PAPERS COMMUNICATED

73. *On the Roots of the Characteristic Equation of a Certain Matrix.*

By Shin-ichi TAKAHASHI.
Shiomi Institute, Osaka.

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The object of this note is to prove the following

THEOREM. Let $A=(a_{ik})$ be a square matrix of order n, whose elements are real or complex. If there exist n real or complex numbers $t_1, t_2, \ldots\ldots, t_n$ such that

$$\sum_{k=1}^{n} |t_k|\,|a_{ik}| \leqq |t_i|, \qquad (i=1, 2, \ldots\ldots, n),$$

then all the roots of the characteristic equation of matrix A have their absolute values not greater than 1.

I prove this theorem by purely algebraical method due to Mr. Rohrbach,[1] who gave a simple proof of a theorem enunciated by Mr. Tambs Lyche.

Proof. First we shall treat the case, where all $|t_i|$ are equal, viz.

$$\sum_{k=1}^{n} |a_{ik}| \leqq 1, \qquad i=1, 2, \ldots\ldots, n.$$

Now let λ be a root of the characteristic equation of matrix A. Then the system of equations

$$(a_{\rho\rho}-\lambda)x_\rho - \sum_{k\neq\rho}^{1,n} a_{\rho k}x_k = 0, \qquad \rho=1, 2, \ldots\ldots, n$$

have the solutions $x_1, x_2, \ldots\ldots, x_n$ not all zero. Suppose now

$$|x_\rho| = Max(|x_1|, |x_2|, \ldots\ldots, |x_n|),$$

then we get

$$|a_{\rho\rho}-\lambda|\,|x_\rho| \leqq \sum_{k\neq\rho}^{1,n} |a_{\rho k}|\,|x_k| \leqq \Big(\sum_{k\neq\rho}^{1,n} |a_{\rho k}|\Big)|x_\rho|,$$

therefore

$$|a_{\rho\rho}-\lambda| \leqq \sum_{k\neq\rho}^{1,n} |a_{\rho k}| \leqq 1-|a_{\rho\rho}|.$$

In case $|a_{\rho\rho}|=1$, the above inequality shows us

$$|\lambda| = |a_{\rho\rho}| = 1.$$

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1) Jahresber. d. D. M. V. **40** (1931), 49.

In other case, we see that λ lies in the circle with centre a_{pp} and the radius $\leqq 1-|a_{pp}|$, and it is evident that this circle is wholly contained in the unit circle with centre at the origin. Thus all the roots of the characteristic equation of matrix A have their absolute values not greater than 1.

From this special case, our theorem is easily deduced.

Let

$$T=\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix},$$

then

$$B=T^{-1}AT=\begin{pmatrix} a_{11} & \frac{t_2}{t_1}a_{12} & \cdots\cdots & \frac{t_n}{t_1}a_{1n} \\ \frac{t_1}{t_2}a_{21} & a_{22} & \cdots\cdots & \frac{t_n}{t_2}a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{t_1}{t_n}a_{n1} & \frac{t_2}{t_n}a_{n2} & \cdots\cdots & a_{nn} \end{pmatrix}=(b_{ik}).$$

Now the matrix B satisfies by our hypothesis the conditions

$$\sum_{k=1}^{n}|b_{ik}|\leqq 1, \qquad i=1, 2, \ldots\ldots, n.$$

Therefore all the roots of the characteristic equation of matrix B have their absolute values not greater than 1.

Since A and B are similar matrices, the roots of the characteristic equation of matrix B coincide with those of A. Our theorem is thus completely proved.

N.B. In case where the system of equations

$$(a_{pp}-\lambda)x_p-\sum_{k\neq p}^{1,n}a_{pk}x_k=0$$

have the solutions $t_1, t_2, \ldots\ldots, t_n$, our theorem is trivial. For then

$$\begin{aligned}|\lambda|\,|x_p| &\leqq \sum_{k=1}^{n}|a_{pk}|\,|x_k| \\ &=\sum_{k=1}^{n}|a_{pk}|\,|t_k| \\ &\leqq |t_p|=|x_p|,\end{aligned}$$

therefore

$$|\lambda|\leqq 1.$$

From our proof it is also evident that we can establish the following theorem:

Let λ be a root of the characteristic equation of a matrix of the n th order $A=(a_{ik})$. Then

$$|\lambda|\leqq M \qquad \text{or} \qquad |\lambda|\leqq N,$$

where

$$|a_{i1}|+|a_{i2}|+\cdots\cdots+|a_{in}|=M_i, \qquad |a_{1k}|+|a_{2k}|+\cdots\cdots+|a_{nk}|=N_k,$$
$$M=Max(M_1, M_2, \ldots\ldots, M_n), \qquad N=Max(N_1, N_2, \ldots\ldots, N_n).$$

This is an extension of Bromwich's inequality

$$|\lambda|\leqq nMax|a_{ik}|, \qquad i, k=1, 2, \ldots\ldots, n.$$

Further applying this theorem to an equality

$$a_0\begin{vmatrix} z+\frac{a_1}{a_0} & \frac{a_2}{a_0} & \frac{a_3}{a_0} & \cdots\cdots & \frac{a_{n-1}}{a_0} & \frac{a_n}{a_0} \\ -1 & z & 0 & \cdots\cdots & 0 & 0 \\ 0 & -1 & z & \cdots\cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots\cdots & -1 & z \end{vmatrix}=a_0z^n+a_1z^{n-1}+\cdots\cdots+a_n,$$

which is easily proved by mathematical induction, we obtain two well-known theorems:

THEOREM A. The absolute values of the roots of an algebraic equation

$$a_0z^n+a_1z^{n-1}+\cdots\cdots+a_n=0, \qquad a_0\neq 0$$

are not greater than

$$Max\left(1, \frac{|a_1|+|a_2|+\cdots\cdots+|a_n|}{|a_0|}\right).$$

THEOREM B. The absolute values of the roots of an algebraic equation

$$a_0z^n+a_1z^{n-1}+\cdots\cdots+a_n=0, \qquad a_0\neq 0$$

are not greater than

$$1+\frac{K}{|a_0|},$$

where

$$K=Max(|a_1|, |a_2|, \ldots\ldots, |a_n|).$$