65. Theory of Connections in a Kawaguchi Space of Higher Order.

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(Comm. by S. KAKEYA, M.I.A., July 12, 1937.)

The object of the present paper is to give the foundation to the geometry in a Kawaguchi space of order m (m: a positive integer) and of dimension n by generalization of the results in the previous paper.[1] An element of this space is a line element of not the mth order but the $(2m-1)$-th.

1. The assumption that the metrics in the space with a point coordinate system x^i $(i=1, 2, \ldots\ldots, n)$:

$$s=\int F(x, x', x'', \ldots\ldots, x^{(m)})\,dt$$

is invariant under any change of parameter t, offers the necessary and sufficient conditions:

$$\sum_{\lambda=a}^{m}\binom{\lambda}{a}F_{(\lambda)i}x^{(\lambda-a+1)i}=\delta^1_aF\,, \tag{1}$$

putting $x^{(\lambda)i}=\dfrac{d^\lambda x^i}{dt^\lambda}$. Owing to (1) it can be derived from the Synge vectors $\overset{a}{E}_i(a=0, 1, \ldots\ldots, m)$ the following intrinsic vectors

$$\overset{a}{\mathfrak{E}}_i=F^{-1}\sum_{\lambda=a}^{m}\overset{\lambda}{E}_iA^a_{\lambda-a+1}\,,\qquad a=0, 1, \ldots\ldots, m\,, \tag{2}$$

where A^a_b are defined by the recurring formulae

$$A^0_1=1\,,\qquad A^a_b=\frac{dA^a_{b-1}}{dt}+A^{a-1}_bF\,,$$

$$A^1_c=F^{(c-1)}\,,\quad A^c_0=0\,,\quad A^0_d=0\,,\quad c=1, 2, \ldots\ldots, m\,; d=2, 3, \ldots\ldots, m\,.$$

We shall assume that the matrix $((mF_{(m)i(m)j}+\overset{m}{\mathfrak{E}}_i\overset{m}{\mathfrak{E}}_j))$ is of rank $n-1$, then the determinant of the intrinsic tensor

$$g_{ij}=mF^{2m-1}F_{(m)i(m)j}+\overset{m}{\mathfrak{E}}_i\overset{m}{\mathfrak{E}}_j+\overset{1}{\mathfrak{E}}_i\overset{1}{\mathfrak{E}}_j \tag{3}$$

is not identically equal to zero, for $g_{ij}x'^j=-F\overset{1}{\mathfrak{E}}_i$. g_{ij} may be functions of a line element of the $(2m-1)$-th order and this tensor can be taken as the fundamental tensor. It follows immediately

$$F^{2m-1}\overset{1}{\mathfrak{E}}_{i(2m-1)j}=g_{ij}-\overset{1}{\mathfrak{E}}_i\overset{1}{\mathfrak{E}}_j\,. \tag{4}$$

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1) A. Kawaguchi, Theory of connections in a Kawaguchi space of order two, Proc. **13** (1937), 6. We adopt here the same notations as in this paper.

Under the assumption that $\sigma \equiv g^{ij}\overset{m}{\mathfrak{E}}_i\overset{m}{\mathfrak{E}}_j \neq 1$, we have a scalar of order $2m-1$:

(5) $$\overset{\tau}{\Psi} = \overset{\tau-1}{\Psi}{}^{(1)} + \sum_{\lambda=1}^{\tau}(A^{m+\lambda(1)}_{\tau-\lambda} - A^{m+\lambda}_{\tau-\lambda+1})\overset{\lambda}{S}, \qquad \tau=1, 2, \ldots\ldots, m-1,$$

which behave under a change of parameter in the same way as $F^{(m+\tau-1)}$, where

$$\overset{0}{\Psi} = F^{(m-1)},$$

$$\overset{\tau}{S} = [\overset{\tau-1}{\Psi}{}^{(1)} + \sum_{\lambda=1}^{\tau-1}(A^{m+\lambda(1)}_{\tau-\lambda} - A^{m+\lambda}_{\tau-\lambda+1})\overset{\lambda}{S}]_{(2m)j}\, mF^{2m-\tau}g^{jk}\Big(\overset{0}{\mathfrak{E}}_k - \frac{1}{1-\sigma}\overset{m}{\mathfrak{E}}_k g^{pq}\overset{m}{\mathfrak{E}}_p\overset{0}{\mathfrak{E}}_q\Big).$$

2. X^i be an arbitrary intrinsic vector, then

$$F^{2m-1}\overset{1}{D}_j(F_{(m)i})X^j = mF^{2m-1}F_{(m)i(m)j}\frac{dX^j}{dt}$$
$$+F^{2m-1}(F_{(m)i(m-1)j} + F^{-1}F^{(1)}F_{(m)i(m)j})X^j,$$

$$\frac{1}{m}F^{2m-2}F_{(m)i}\overset{1}{D}_j(F)X^j = F^{2m-2}F_{(m)i}F_{(m)j}\frac{dX^j}{dt}$$
$$+\frac{1}{m}F^{2m-2}F_{(m)i}(F_{(m-1)j} + F^{-1}F^{(1)}F_{(m)j})X^j,$$

$$\overset{1}{\mathfrak{E}}_i\Big\{(\overset{1}{\mathfrak{E}}_jX^j)^{(1)} - \Big(m\overset{0}{E}_j + \frac{m}{1-\sigma}\overset{m}{\mathfrak{E}}_k\overset{0}{E}_l g^{kl}\overset{m}{\mathfrak{E}}_j\Big)X^j\Big\}$$
$$= \overset{1}{\mathfrak{E}}_i\overset{1}{\mathfrak{E}}_j\frac{dX^j}{dt} - \overset{1}{\mathfrak{E}}_i\Xi_jX^j$$

are all geometrical vectors of class 1 and order $2m-1$, where Ξ_j are functions of a line element of the $(2m-1)$-th order. From these vectors it follows a covariant differentiation of a vector X^i, which is a geometrical vector of class 1 and of order $2m-1$:

(6) $$\frac{\delta X^i}{dt} = \frac{dX^i}{dt} + \Gamma^i_jX^j,$$

where

(7) $$\Gamma^i_j = g^{ik}\Big(F^{2m-1}F_{(m)k(m-1)j} + \frac{1}{m}F^{2m-2}F_{(m)k}F_{(m-1)j}\Big)$$
$$+\frac{1}{m}\frac{F^{(1)}}{F}\Big(\delta^i_j + \frac{1}{F}x'^i\overset{1}{\mathfrak{E}}_j\Big) + \frac{x'^i}{F}\Xi_j$$

is a geometrical quantity of class 1 and order $2m-1$. Put

(8) $$\overset{p}{D}\Gamma^i_j = \sum_{\lambda=p}^{m}\binom{\lambda}{p}\Gamma^i_{j(\lambda)k}dx^{(\lambda-p)k}, \qquad p=1, 2, \ldots\ldots, m,$$

then $\overset{p}{D}\Gamma^i_j$ are all tensors except $\overset{1}{D}\Gamma^i_j$. It can be proved after some calculations that

$$F^{-1}\Big(\sum_{p=1}^{m} F^{(p-1)}\overset{p}{D}\Gamma^i_j + \sum_{p=1}^{m-1}\overset{p}{\Psi}\overset{m+p}{D}\Gamma^i_j\Big) \equiv \sum_{a=0}^{2m-2}\overset{a}{\Gamma^i_{jk}}dx^{(a)k} \tag{9}$$

is an intrinsic quantity of order $2m-1$ and is transformed by a coordinate transformation in the way

$$\sum_{a=0}^{2m-2}\overset{a}{\Gamma^{\lambda}_{\mu\nu}}dx^{(a)\nu} = \sum_{a=0}^{2m-2}\overset{a}{\Gamma^i_{jk}}dx^{(a)k}\frac{\partial x^{\lambda}}{\partial x^i}\frac{\partial x^j}{\partial x^{\mu}} - \frac{\partial^2 x^{\lambda}}{\partial x^i \partial x^j}\frac{\partial x^i}{\partial x^{\mu}}dx^j , \tag{10}$$

from which an intrinsic covariant differential of an intrinsic vector X^i of order $2m-1$ follows immediately:

$$\delta X^i = dX^i + \sum_{a=0}^{2m-2}\overset{a}{\Gamma^i_{jk}}X^j dx^{(a)k} . \tag{11}$$

From this differential one can define a covariant differential of an arbitrary tensor by the usual method.

3. The base connections are defined by

$$F^{2m-1}g^{ij}\delta\overset{1}{\mathfrak{C}}_i \equiv \delta x^{(2m-1)j} = \Big(\delta^j_i + \overset{1}{\mathfrak{C}}_i\frac{x'^j}{F}\Big)dx^{(2m-1)i} + \sum_{a=0}^{2m-2}\overset{a}{\underset{2m-1}{A^j_i}}dx^{(a)i} , \tag{12}$$

$$\binom{2m-1}{p}^{-1}F^{2m-p-1}g^{ij}\sum_{\mu=p}^{2m-1}A^p_{\mu-p+1}\sum_{\lambda=\mu}^{2m-1}\binom{\lambda}{\mu}\overset{1}{\mathfrak{C}}_{i(\lambda)k}dx^{(\lambda-\mu)k} \equiv \delta x^{(2m-p-1)j} \tag{13}$$

$$= \Big(\delta^j_i + \overset{1}{\mathfrak{C}}_i\frac{x'^j}{F}\Big)dx^{(2m-p-1)i} + \sum_{a=0}^{2m-p-2}\overset{a}{\underset{2m-p-1}{A^j_i}}dx^{(a)i} ,$$

$$p = 1, 2, \ldots\ldots, 2m-2 ,$$

where $F^{(m+\tau-1)}(\tau = 1, \ldots\ldots, m-1)$ in A's should be replaced by $\overset{\tau}{\Psi}$ respectively, and the following relation holds good for any intrinsic vector X^i

$$\delta X^i = \sum_{a=0}^{2m-1}\nabla^{(a)}_j X^i \cdot \delta x^{(a)j} , \qquad (\delta x^{(0)j} \equiv dx^j) , \tag{14}$$

where

$$\begin{cases} \nabla^{(2m-1)}_j X^i = X^i_{(2m-1)j} , \\ \nabla^{(p)}_j X^i = X^i_{(p)j} - \sum_{\lambda=p+1}^{2m-1}\nabla^{(\lambda)}_k X^i \cdot \underset{\lambda}{\overset{p}{A^k_j}} + \overset{p}{\Gamma^i_{kj}}X^k , \end{cases} \tag{15}$$

$$p = 0, 1, \ldots\ldots, 2m-2$$

are the covariant derivatives of X^i. $\nabla^{(p)}_j X^i$ is a geometrical tensor of class a. One can easily verify that

$$x'^j\nabla^{(p)}_j X^i = 0 \qquad \text{for } p = 1, 2, \ldots\ldots, 2m-1 . \tag{16}$$

The curvature and torsion tensors are calculated form Γ's and A's and fundamental theorems can be proved by a similar method as in the case of order 2.

4. The connection defined by (11) is not metric, i. e. $\delta g_{ij} \neq 0$. But it is very easy to derive a metric connection from (11), in fact,

$$\theta X^i = dX^i + \sum_{a=0}^{2m-1} \overset{a}{\Pi}{}^i_{jk} X^j dx^{(a)k}, \tag{17}$$

$$\sum_{a=0}^{2m-1} \overset{a}{\Pi}{}^i_{jk} dx^{(a)k} = \sum_{a=0}^{2m-1} \overset{a}{\Gamma}{}^i_{jk} dx^{(a)k} + \frac{1}{2} g^{ih} \delta g_{hj}$$

$$= \sum_{a=0}^{2m-2} \frac{1}{2} g^{ih} (g_{hj(a)k} - \overset{a}{\Gamma}{}^l_{hk} g_{lj} + \overset{a}{\Gamma}{}^l_{jk} g_{lh}) dx^{(a)k}$$

$$+ \frac{1}{2} g^{ih} g_{hj(2m-1)k} dx^{(2m-1)k}$$

defines a metric connection, i. e. $\theta g_{ij} = 0$, which is easily verified.