Existence of vector bundles of rank two with fixed determinant and sections

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Abstract: Consider the scheme $B^k_{2,L}$ of stable vector bundles of rank two and fixed determinant L which have at least k sections. Under suitable numerical conditions and for generic L, we show the existence of a component of the expected dimension of $B^k_{2,L}$.

Key words: Vector bundle; Brill-Noether.

1. Introduction. The determinant of a vector bundle is the line bundle obtained as its highest wedge power. Given a projective non-singular curve of genus g and a line bundle L on it, there exists a moduli space of dimension $(r^2-1)(g-1)$ parameterizing vector bundles of fixed rank and with the given line bundle as determinant. We are interested in the Brill-Noether locus $B^k_{r,L}$ consisting of stable vector bundles of rank r with determinant L with at least k independent sections. As in the case that one fixes only the degree and not the determinant (see [GT] for an overview of results in that case), this locus can be represented as a determinatal variety. Therefore, its expected dimension is given by the Brill-Noether number

$$\rho^k_{r,L} = (r^2-1)(g-1) - k(k-d+r(g-1)).$$

While there has been substantial progress in our knowledge of the non-fixed determinant case, no work has been done so far in trying to see how far reality is from this expectation. The purpose of this paper is to partially fill this gap by showing existence of a component of the expected dimension for stable vector bundles of rank two with fixed generic determinant and a preassigned number of sections if the degree is sufficiently large.

For small degree d or for special line bundles, things are expected to behave differently. It is known that when the determinant is the canonical, the locus has dimension larger than the number above (see [M1,M2,BF,T4,T5]). Osserman recently extended this set up to other special line bundles [O].

In this paper, we deal with the case in which $r=2$ The result that we obtain is the following

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1.1. Theorem. *Let C be a generic curve of genus g and L a generic line bundle on C, $B^k_{2,L}$ is non-empty and has a component of the expected dimension ρ if $\rho \geq g-3$ for k even. For $k = 2k_1+1$ odd, $B^k_{2,L}$ is non-empty and has a component of dimension the expected dimension if $(k_1+1)(k-d+2(g-1)) \leq g-1$.*

2. Review of some technical results. The main point of the proof is the following fact (a similar argument was already used in [T1,T5]): the dimension of $B^k_{2,L}$ at any point is at least ρ. One can also consider the case of a family of curves

$$\mathcal{C} \to T$$

and a line bundle $\mathcal{L}$ on $\mathcal{C}$ of degree d on every fiber. Define

$$\mathcal{B}^k_{2,\mathcal{L}} = \{(b,E) | b \in B,\ E \in B^k_{2,\mathcal{L}_b}(C_b)\}.$$

Then,

$$dim \mathcal{B}^k_{2,\mathcal{L}} \geq \rho^k_{2,L} + dim B$$

at every point. If one can find a particular curve C_0 such that the dimension of $B^k_{2,L}(C_0)$ is ρ, then the dimension of the generic fiber of the map $\mathcal{B}^k_{2,L} \to T$ attains its minimum in a neighborhood of the curve. Hence, for a generic curve C in an open neighborhood of C_0, the dimension of $B^k_{2,L}(C)$ is ρ (and the locus is non-empty). We only need to explicitly exhibit a curve C_0 and the corresponding family of vector bundles in $B^k_{2,L}(C_0)$. Our C_0 is a reducible curve that we define as follows:

2.1. Definition. *Let $C_1 \dots C_g$ be elliptic curves. Let P_i, Q_i be generic points in C_i. Then C_0 is the chain obtained by gluing the elliptic curves when identifying the point Q_i in C_i to the point P_{i+1} in C_{i+1}, $i = 1 \dots g-1$.*

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When dealing with reducible curves, the notion of a line bundle and a space of its sections needs to be replaced by the analogous concept of limit linear series as introduced by Eisenbud and Harris (cf [EH]). A similar definition can be given for vector bundles (cf [T1,T6]). For the convenience of the reader, we reproduce this definition here

2.2. Limit linear series. *A limit linear series of rank r, degree d and dimension k on a chain of M* (*not necessarily elliptic*) *curves consists of data* I, II *below for which data* III, IV *exist satisfying conditions a*)*–c*)

(I) *For every component C_i, a vector bundle E_i of rank r and degree d_i and a k-dimensional space V_i of sections of E_i.*

(II) *For every node obtained by gluing Q_i and P_{i+1}, an isomorphism of the projectivisation of the fibers $(E_i)_{Q_i}$ and $(E_{i+1})_{P_{i+1}}$.*

(III) *A positive integer b.*

(IV) *For every node obtained by gluing Q_i and P_{i+1} basis $s^t_{Q_i}, s^t_{P_{i+1}}$, $t = 1 \dots k$ of the vector spaces V_i, V_{i+1} in* (I).

Subject to the conditions

(*a*) $\sum_{i=1}^{M} d_i - r(M-1)b = d$.

(*b*) *The orders of vanishing at Q_i, P_{i+1} of the sections of the chosen basis satisfy $ord_{Q_i} s^t_{Q_i} + ord_{P_{i+1}} s^t_{P_{i+1}} \geq b$.*

(*c*) *Sections of the vector bundles $E_i(-bP_i)$, $E_i(-bQ_i)$ are completely determined by their value at the nodes.*

Notation. We shall denote by u^i_j, v^i_j $j = 1 \dots k$, $i = 1 \dots g$ the value of these vanishing. We shall always assume that the u^i_j are in increasing order and the v^i_j are in decreasing order, namely

$$u^i_1 \leq u^i_2 \leq \dots \leq u^i_k,\ v^i_1 \geq v^i_2 \geq \dots \geq v^i_k.$$

At most two of the u^i can take a given value (say $u^i_{j-1} < u^i_j = u^i_{j+1} < u^i_{j+2}$ and when this happens, there are two linearly independent sections of $E_i(-u^i_j P_i)$ that vanish at P_i with multiplicity exactly u^i_j and generate the fiber of $E_i(-u^i_j P_i)$ at this point. Analogous statements can be given for the vanishing at Q_i.

2.3. Notation. We shall write

$$d = 2d_1 + \epsilon,\ k = 2k_1 + \delta,\ 0 \leq \epsilon,\ \delta \leq 1.$$

We shall use $b = d_1$.

To simplify notations, we shall omit the superindex i when the curve is clear. In order to optimize the vanishing, we always take the u^1_j at P_1 to be the smallest possible, namely $(0, 0, 1, 1, 2, 2 \dots)$ and for $i > 1$, take $u^i_j = b - v^{i-1}_j = d_1 - v^{i-1}_j$ where b is as defined in 2.2 III.

2.4. Remark. We need to prove that some vector bundles we construct on the reducible curves are stable. From [T2] Step 2, p. 342 and 2.4 Prop 1.2 it is enough to see that the restrictions to each of the components is semistable and at least one of them is stable or in the case they are all strictly semistable, the destabilizing subbundles do not glue with each other.

3. Even degree and number of sections. We start by considering the case of even degree and even number of sections. So

$$d = 2d_1,\ k = 2k_1.$$

We define

$$\alpha = k_1 - d_1 + g - 1.$$

The condition $\rho \geq g - 3$ can then be written as $2k_1(k_1 - d + g - 1) \leq g$ or equivalently

$$(*) \quad 2k_1\alpha \leq g.$$

We construct a vector bundle on a curve C_0 as in 2.1 with fixed determinant and a k-dimensional limit linear series.

Define, for $i > 1$

$$u^i_j = d_1 - v^{i-1}_j, j = 1 \dots k.$$

On the curve C_1, take the vector bundle

$$(\mathcal{O}(d_1 Q_1)) \oplus L_1$$

where L_1 has been chosen so that the determinant is as preassigned (and by our genericity assumption for the determinant, this implies it is generic).

On the curve C_i, $i = k_1 t + j$, $j = 2 \dots k_1$, $t = 0, \cdots 2\alpha - 2$, take the vector bundle to be

$$(\mathcal{O}(u^i_{2j-1} P_i + (d_1 - u^i_{2j-1}) Q_i)) \oplus L_i$$

where L_i has been chosen so that the determinant is as preassigned. Glue this curve to the previous one so that L_i glues with the direction of the section on C_{i-1} with vanishing v^{i-1}_{2j-2} at Q_{i-1}. Then

$$v^i_t = d_1 - u^i_t - 1,\ t \neq 2j - 1$$

and

$$v^i_{2j-1} = d_1 - u^i_{2j-1}.$$

On the curve C_i, $i = k_1 t + 1$, $t = 1, \cdots 2\alpha - 2$, take the vector bundle to be the unique indecomposable vector bundle of rank two of degree $2d_1+1$ with preassigned determinant if t is odd and of degree $2d_1-1$ with preassigned determinant if t is even. This vector bundle has a unique section that vanishes at P_i with multiplicity u_1^i and at Q_i with multiplicity $d_1-\epsilon-u_1^i$ where $\epsilon=1$ if t is even, $\epsilon=0$ if t is odd. It has a second section that vanishes at P_i with multiplicity $u_{2k_1}^i$ and at Q_i with multiplicity $d_1-\epsilon-u_{2k_1}^i$ where $\epsilon=1$ if t is even. Glue these two sections with the directions of the sections on C_{i-1} that vanish to order v_1^{i-1} and $v_{2k_1}^{i-1}$ respectively. Then if t is even or $t=1$

$$v_t^i = d_1 - u_t^i - \epsilon$$

and for odd $t \neq 1$

$$v_t^i = d_1 - u_t^i - 1.$$

On the curve $C_i, i=k_1(2\alpha-1)+j,\ j=1\dots k_1-1$, take the vector bundle to be

$$(\mathcal{O}(u_{2j}^i P_i + (d_1-u_{2j}^i)Q_i)) \oplus L_i$$

where L_i has been chosen so that the determinant is as preassigned. Glue this curve to the previous one so that L_i glues with the direction of the section on C_{i-1} with vanishing v_{2j-1}^{i-1} at Q_{i-1}. Then

$$v_t^i = d_1 - u_t^i - 1,\ t \neq 2j$$

and

$$v_{2j}^i = d_1 - u_{2j}^i.$$

On the curve $C_i,\ i=k_1(2\alpha-1)+k_1=2k_1\alpha$, take the vector bundle to be

$$(\mathcal{O}(u_{2k_1}^i P_i + (d_1-u_{2k_1}^i)Q_i)) \oplus L_i$$

where L_i has been chosen so that the determinant is as preassigned. Glue this curve to the previous one so that L_i glues with the direction of the section on C_{i-1} with vanishing $v_{2k_1-1}^{i-1}$ at Q_{i-1} and $(\mathcal{O}(u_{2k_1}^i P_i + (d_1-u_{2k_1}^i)Q_i))$ glues with the direction with vanishing $v_{2k_1}^{i-1}$ at Q_{i-1}. Then

$$v_t^i = d_1 - u_t^i - 1,\ t \neq 2k_1$$

and

$$v_{2k_1}^i = d_1 - u_{2k_1}^i.$$

Note that, by our assumptions (se $(*)$), $g-2k_1\alpha \geq 0$.

On the remaining $g-2k_1\alpha$ components, take the vector bundle to be the direct sum of two line bundles of degree d_1 such that their tensor product is the preassigned determinant. Take the gluing among these components to be generic

$$v_j^i = d_1 - u_j^i - 1.$$

Hence,

$$\begin{aligned}(v_1^g, v_2^g, \dots v_{k-1}^g, v_k^g) &= (d_1-g+\alpha, d_1-g+\alpha, \dots\\ &\quad d_1-g+\alpha-(k_1-1), d_1-g+\alpha-(k_1-1))\\ &= (k_1-1, k_1-1, \dots 0, 0).\end{aligned}$$

The vanishings at Q_g are the smallest possible. Hence, we cannot make the vector bundles more general and still obtain a linear series of dimension k.

We now compute the number of moduli of such a family. The restrictions of the vector bundles to the first $2k_1\alpha$ components are completely determined. On the remaining $g-2k_1\alpha$ components, the restriction depends on one parameter (see [A]). The gluing at each of the last $g-2k_1\alpha$ nodes is generic and therefore depends on four parameters. At the nodes $k_1t,\ 1\leq t\leq 2\alpha-2$, they depend on two parameters while at the remaining nodes, they depend on three parameters. Each of the vector bundles obtained as the restriction to a component has a two dimensional family of automorphisms except for the vector bundles on the components $k_1t+1,\ 1\leq t\leq 2\alpha-2$ which have only one. The resulting vector bundle on the reducible curve has a one-dimensional family of automorphisms, as it is stable. Hence, the number of moduli for the family is

$$\begin{aligned}&g-2k_1\alpha+4(g-2k_1\alpha)+2(2\alpha-2+1)\\ &\quad +3(2k_1\alpha-(2\alpha-2)-2)-2(g-(2\alpha-2))\\ &\quad -(2\alpha-2)+1=\rho.\end{aligned}$$

4. Odd degree and even number of sections. Now

$$d=2d_1+1,\ k=2k_1.$$

We define

$$\alpha = k_1 - d_1 + g - 1.$$

Now the condition $\rho \geq g-3$ can then be written as

$$(*) \quad k_1(2\alpha-1)\leq g.$$

This case is very similar to the previous one, except that we include an odd number of vector bundles of odd degree.

On the curve C_1, take the vector bundle

$$(\mathcal{O}(d_1Q_1)) \oplus L_1$$

where L_1 has been chosen so that the determinant is as preassigned (and by our genericity assumption for the determinant, this implies it is generic).

On the curve $C_i, i = k_1t + j,\quad j = 2 \dots k_1$, $t = 0, \cdots 2\alpha - 3$, take the vector bundle to be

$$(\mathcal{O}(u^i_{2j-1}P_i + (d_1 - u^i_{2j-1})Q_i)) \oplus L_i$$

where L_i has been chosen so that the determinant is as preassigned. Glue this curve to the previous one so that L_i glues with the direction of the section on C_{i-1} with vanishing v^{i-1}_{2j-2} at Q_{i-1}. Then

$$v^i_l = d_1 - u^i_l - 1,\ l \neq 2j - 1$$

and

$$v^i_{2j-1} = d_1 - u^i_{2j-1}.$$

On the curve $C_i, i = k_1t + 1, t = 1, \cdots 2\alpha - 1$, take the vector bundle to be the unique indecomposable vector bundle of rank two and degree $2d_1 + 1$ with preassigned determinant if t is odd and of degree $2d_1 - 1$ with preassigned determinant if t is even. This vector bundle has a unique section that vanishes at P_i with multiplicity u^i_1 and at Q_i with multiplicity $d_1 - \epsilon - u^i_1$ where $\epsilon = 1$ if t is even, $\epsilon = 0$ if t is odd. It has a second section that vanishes at P_i with multiplicity $u^i_{2k_1}$ and at Q_i with multiplicity $d_1 - \epsilon - u^i_{2k_1}$ where $\epsilon = 1$ if t is even. Glue these two sections with the directions of the sections on C_{i-1} that vanish to order v^{i-1}_1 and $v^{i-1}_{2k_1}$ respectively. Then if l is even or $l = 1$

$$v^i_l = d_1 - u^i_l - \epsilon$$

and for odd $l \neq 1$

$$v^i_l = d_1 - u^i_l - 1.$$

On the curve $C_i, i = k_1(2\alpha - 2) + j,\ j = 1 \dots k_1 - 1$, take the vector bundle to be

$$(\mathcal{O}(u^i_{2j}P_i + (d_1 - u^i_{2j})Q_i)) \oplus L_i$$

where L_i has been chosen so that the determinant is as preassigned. Glue this curve to the previous one so that L_i glues with the direction of the section on C_{i-1} with vanishing v^{i-1}_{2j-1} at Q_{i-1}. Then

$$v^i_t = d_1 - u^i_t - 1,\ t \neq 2j$$

and

$$v^i_{2j} = d_1 - u^i_{2j}.$$

On the curve $C_i, i = k_1(2\alpha - 2) + k_1 = k_1(2\alpha - 1)$, take the vector bundle to be

$$(\mathcal{O}(u^i_{2k_1}P_i + (d_1 - u^i_{2k_1})Q_i)) \oplus L_i$$

where L_i has been chosen so that the determinant is as preassigned. Glue this curve to the previous one so that L_i glues with the direction of the section on C_{i-1} with vanishing $v^{i-1}_{2k_1-1}$ at Q_{i-1} and $(\mathcal{O}(u^i_{2j}P_i + (d_1 - u^i_{2j})Q_i))$ glues with the direction with vanishing $v^{i-1}_{2k_1}$ at Q_{i-1}. Then

$$v^i_l = d_1 - u^i_l - 1,\ t \neq 2k_1$$

and

$$v^i_{2k_1} = d_1 - u^i_{2k_1}.$$

Note that by our assumptions (see $(*)$), $g - k_1(2\alpha - 1) \geq 0$. On the remaining $g - k_1(2\alpha - 1)$ components, take the vector bundle to be the direct sum of two line bundles of degree d_1 such that their tensor product is the preassigned determinant. Take the gluing among these components to be generic

$$v^i_j = d_1 - u^i_j - 1.$$

Hence,

$$\begin{aligned}(v^g_1, v^g_2, \dots v^g_{k-1}, v^g_k) &= (d_1 - g + \alpha, d_1 - g + \alpha, \dots \\ &\quad d_1 - g + \alpha - (k_1 - 1), d_1 - g + \alpha - (k_1 - 1)) \\ &= (k_1 - 1, k_1 - 1, \dots 0, 0).\end{aligned}$$

The vanishings at Q_g are the smallest possible. Hence, we cannot make the vector bundles more general and still obtain a linear series of dimension k.

We now compute the number of moduli of such a family. The restrictions of the vector bundles to the first $k_1(2\alpha - 1)$ components are completely determined. On the remaining $g - k_1(2\alpha - 1)$ components, the restriction depends on one parameter. The gluing at each of the last $g - k_1(2\alpha - 1)$ nodes is generic and therefore depends on four parameters. At the nodes k_1t, $1 \leq t \leq 2\alpha - 3$ or the node $k_1(2\alpha - 1) - 1$, they depend on two parameters while at the remaining nodes, they depend on three parameters. Each of the vector bundles obtained as the restriction to a component has a two dimensional family of automorphisms except for the vector bundles on the components $k_1t + 1$, $1 \leq t \leq 2\alpha - 3$ which have only one. The resulting vector bundle on the reducible curve has a one-dimensional family of automorphisms, as it is stable. Hence, the number of moduli for the family is

$$g - k_1(2\alpha - 1) + 4(g - k_1(2\alpha) - 1)$$
$$+2(2\alpha - 2) + 3(k_1(2\alpha - 1) -$$
$$-(2\alpha - 2) - 1) - 2(g - 2(g - (2\alpha - 3))$$
$$-(2\alpha - 3) + 1 = \rho.$$

5. Even degree and odd number of sections. Write

$$d = 2d_1,\ k = 2k_1 + 1,\ \alpha = k_1 - d_1 + g - 1.$$

On the curve $C_i, i = (k_1 + 1)t + 1,\ \ t = 0, \cdots 2\alpha + 1$, take the vector bundle to be the unique indecomposable vector bundle of rank two and degree $2d_1 + 1$ with preassigned determinant if t is even and of degree $2d_1 - 1$ with preassigned determinant if t is odd. This vector bundle has a unique section that vanishes at P_i with multiplicity $u^i_{2k_1+1}$ and at Q_i with multiplicity $d_1 - \epsilon - u^i_{2k_1}$ where $\epsilon = 1$ if t is odd. Glue this section with the directions of the sections on C_{i-1} that vanishes to order $v^{i-1}_{2k_1+1}$. Then if l is odd

$$v^i_l = d_1 - u^i_l - \epsilon$$

and for even l

$$v^i_l = d_1 - u^i_l - 1 - \epsilon.$$

On the curve $C_i,\ i = (k_1 + 1)t + j,\ j = 2 \ldots k_1 + 1, t = 0, \cdots 2\alpha$, take the vector bundle to be

$$(\mathcal{O}(u^i_{2j-2}P_i + (d_1 - u^i_{2j-2})Q_i)) \oplus L_i$$

where L_i has been chosen so that the determinant is as preassigned. Glue this curve to the previous one so that L_i glues with the direction of the section on C_{i-1} with vanishing v^{i-1}_{2j-3} at Q_{i-1}. Then

$$v^i_l = d_1 - u^i_l - 1,\ l \neq 2j - 2$$

and

$$v^i_{2j-2} = d_1 - u^i_{2j-1}.$$

The condition that we are assuming, namely $(k_1 + 1)(k - d + 2(g - 1)) \leq g - 1$ can be written as $g - (k_1 + 1)(2\alpha + 1) - 1 \geq 0$.

On the remaining $g - (k_1 + 1)(2\alpha + 1) - 1$ components, take the vector bundle to be the direct sum of two line bundles of degree d_1 such that their tensor product is the preassigned determinant. Take the gluing among these components to be generic. Then,

$$v^i_j = d_1 - u^i_j - 1.$$

Hence,

$$(v^g_1, v^g_2, \ldots v^g_{k-1}, v^g_k) = (d_1 - g + \alpha, d_1 - g + \alpha, \ldots$$
$$d_1 - g + \alpha - (k_1 - 1), d_1 - g + \alpha - (k_1 - 1))$$
$$= (k_1 - 1, k_1 - 1, \ldots 0, 0).$$

The vanishings at Q_g are the smallest possible. Hence, we cannot make the vector bundles or gluing more general and still obtain a linear series of dimension k.

We now compute the number of moduli of such a family. The restrictions of the vector bundles to the first $(k_1 + 1)(2\alpha + 1) + 1$ components are completely determined. On the remaining $g - (k_1 + 1)(2\alpha + 1) - 1)$ components, the restriction depends on one parameter. The gluing at each of the last $g - (k_1 + 1)(2\alpha + 1) - 1)$ nodes is generic and therefore depends on four parameters. At the remaining nodes it depends on three parameters. Each of the vector bundles obtained as the restriction to a component has a two dimensional family of automorphisms except for the vector bundles on the components $(k_1 + 1)t + 1,\ 0 \leq t \leq 2\alpha + 1$ which have only one. The resulting vector bundle on the reducible curve has a one-dimensional family of automorphisms, as it is stable (see 2.4.). Hence, the number of parameters for the family is

$$g - (k_1 + 1)(2\alpha + 1)$$
$$-1 + 4(g - (k_1 + 1)(2\alpha + 1)) - 1)$$
$$+3((k_1 + 1)(2\alpha + 1) -$$
$$-2(g - 2\alpha - 2) - (2\alpha + 2) + 1 = \rho.$$

6. Odd degree and odd number of sections. This case is similar to the previous one. Write

$$d = 2d_1 + 1,\ k = 2k_1 + 1,\ \alpha = k_1 - d_1 + g - 1.$$

On the curve $C_i,\ i = (k_1 + 1)t + 1, t = 0, \cdots 2\alpha$, take the vector bundle to be the unique indecomposable vector bundle of rank two and degree $2d_1 + 1$ with preassigned determinant if t is even and of degree $2d_1 - 1$ with preassigned determinant if t is odd. This vector bundle has a unique section that vanishes at P_i with multiplicity $u^i_{2k_1+1}$ and at Q_i with multiplicity $d_1 - \epsilon - u^i_{2k_1}$ where $\epsilon = 1$ if t is odd. Glue this section with the directions of the sections on C_{i-1} that vanishes to order $v^{i-1}_{2k_1+1}$. Then if l is odd

$$v^i_l = d_1 - u^i_l - \epsilon$$

and for even l

$$v_l^i = d_1 - u_l^i - 1 - \epsilon.$$

On the curve C_i, $i = (k_1+1)t + j$, $j = 2 \ldots k_1+1, t = 0, \cdots 2\alpha - 1$, take the vector bundle to be

$$(\mathcal{O}(u_{2j-2}^i P_i + (d_1 - u_{2j-2}^i)Q_i)) \oplus L_i$$

where L_i has been chosen so that the determinant is as preassigned. Glue this curve to the previous one so that L_i glues with the direction of the section on C_{i-1} with vanishing v_{2j-3}^{i-1} at Q_{i-1}. Then

$$v_l^i = d_1 - u_l^i - 1,\ l \neq 2j-2$$

and

$$v_{2j-2}^i = d_1 - u_{2j-1}^i.$$

The condition that we are assuming, namely $(k_1+1)(k-d+2(g-1)) \leq g-1$ can be written as $g-(k_1+1)2\alpha - 1 \geq 0$.

On the remaining $g-(k_1+1)2\alpha-1$ components, take the vector bundle to be the direct sum of two line bundles of degree d_1 such that their tensor product is the preassigned determinant. Take the gluing among these components to be generic. Then,

$$v_j^i = d_1 - u_j^i - 1.$$

Hence,

$$(v_1^g, v_2^g, \ldots v_{k-1}^g, v_k^g) = (d_1 - g + \alpha, d_1 - g + \alpha, \ldots$$
$$d_1 - g + \alpha - (k_1 - 1), d_1 - g + \alpha - (k_1 - 1))$$
$$= (k_1 - 1, k_1 - 1, \ldots 0, 0).$$

The vanishings at Q_g are the smallest possible. Hence, we cannot make the vector bundles or gluing more general and still obtain a linear series of dimension k.

We now compute the number of moduli of such a family. The restrictions of the vector bundles to the first $(k_1+1)2\alpha+1$ components are completely determined. On the remaining $g-(k_1+1)2\alpha-1$ components, the restriction depends on one parameter. The gluing at each of the last $g-(k_1+1)2\alpha-1$ nodes is generic and therefore depends on four parameters. At the remaining nodes it depends on three parameters. Each of the vector bundles obtained as the restriction to a component has a two dimensional family of automorphisms except for the vector bundles on the components $(k_1+1)t+1$, $0 \leq t \leq 2\alpha$ which have only one. The resulting vector bundle on the reducible curve has a one-dimensional family of automorphisms, as it is stable (see 2.4). Hence, the number of parameters for the family is

$$g - (k_1+1)2\alpha - 1 + 4(g - (k_1+1)(2\alpha) - 1)$$
$$+3((k_1+1)(2\alpha) -$$
$$-2(g - (2\alpha+1)) - (2\alpha+1) + 1 = \rho.$$

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