PAPERS COMMUNICATED

116. On the Convergency of the Series Summable (C, r).

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1. In the Tôhoku Mathematical Journal, 33 (1930) Mr. Izumi treated of the condition for the convergency of the series summable (C, r), and gave a simple proof for Hardy-Landau's theorem in the following generalized form:

A. *If the series is summable* (C, r) $(r>0)$, *and*

$$\liminf (s_m - s_n) \geqq 0, \quad \textit{for} \quad m>n,\ n\to\infty,\ \frac{m}{n}\to 1,$$

where s_n denotes the sum of the first $n+1$ terms of the series, then the series is convergent.

In the proof for the case $r=2$, he started from

$$V_{mn}=S_m^{(2)}-S_n^{(2)}-(m-n)S_n^{(1)}-\frac{1}{2!}(m-n)(m-n+1)S_n^{(0)},$$

$$W_{mn}=S_m^{(2)}-S_n^{(2)}-(m-n)S_m^{(1)}+\frac{1}{2!}(m-n)(m-n-1)S_n^{(0)}.$$

If we take instead of V_{mn}, W_{mn}

$$U_{mn}=S_m^{(2)}-2S_\mu^{(2)}+S_n^{(2)}-\left(\frac{m-n}{2}\right)^2 S_n^{(2)}, \quad \left(\mu=\frac{m+n}{2}\right)$$

where $m-n$ is an even number, the proof will be much simpler.

For the general case, where r denotes any positive integer, we have to put

$$U_{mn}^{(r)}=S_{n+rl}^{(r)}-\binom{r}{1}S_{n+(r-1)l}^{(r)}+\binom{r}{2}S_{n+(r-2)l}^{(r)}-\cdots\cdots+(-1)^r S_n^{(r)}-l^r S_n^{(0)},$$

$$(m=n+rl).$$

2. In the following lines I wish to give the proof of the theorem in more general form.

Suppose that the series $\sum a_n$ is summable (C, r) to the sum s, where r denotes any positive integer. Then it is well known that

$$\lim_{n\to\infty} S_n^{(r)}/\binom{n+r}{r}=s,$$

where $\quad S_n^{(\rho)}=S_0^{(\rho-1)}+S_1^{(\rho-1)}+S_2^{(\rho-1)}+\cdots\cdots+S_n^{(\rho-1)} \quad (\rho=1, 2, \ldots\ldots, r),$

and $S_n^{(0)}=s_n$ is the sum of the first $n+1$ terms of the series.

Let us now put $S_n^{(r)}-\binom{n+r}{r}s=f(n)$,

$$\Delta f(n)=\Delta^1 f(n)=f(n+1)-f(n),$$

$$\Delta^\rho f(n)=\Delta^{\rho-1}f(n+1)-\Delta^{\rho-1}f(n) \qquad (\rho\geqq 2);$$

then we get
$$\lim_{n\to\infty} f(n)/n^r=0$$
and
$$\Delta^\rho f(n)=S_{n+\rho}^{(r-\rho)}-\binom{n+r}{r-\rho}s \qquad (\rho=1, 2, \ldots\ldots, r),$$

$$\Delta^r f(n)=s_{n+r}-s,$$

so that the theorem A is nothing but a special case $M=0$ of the following theorem:

B. *If* $$f(n)=o(n^r),$$

and $\liminf (\Delta^r f(m)-\Delta^r f(n))\geqq -M,$ *for* $m>n,\ n\to\infty,\ \dfrac{m}{n}\to 1,$

where M denotes any constant $\geqq 0$, then

$$\limsup_{n\to\infty} |\Delta^r f(n)|\leqq M.$$

3. To prove the theorem B we need the following lemma:

LEMMA. *Let* $\Delta^1_{(\Delta x=\xi)}f(x)=f(x+\xi)-f(x)$,

$$\Delta^r_{(\Delta x=\xi)}f(x)=\Delta^{r-1}_{(\Delta x=\xi)}f(x+\xi)-\Delta^{r-1}_{(\Delta x=\xi)}f(x) \qquad (r\geqq 2),$$

then for a positive integer l we have

$$\Delta^r_{(\Delta x=l\xi)}f(x)=\sum_{\lambda=0}^{r(l-1)}k_{r\lambda}\Delta^r_{(\Delta x=\xi)}f(x+\lambda\xi), \tag{1}$$

where the coefficients $k_{r\lambda}$ are positive integers depending on l, and

$$\sum_{\lambda=0}^{r(l-1)}k_{r\lambda}=l^r. \tag{2}$$

Putting $\xi=1$, we will prove this lemma by the mathematical induction.

For the case $r=1$, we have

$$\Delta_{(\Delta x=l)}f(x)=f(x+l)-f(x)$$
$$=\Delta f(x)+\Delta f(x+1)+\cdots\cdots+\Delta f(x+l-1).$$

Thus $k_{1\lambda}=1$ and (2) holds good.

Now suppose that (1) and (2) hold good for a positive integer r, then we have

$$\Delta^r_{(\Delta x=l)}f(x)=\sum_{\lambda=0}^{r(l-1)}k_{r\lambda}\Delta^r f(x+\lambda),$$

$$\Delta^r_{(\Delta x=l)}f(x+l)=\sum_{\lambda=0}^{r(l-1)}k_{r\lambda}\Delta^r f(x+l+\lambda),$$

so that
$$\Delta^{r+1}_{(\Delta x=l)}f(x)=\sum_{\lambda=0}^{r(l-1)} k_{r\lambda}(\Delta^r f(x+l+\lambda)-\Delta^r f(x+\lambda))$$
$$=\sum_{\lambda=0}^{r(l-1)} k_{r\lambda}\sum_{\mu=\lambda}^{\lambda+l-1}\Delta^{r+1}f(x+\mu), \qquad (3)$$

that is
$$\Delta^{r+1}_{(\Delta x=l)}f(x)=\sum_{\mu=0}^{(r+1)(l-1)} k_{(r+1)\mu}\Delta^{r+1}f(x+\mu), \qquad (4)$$

where $k_{(r+1)\mu}$ is the sum of $k_{r\lambda}$ for some values of λ and hence a positive integer; putting $\Delta^{r+1}f(x+\mu)\equiv 1$ in (3) and (4) we get
$$\sum_{\mu=0}^{(r+1)(l-1)} k_{(r+1)\mu}=\sum_{\lambda=0}^{r(l-1)} k_{r\lambda}\cdot l=l^{r+1}.$$

Hence (1) and (2) hold good for $r+1$. The lemma is thus proved.

4. *Proof of Theorem B.* By the lemma we have
$$\Delta^r_{(\Delta n=l)}f(n)-l^r\Delta^r f(n)=\sum_{\lambda=0}^{r(l-1)} k_{r\lambda}(\Delta^r f(n+\lambda)-\Delta^r f(n))\text{[1)]}$$

and
$$l^r\Delta^r f(m)-\Delta^r_{(\Delta n=l)}f(n)=\sum_{\lambda=0}^{r(l-1)} k_{r\lambda}(\Delta^r f(m)-\Delta^r f(n+\lambda)),\text{[2)]} \quad m=n+rl.$$

By the hypothesis $\liminf\,(\Delta^r f(m)-\Delta^r f(n))\geqq -M$
we can find an integer N and a positive number δ, for any assigned positive number ε, such that
$$\Delta^r f(n+\lambda)-\Delta^r f(n)>-(M+\varepsilon),$$
$$\Delta^r f(m)-\Delta^r f(n+\lambda)>-(M+\varepsilon),$$
$$m-n>\lambda>0,\quad n>N,\quad m/n\leqq 1+\delta.$$

Therefore, as $k_{r\lambda}>0$, we get
$$\Delta^r_{(\Delta n=l)}f(n)-l^r\Delta^r f(n)>-(M+\varepsilon)\sum_{\lambda=0}^{r(l-1)} k_{r\lambda},$$

so that $\Delta^r f(n)<(M+\varepsilon)+l^{-r}\Delta^r_{(\Delta n=l)}f(n)$,

and similarly $\Delta^r f(m)>-(M+\varepsilon)+l^{-r}\Delta^r_{(\Delta n=l)}f(n)$,
$$m=n+rl,\quad n>N,\quad (m-n)/n\leqq\delta.$$

Now let us take $l=(m-n)/r=[n\delta/r]$ (integral part of $n\delta/r$), then by the hypothesis we have
$$f(n+\rho l)=o(l^r(\frac{r}{\delta}+\rho)^r)=o(l^r) \qquad (\rho=0,1,2,\ldots\ldots,r),$$

so that
$$\Delta^r_{(\Delta n=l)}f(n)=\sum_{\rho=0}^{r}(-1)^{r-\rho}\binom{r}{\rho}f(n+\rho l)=o(l^r).$$

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1) 2) Compare the left-hand sides with $U^{(r)}_{mn}$ and $(-1)^{r-1}U^{(r)}_{mn}$ in §1.

Thus we get $$\limsup_{n\to\infty} \Delta^r f(n) \leqq M+\varepsilon,$$

$$\liminf_{m\to\infty} \Delta^r f(m) \geqq -(M+\varepsilon),$$

that is $$\limsup_{n\to\infty} |\Delta^r f(n)| \leqq M+\varepsilon.$$

Since ε may be as small as we please, we have

$$\limsup_{n\to\infty} |\Delta^r f(n)| \leqq M.$$

Thus the proof is completed.