# Extension of the Beurling's Theorem

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**Abstract:** Under some conditions on a Hilbert space $H$ of analytic functions on the open unit disc we will show that for every nontrivial invariant subspace $\mathcal{M}$ of $H$, there exists a unique nonconstant inner function $\varphi$ such that $\mathcal{M} = \varphi H$. This extends the Beurling's Theorem.

**Key words:** Invariant subspaces; reproducing kernels; inner functions; multipliers.

**1. Introduction.** Consider a Hilbert space $H$ of functions analytic on a plane domain $\Omega$, such that $H$ contains the constant functions and for each $\lambda \in \Omega$ the linear functional $e_\lambda$ of evaluation at $\lambda$ is bounded on $H$. The continuity of point evaluations along with the Riesz representation theorem imply that for each $\lambda \in \Omega$ there is a unique function $K_\lambda \in H$ such that $f(\lambda) = < f, K_\lambda >$, $f \in H$. The function $K_\lambda$ is the reproducing kernel for the point $\lambda$. For a good source on this topic see [1,2].

A complex valued function $\varphi$ on $\Omega$ for which $\varphi f \in H$ for every $f \in H$ is called a multiplier of $H$ and the collection of all multipliers is denoted by $M(H)$. Each multiplier $\varphi$ of $H$ determines a multiplication operator $M_\varphi$ on $H$ by $M_\varphi f = \varphi f$, $f \in H$. Clearly $M(H) \subset H^\infty(\Omega)$, i.e., each multiplier is a bounded analytic function on $\Omega$. In fact $\|\varphi\|_\Omega \le \|M_\varphi\|$. A good source on this topic is [6].

Thirty seven years after the appearance of [6] it is reasonable to expect some words explaining the motivation of such a study and of any developments in the area. The description of invariant subspaces in abstract spaces has in fact appeared under some additional hypotheses and one of the first results seems to be [5].

In his paper Shapiro [5] uses a construction and the idea of which is employed in the main theorem of this paper. For a good source of invariant subspaces see [3,4]. We studied some properties of the multiplication operators on Hilbert spaces of analytic functions in [7,8] and now we want to investigate the invariant subspaces of the multiplication operator $M_z$ on such Hilbert spaces.

**2. Main results.** Through this paper $U$ denotes the open unit disc. We consistently use letters $z, w, \lambda$ to denote points of $U$, and $t$ to denote points of $\partial U$. We denote by $H$ a Hilbert space that is contained in the Hardy space $H^2$. The inner products of $f$ and $g$ on $H$ and $H^2$ are denoted by $< f, g >$ and $< f, g >_{H^2}$ respectively. We assume further that $H$ satisfies the following axioms:

**A1.** For every four functions $f_1, f_2, g_1, g_2 \in H$ which satisfy $< f_1, g_1 > = < f_2, g_2 >$, then we have $< f_1, g_1 >_{H^2} = < f_2, g_2 >_{H^2}$.

**A2.** Inner functions preserves inner products on $H$, i.e., if $\varphi$ is an inner function, then $< \varphi f, \varphi g > = < f, g >$ for all $f, g \in H$.

Note that by axiom A1, if $\|f\|_H = 0$ then $\|f\|_{H^2} = 0$ and this says that $H$ is contained in $H^2$ continuously.

**Example 1.** Let $b$ be an inner function and $H = bH^2$. Then $H$ satisfies the axioms A1 and A2.

**Example 2.** Let $0 < k \ne 1$ and $b_0$ be a nonconstant function in $H^\infty$. Put $H = b_0 H^2$ and define the inner product on $H$ by

$$< f, g >_H = k < f, g >_{H^2}$$

for all $f$ and $g$ in $H$. Then $H \ne H^2$ and it satisfies the axioms A1 and A2.

In the following $m$ denotes the Lebesgue measure on the unit circle $\partial U$.

**Theorem 3.** *Let $H$ be a Hilbert space of analytic functions satisfying the the above conditions. Let $\mathcal{M}$ be a nontrivial subspace of $H$ that is invariant under the multiplication operator $M_z$. If $M(H) = H^\infty$, then there exists a unique nonconstant inner function $\varphi$ such that $\mathcal{M} = \varphi H$.*

*Proof.* First we note that since $H$ is continuously contained in $H^2$, every function of $H$ has

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nontangential limit a.e.$[m]$ on $\partial U$. Also every point of $U$ is a bounded point evaluation on $H$, since the functional of point evaluations are continuous on $H^2$. For $\lambda \in U$ we denote the reproducing kernels of $\mathcal{M}$ and $H$ by $k_\lambda$ and $K_\lambda$ respectively. Define

$$\varphi_\lambda(z) = \frac{k_\lambda(z)}{K_\lambda(z)}, \quad z \in U.$$

Since the sets of $\lambda$ for which $K_\lambda(z) \equiv 0$ and $k_\lambda(z) \equiv 0$ are at most countable, $\varphi_\lambda(z)$ is analytic in $U$ except for a countable set of $\lambda$. So there exists $\lambda_0 \in U$ such that $\varphi_{\lambda_0}$ is analytic and also nonzero (because $\mathcal{M}$ is nontrivial). Let

$$f(z) = k_{\lambda_0}(w)K_{\lambda_0}(z); \quad F(z) = K_w(z),$$
$$g(z) = K_{\lambda_0}(w)k_{\lambda_0}(z); \quad G(z) = k_w(z),$$

where $z, w \in U$. Now we have

$$< z^n f, F > = < z^n g, G > = w^n k_{\lambda_0}(w)K_{\lambda_0}(w)$$

for all nonnegative integers $n$. Thus by axiom A1 we get:

$$\int_{\partial U} t^n(f\bar{F} - g\bar{G})(t)dm(t)$$
$$= < z^n f, F >_{H^2} - < z^n g, G >_{H^2} = 0.$$

Also since $K_w(\lambda_0) = \overline{K_{\lambda_0}(w)}$, we have

$$< z^n F, f > = < z^n G, g > = {\lambda_0}^n k_w(\lambda_0)K_w(\lambda_0)$$

for all nonnegative integers n. Again by using the Axiom 1 and taking a conjugate we get

$$\int_{\partial U} \bar{t}^n(f\bar{F} - g\bar{G})(t)dm(t) = 0$$

for all $n \geq 0$. Put

$$dv = (f\bar{F} - g\bar{G})dm.$$

Then we get

$$\int_{\partial U} p(t, \bar{t})dv(t) = 0$$

for all polynomials $p(t, \bar{t})$ and so

$$\int_{\partial U} h dv = 0$$

for every continuous function $h$ on the unit circle $\partial U$. This implies that $f\bar{F} - g\bar{G} = 0$ a.e.$[m]$ and so

(1) $$\varphi_{\lambda_0}(w) = \varphi_{\lambda_0}(t)\overline{\varphi_w(t)} \quad \text{a.e.}[m].$$

Note that $| < f, K_\lambda > | \leq \|f\|\|K_\lambda\|$ for all $f \in H$. If we set $f = k_\lambda$, then $k_\lambda(\lambda) \leq K_\lambda(\lambda)$. Putting $w = \lambda_0$ in (1) we get

(2) $$|\varphi_{\lambda_0}(t)| \leq 1 \quad \text{a.e.}[m].$$

Note that the set of measure zero in (2) depends on $\lambda_0$. To find a fixed set of measure zero let $D$ be a countable dense subset of $U$ such that for all $\lambda$ in $D$, $k_\lambda \not\equiv 0$ and $K_\lambda \not\equiv 0$. Then (2) holds for all $\lambda$ in $D$ and for all $t \in \partial U$ except on a fixed set of Lebesgue measure zero. Thus, given any fixed $\lambda \in D$, for all $t$ in a set whose complement is of measure zero the relations $|\varphi_\lambda(t)| \leq 1$ as well as

(3) $$|\varphi_w(t)| \leq 1; \quad w \in D,$$

(4) $$\varphi_\lambda(w) = \varphi_\lambda(t)\overline{\varphi_w(t)}; \quad w \in D,$$

are valid. Choosing such a $t$-value we then see from (3) and (4) that

(5) $$|\varphi_\lambda(w)| \leq 1; \quad w \in D,$$

and since $D$ is dense in $U$, by (5) we get $|\varphi_\lambda(z)| \leq 1$ for all $z$ in $U$. On the other hand $\varphi_\lambda(z)$ is analytic in $U$ and so indeed $\varphi_\lambda \in H^\infty$.

Now exactly by the same method used in [5, Theorem 1, p. 451] there exists an analytic function $\varphi(z)$ satisfying

(6) $$\varphi_\lambda(w) = \overline{\varphi(\lambda)}\varphi(w); \quad \lambda, w \in U,$$

(7) $$\varphi_\lambda(\lambda) = |\varphi(\lambda)|^2; \quad \lambda \in U,$$

(8) $$\varphi_\lambda(\lambda) = |\varphi_\lambda(t)|^2 \quad \text{a.e.}[m]; \quad \lambda \in U$$

and also

$$|\varphi(t)| = 1 \quad \text{a.e.}[m].$$

So $\varphi$ is indeed an inner function (i.e., $\varphi \in H^\infty$ and $|\varphi(t)| = 1$ a.e.$[m]$). Now from (6) we obtain

$$\overline{\varphi(\lambda)}\varphi(z) = \varphi_\lambda(z) = \frac{k_\lambda(z)}{K_\lambda(z)}$$

which implies that

$$k_\lambda(z) = \overline{\varphi(\lambda)}\varphi(z)K_\lambda(z)$$

that is a function of $z$ in the space $\mathcal{M} \cap \varphi H$. Note that $\varphi H \subseteq H$ since $\varphi \in H^\infty$ and $M(H) = H^\infty$. Moreover, for every $g = \varphi f$ in $\varphi H$, by axiom A2, we have

$$\begin{aligned} < g, k_\lambda > &= < \varphi f, \overline{\varphi(\lambda)}\varphi K_\lambda > \\ &= \varphi(\lambda) < \varphi f, \varphi K_\lambda > \\ &= \varphi(\lambda) < f, K_\lambda > \\ &= \varphi(\lambda)f(\lambda) = g(\lambda). \end{aligned}$$

Hence $k_\lambda(z)$ is a reproducing kernel for $\varphi H$. Since a subspace is determined by its reproducing kernels, we get $\mathcal{M} = \varphi H$. The uniqueness of $\varphi(z)$ is immediate from the relation (7), which shows that $|\varphi(z)|$ is uniquely determined for all $z \in U$. Also note that if $\varphi$ is constant, by (7), $\varphi_\lambda(\lambda)$ is constant and by (8), $|\varphi_\lambda(t)|$ and hence $\varphi_\lambda(z)$ is constant. But $\varphi_\lambda(z) = \frac{k_\lambda(z)}{K_\lambda(z)}$ and so $\mathcal{M}$ can not be nontrivial that is a contradiction. This completes the proof. $\square$

**Corollary 4.** *Let $\mathcal{M}$ and $H$ satisfy all conditions of Theorem* 3 *except the conditions $M(H) = H^\infty$ and axiom A*2. *Then there exists a unique inner function $\varphi$ such that $\mathcal{M} \subseteq closure(\varphi H)$ where the closure is in the Hardy space $H^2$.*

*Proof.* By the proof of Theorem 3, there exists a unique nonconstant inner function $\varphi$ such that the relation

$$k_\lambda(z) = \overline{\varphi(\lambda)}\varphi(z)K_\lambda(z); \quad \lambda \in U, \ z \in U$$

holds. Note that

$$k_\lambda(z) \in \mathcal{M}, \quad \overline{\varphi(\lambda)}\varphi(z)K_\lambda(z) \in \varphi H$$

where $\varphi \in H^\infty = M(H^2)$ and it may be $\varphi \notin M(H)$. Let $\mathcal{N}$ be the closure of the closed linear span of $\{k_\lambda : \lambda \in U\}$ in the Hardy space $H^2$. Then $\mathcal{M} \subseteq \mathcal{N} \subseteq closure(\varphi H)$ where the closure is in the Hardy space. This completes the proof. $\square$

**Corollary 5.** *Under the conditions of Corollary* 4, *if moreover polynomials are in $H$, then $\mathcal{M} \subseteq closure(\varphi H^2)$ where $\varphi$ satisfies in Corollary* 4 *and the closure is in the Hardy space.*

*Proof.* Since polynomials are in $H$ and they are dense in $H^2$, clearly we have $closure(\varphi H) = \varphi H^2$ where the closure is in the Hardy space and so the proof is complete. $\square$

**Corollary 6** (Beurling's Theorem)**.** *If $S$ is the unilateral shift and $\mathcal{M}$ is a non-zero invariant subspace of $S$, then there is a function $\varphi$ in $H^\infty$ such that $|\varphi| = 1$ and $M = \varphi H^2$.*

*Proof.* It is an immediate consequence of Theorem 3. $\square$

**Question 7.** Is there an example of a Hilbert space $H$ that satisfies the axioms A1 and A2 that is different from the examples 1 and 2 ?

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