 1$, denote $J_\alpha(v)$ by $J(v)$. Then there is a constant $c(n,q)$ depending only on q, n, such that if

$$|\Omega| \le c(n,q)\left(M_2 M_1^{q-1}\right)^{-\frac{n}{2q}}, \tag{9}$$

we have

$$J(v) \ge 0 \quad \text{for all} \quad v \in H_0^1(\Omega) \quad \text{with} \quad \|v\| = \rho \ \text{ given in } \ (6).$$

On the other hand, since $f^+(x_1, 0) > 0$ and $\alpha > 0$, we see that $f_1(x_1, v) \approx cv^{1-\frac{1}{\alpha}}$ for $v > 0$ small. Hence we can choose $v_1 \in H_0^1(\Omega)$ such that $\|v_1\| < \frac{1}{2}\rho$ and

$$J(v_1) < 0. \tag{10}$$

Now a standard argument in critical point theory (see [**2**] or [**6**]) implies that $J(v)$ has at least one nontrivial critical point v_2 (such that $J(v_2) < 0$).

Step 2: Existence of a sub- solution u_2.

This part is almost identical to Step 1. We just sketch here.

We may assume $f^-(x_2, 0) < 0$ for some $x_2 \in \Omega$, otherwise $u_2 = 0$ is a sub-solution.

Let $\alpha \ge \max\left\{\frac{(q-1)(n-2)}{4}, 1\right\}$. The exact value of α will be determined later. Consider

$$\begin{cases} -\Delta u_2 = f^-(x, u_2) + \dfrac{\alpha - 1}{u_2}|\nabla u_2|^2 & \text{in} \quad \Omega; \\ u_2 < 0 & \text{in} \quad \Omega; \\ u_2 = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{11}$$

Change variable $v = \frac{1}{\alpha}|u_2|^{\alpha-1}u_2$ in (11), then v satisfies

$$\begin{cases} -\Delta v = f^-\left(x, -(\alpha|v|)^{\frac{1}{\alpha}}\right)(\alpha|v|)^{\frac{(\alpha-1)}{\alpha}} & \text{in} \quad \Omega; \\ v < 0 & \text{in} \quad \Omega; \\ v = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{12}$$

It is clear that every solution of (12) corresponds to a solution of (11).

Let $f_2(x,v) = f^+\left(x, -(\alpha|v|)^{\frac{1}{\alpha}}\right)(\alpha|v|)^{\frac{(\alpha-1)}{\alpha}}$. Then $f_2(x,v) \le 0$ for all v and is Hölder continuous about v. (†) implies that for all v

$$0 \ge f_2(x,v) \ge -M_1\alpha^{\frac{(\alpha-1)}{\alpha}}|v|^{\frac{\alpha-1}{\alpha}} - M_2\alpha^{\frac{(q-1)}{\alpha}}|v|^{\frac{(q+\alpha-1)}{\alpha}}. \tag{13}$$

Once again we notice that $\frac{q+\alpha-1}{\alpha} < \frac{n+2}{n-2}$ when $\alpha > \frac{(q-1)(n-2)}{4}$. Thus $f_2(x,v)$ has sub- critical growth in v if $\alpha > \frac{(q-1)(n-2)}{4}$.

Consider the functional

$$I_\alpha(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega F_2(x,v)dx, \qquad v \in H_0^1(\Omega),$$

where $F_2(x,v) = \int_0^v f_2(x,s)ds$.

We shall show that $I_\alpha(v)$ has a nontrivial critical point for suitable value α (and under the assumptions of Theorem 1). Then the maximum principle implies that the non-trivial point is a negative solution of (12).

For $v \in H_0^1(\Omega)$, by (13), we have

$$\int_\Omega F_2(x,v)dx \le \frac{\alpha^{2-\frac{1}{\alpha}}}{2\alpha-1}M_1\int_\Omega |v|^{\frac{2\alpha-1}{\alpha}}dx + \frac{1}{q+2\alpha-1}\alpha^{\frac{q+2\alpha-1}{\alpha}}M_2\int_\Omega |v|^{\frac{q+2\alpha-1}{\alpha}}dx.$$

Thus

$$\begin{aligned} I_\alpha(v) \ge \frac{1}{2}\int_\Omega |\nabla v|^2 dx &- \frac{\alpha^{2-\frac{1}{\alpha}}}{2\alpha-1}M_1\int_\Omega |v|^{\frac{2\alpha-1}{\alpha}}dx \\ &- \frac{1}{q+2\alpha-1}\alpha^{\frac{q+2\alpha-1}{\alpha}}M_2\int_\Omega |v|^{\frac{q+2\alpha-1}{\alpha}}dx. \end{aligned}$$

As we did in Step 1, we choose $\alpha = \frac{(q-1)(n-2)}{4} + 1$. Then there is a constant $c(n,q)$ depending only on q, n, such that if $|\Omega| \le c(n,q)\left(M_2M_1^{q-1}\right)^{-\frac{n}{2q}}$, (here $I_\alpha(v)$ is denoted by $I(v)$),

$$I(v) \ge 0 \quad \text{for all} \quad v \in H_0^1(\Omega) \quad \text{with} \quad \|v\| = \rho \text{ given in } (6).$$

Since $f^-(x_2,0) < 0$ and $\alpha > 0$, we see that $f_2(x_2,v) \approx c|v|^{-\frac{1}{\alpha}}v$ for $v < 0$ small. Hence we can choose $v_3 \in H_0^1(\Omega)$ such that $\|v_3\| \le \frac{1}{2}\rho$ and

$$I(v_3) < 0.$$

Thus $I(v)$ has at least one nontrivial critical point v_4.

Step 3: Existence of at least one solution.

Since $u_2 \le u_1$ is a pair of super- sub- solutions to (D_0), (D_0) has a solution by Theorem 6.5 in [4]. □

Remark 1. From the proof we see that the choice of α is not unique. The choice of α will certainly have impact on the magnitude of the constant $c(n,q)$ in (9). Naturally one interesting question is for which value of α is the constant $c(n,q,\alpha)$ in (8) maximized. It is easy to check that the constant $c(n,q,\alpha)$ defined in (8) will tend to zero as $\alpha \longrightarrow \infty$, so one might think that $c(n,q,\alpha)$ attains the maximum value when α is small. The smallest value that α can take is $\max\left\{\frac{(q-1)(n-2)}{4}, 1\right\}$ if $q \neq \frac{n+2}{n-2}$. And if $q = \frac{n+2}{n-2}$, then α can take any value arbitrary close to 1 (but greater than 1). It is not difficult to see that in any case the constant $c(n,q)$ in Theorem 1 can be obtained by choosing $\alpha = \max\left\{\frac{(q-1)(n-2)}{4}, 1\right\}$ in $c(n,q,\alpha)$.

The proof of Theorem 1 can be modified to obtain a more general version. Let $F(x,t) = \int_0^t f(x,s)ds$, $\Omega_1 = \{x | F(x,t) \neq 0 \text{ for some } t > 0\}$, $\Omega_2 = \{x | F(x,t) \neq 0 \text{ for some } t < 0\}$, We now impose the growth conditions on $f(x,t)$ and $F(x,t)$.

(F_+) There are positive constants M_1, M_2, $q_1 \geq 1$, such that

$$\limsup_{t \longrightarrow +\infty} \frac{|f(x,t)|}{t^{q_1}} < +\infty, \tag{14}$$

and

$$|F(x,t)| \leq M_1|t| + M_2|t|^{q_1+1} \qquad \text{for all } x \in \overline{\Omega},\ t \geq 0. \tag{15}$$

(F_-) There are positive constants m_1, m_2, $q_2 \geq 1$, such that

$$\limsup_{t \longrightarrow -\infty} \frac{|f(x,t)|}{|t|^{q_2}} < +\infty, \tag{16}$$

and

$$|F(x,t)| \leq m_1|t| + m_2|t|^{q_2+1} \qquad \text{for all } x \in \overline{\Omega},\ t \leq 0. \tag{17}$$

Then we have

Theorem 1*. *There are constants $c_1(n,q_1)$, $c_2(n,q_2)$ depending only on q_1, q_2 and n, such that if we assume*
(1) *(F_+) and $|\Omega_1| \leq c_1(n,q_1)\left(M_2 M_1^{q_1-1}\right)^{-\frac{n}{2q_1}}$;*
(2) *(F_-) and $|\Omega_2| \leq c_2(n,q_2)(m_2 m_1^{q_2-1})^{-\frac{n}{2q_2}}$,*
then (D_0) has a solution.

Proof. The proof here is more or less the same as that for Theorem 1. We only indicate the necessary changes here.

Once again, we may assume that $u = 0$ is not a solution, otherwise there is nothing to prove.

Let $\phi(t)$ be a smooth function defined by $\phi(t) = 0$ if $t < 1$, $\phi(t) = 1$ if $t > 2$, and $0 \leq \phi(t) \leq 1$ on $1 \leq t \leq 2$. For any small positive constant $0 < \delta < 1$, set $f_3(x,t) = f^+(x,t) + \phi(\frac{t}{\delta})f^-(x,t)$ if $t > 0$ and $f_3(x,t) = f^+(x,0)$ if $t \leq 0$; $f_4(x,t) = f^-(x,t) + \phi(-\frac{t}{\delta})f^+(x,t)$ if $t < 0$ and $f_4(x,t) = f^-(x,0)$ if $t \geq 0$. Then $f_3(x,t) = f(x,t)$ if $t \geq 2\delta$ and $f_4(x,t) = f(x,t)$ if $t \leq -2\delta$. Consider

$$(18) \qquad \begin{cases} -\Delta u_1 = f_3(x,u_1) + \dfrac{\alpha - 1}{u_1}|\nabla u_1|^2 & \text{in } \Omega; \\ u_1 > 0 & \text{in } \Omega; \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$(19) \qquad \begin{cases} -\Delta u_2 = f_4(x,u_2) + \dfrac{\alpha - 1}{u_2}|\nabla u_2|^2 & \text{in } \Omega; \\ u_2 < 0 & \text{in } \Omega; \\ u_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

It is clear that any solution of (18) is a super- solution of (D_0) and any solution of (19) is a sub- solution of (D_0). Since $u_2 < u_1$ for any solutions u_2 and u_1 of (19) and (18) respectively, we only have to show that (18) and (19) have solutions.

Here we shall sketch the proof that (18) has a solution (under the assumption that $f^+(x,0)$ is not identically zero, otherwise 0 is a super- solution). (The proof that (19) has a solution is similar.)

Change variable $v = \frac{1}{\alpha}|u_1|^\alpha$ in (18), then v satisfies

$$(20) \qquad \begin{cases} -\Delta v = f_3\left(x, (\alpha|v|)^{\frac{1}{\alpha}}\right)(\alpha|v|)^{\frac{(\alpha-1)}{\alpha}} & \text{in } \Omega; \\ v > 0 & \text{in } \Omega; \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Consider the functional

$$J_{\alpha,\delta}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega F_3(x,v)dx, \qquad v \in H_0^1(\Omega),$$

where $F_3(x,v) = \int_0^v f_3\left(x, (\alpha|s|)^{\frac{1}{\alpha}}\right)(\alpha|s|)^{\frac{(\alpha-1)}{\alpha}} ds$.

Since $f_3(x,v) \geq 0$ when $v \leq \frac{\delta^\alpha}{\alpha}$, the maximum principle concludes that any non-trivial critical point of $J_{\alpha,\delta}(v)$ is a positive solution to (20).

Now let us show that $J_{\alpha,\delta}(v)$ has a non-trivial critical point for some small δ and $\alpha = \max\left\{\frac{(q_1-1)(n-2)}{4}, 1\right\}$.

For $v > 0$,

$$
\begin{aligned}
F_3(x,v) &= \int_0^v f_3(x,(\alpha|s|)^{\frac{1}{\alpha}})(\alpha|s|)^{\frac{(\alpha-1)}{\alpha}}\,ds \\
&= \int_0^v \left\{ f^+\left(x,(\alpha|s|)^{\frac{1}{\alpha}}\right) + \phi\left(\frac{(\alpha|s|)^{\frac{1}{\alpha}}}{\delta}\right) f^-\left(x,(\alpha|s|)^{\frac{1}{\alpha}}\right)\right\}(\alpha|s|)^{\frac{(\alpha-1)}{\alpha}}\,ds \\
&= \int_0^v f\left(x,(\alpha|s|)^{\frac{1}{\alpha}}\right)(\alpha|s|)^{\frac{(\alpha-1)}{\alpha}}\,ds \\
&\quad + \int_0^v \left(\phi\left(\frac{(\alpha|s|)^{\frac{1}{\alpha}}}{\delta}\right) - 1\right) f^-\left(x,(\alpha|s|)^{\frac{1}{\alpha}}\right)(\alpha|s|)^{\frac{(\alpha-1)}{\alpha}}\,ds \\
&\le \int_0^v f\left(x,(\alpha|s|)^{\frac{1}{\alpha}}\right)(\alpha|s|)^{\frac{(\alpha-1)}{\alpha}}\,ds + c(f,n,q_1)\delta^\alpha \\
&= \int_0^{(\alpha v)^{\frac{1}{\alpha}}} f(x,z)z^{2(\alpha-1)}\,dz + c(f,n,q_1)\delta^\alpha \\
&= F\left(x,(\alpha v)^{\frac{1}{\alpha}}\right)(\alpha v)^{\frac{2(\alpha-1)}{\alpha}} \\
&\quad - 2(\alpha-1)\int_0^{(\alpha v)^{\frac{1}{\alpha}}} F(x,z)z^{2\alpha-3}\,dz + c(f,n,q_1)\delta^\alpha \\
&\le F\left(x,(\alpha v)^{\frac{1}{\alpha}}\right)(\alpha v)^{\frac{2(\alpha-1)}{\alpha}} \\
&\quad + 2(\alpha-1)\int_0^{(\alpha v)^{\frac{1}{\alpha}}} |F(x,z)|z^{2\alpha-3}\,dz + c(f,n,q_1)\delta^\alpha \\
&\le \left(M_1(\alpha v)^{\frac{1}{\alpha}} + M_2(\alpha v)^{\frac{q_1+1}{\alpha}}\right)(\alpha v)^{\frac{2\alpha-2}{\alpha}} \\
&\quad + 2(\alpha-1)\int_0^{(\alpha v)^{\frac{1}{\alpha}}} (M_1 z + M_2 z^{q_1+1})z^{2\alpha-3}\,ds + c(f,n,q_1)\delta^\alpha \\
&= \frac{4\alpha-3}{2\alpha-1}M_1(\alpha v)^{\frac{2\alpha-1}{\alpha}} + \frac{q_1+4\alpha-3}{q_1+2\alpha-1}M_2(\alpha v)^{\frac{q_1+2\alpha-1}{\alpha}} + c(f,n,q_1)\delta^\alpha.
\end{aligned}
$$

Now as we did in the proof of Theorem 1, it follows that there are constants $c(n,q_1)$ and ρ_1 depending only on n, q_1, such that

$$
\begin{aligned}
&\text{if}\quad |\Omega_1| \le c(n,q_1)\left(M_2 M_1^{q_1-1}\right)^{-\frac{n}{2q_1}}, \\
&J(v) \ge -c(f,n,q_1)\delta^\alpha \qquad \text{for all}\quad v\in H_0^1(\Omega)\ \text{ with }\ \|v\| = \rho_1.
\end{aligned}
$$

On the other hand, $f^+(x_1,0)\neq 0$ for some $x_1\in\Omega$ implies that we can choose a v_5 independent of δ, such that $\|v_5\| < \frac{1}{2}\rho_1$ and $J(v_5) < 0$. Now if we choose a $\delta > 0$ such that

$$-c(f, n, q_1)\delta^\alpha > J(v_5),$$

we see that $J(v)$ has a nontrivial critical point v_6 such that $\|v_6\| < \rho_1$ and $J(v_6) < J(v_5) < 0$. Thus there is a solution to (18).

The rest of the proof is clear. □

Remark 2. Since conditions (F_+) and (F_-) are imposed on $F(x,t)$, the behavior of $f(x,t)$ can be quite different. Furthermore the q_1 in (14) and (15) and the q_2 in (16) and (17)) can be two different numbers. That is, $f(x,t)$ and $F(x,t)$ can have different growth rates. If this is the case, the constant $c(n, q_1)$ will be changed accordingly. Finally if $q_1 < \frac{n+2}{n-2}$, we can take $\alpha = 1$ in the proof and replace $F(x,t)$ by $F^+(x,t) = \max\{F(x,t), 0\}$ in (14). Thus we have recovered the main result in [**3**].

When $f(x,0) = 0$, (D_0) has a trivial solution $u = 0$. Then the main interest in this case is in non-trivial solutions. On the other hand, the conditions in Theorem 1 are not enough to assure a nontrivial solution. Indeed, if $f(x,t) = |t|^{q-1}t$ with $q > \frac{n+2}{n-2}$, the well known Pohozaev identity [**5**] concludes that (D_0) does not have any non-trivial solutions for any ball Ω. To get a nontrivial solution for (D_0), we use an additional condition 3) in Theorem 2. Basically 3) in Theorem 2 assures that (D_0) has a very small positive sub- solution.

2.2. Proof of Theorem 2. Since $\lim_{t\longrightarrow 0^+} \frac{f(x,t)}{t} > \lambda_1$, there is a $d > 0$, such that $f(x,t) > \lambda_1 t$ for $0 < t < d$. Then for any $0 < \delta < d$, $u_2 = \delta\varphi(x)$ is a sub- solution for (D_0), where $\varphi(x)$ is the positive first eigenfunction of $-\Delta$ on Ω with Dirichlet boundary conditions and $\max_{\{x\in\Omega\}}\varphi(x) = 1$.

Now define

$$f^*(x,t) = \begin{cases} f(x,0) & \text{if } \ t \le 0; \\ f(x,t) & \text{if } \ t > 0. \end{cases}$$

Then $f^*(x,t)$ satisfies (†) with the same constants M_1 and M_2 as used by $f(x,t)$.

Consider

$$(P^*) \qquad \begin{cases} -\Delta v = f^*(x,v) + \dfrac{\alpha - 1}{v}|\nabla u|^2 & \text{in } \ \Omega; \\ v = 0 & \text{on } \ \partial\Omega. \end{cases}$$

As we did in the Step 1 of the proof of Theorem 1, (P^*) has a positive solution $v > 0$ (under the assumptions (1) and (2) of Theorem 2, and we shall use $\lim_{t\longrightarrow 0^+} \frac{f(x,t)}{t} > \lambda_1$ to find v_1 satisfying (10)). In particular v is a super- solution for (D_0). Since $f(x,t) > 0$ for $t > 0$ small, an application of maximum principle implies that $v(x) \geq \delta_1\varphi(x)$ on Ω for some positive constant δ_1.

Now fix a $0 < \delta < \delta_1$, then $u_2 = \delta\varphi(x) < v$, and u_2, v is a pair of super-sub- solutions. Therefore (D_0) has a positive solution. $\square$

Remark 3. If $f(x,t)$ is C^1 near $t = 0$ in Theorem 2, we see that (D_0) has two solutions $u_1 > 0$ and $u_2 < 0$.

Remark 4. It is straightforward to modify the method used here to obtain similar results for Dirichlet problems for a second order elliptic equations in divergent form

$$\begin{cases} -\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = f(x,u) & \text{in} \quad \Omega; \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

but now the constant $c(n,q)$ will depends on the dimension n, growth exponent q and the smallest eigenvalue of the positive matrix $(a_{ij}(x))$ on $\overline{\Omega}$.

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Received August 31, 1994 and revised January 26, 1995.

Wichita State University
Wichita, KS 67260-0033
E-mail address: zhiren@cs.twsu.edu