80. A Relation between the Length of a Plane Curve and Angles Stretched by it.

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Let us consider a continuous plane curve C, defined by the equation: $x=\varphi(t)$, $y=\psi(t)$ where $\varphi(t)$ and $\psi(t)$ are one-valued and continuous functions of t in $[0,1]=[0\leqq t\leqq 1]$. For brevity, let us call a continuous plane curve without double point a *Jordan curve*. A Jordan curve is therefore a continuous transformation of the segment $[0,1]$ which is biuniform with a possible exception: $\varphi(0)=\varphi(1)$, $\psi(0)=\psi(1)$. We shall denote this transformation by $(x,y)=f(t)$. Let T be a set of numbers contained in $[0,1]$. As t ranges over T, the point (x,y) ranges over a sub-set A of C. A will be called a *generalized arc of* C, and denoted by $f(T)$. The minimum value τ_0 and the maximum value τ_1 of $\overline{T}$[1] are called *extreme values of* A.

Let now C be rectifiable, and let $U=[a,b]$ be an interval contained in $[0,1]$. The arc $f\{U\}$ will be denoted by $\mathfrak{A}(a,b)$. Devide $[a,b]$ into n sub-intervals by the points $t_0=a<t_1<t_2<\cdots\cdots<t_n=b$. To the value $t_m (m=0,1,2,\ldots\ldots,n)$, there corresponds the point $p_m=f(t_m)$ of the curve. Denote by (p_{m-1},p_m) the segment between two points p_{m-1}, p_m, and by $(\overline{p_{m-1},p_m})$ its length. We define as usual the length of the arc $\mathfrak{A}(a,b)$ as the limit of the sum $\sum_{m=1}^{n}(\overline{p_{m-1},p_m})$ (the length of the polygon whose vertices are p_m), when the greatest of the lengths t_m-t_{m-1} tends to 0, and denote it by $l(a,b)$. The complement of $\overline{T}$ (in regard of $[\tau_0,\tau_1]$) is decomposed into at most enumerable infinity of contiguous intervals (a_r,b_r). We call $l(A)=l(\tau_0,\tau_1)-\sum_{r=1}^{\infty}l(a_r,b_r)$ *the length of the generalized arc* A.[2]

If $A_1, A_2, \ldots\ldots, A_k$ are a finite number of generalized arcs of C such as $A=\sum_{\nu=1}^{k}A_\nu$, we have, as usual

$$l(A)\leqq l(A_1)+l(A_2)+\cdots\cdots+l(A_k).$$

We can divide the interval $[\tau_0,\tau_1]$ into n sub-intervals by the points of T: $t_0<t_1<t_2<\cdots\cdots<t_n$, so that the sum $\sum_{m=1}^{n}(\overline{p_{m-1},p_m})$ converges to $l(\tau_0,\tau_1)-\sum_{r=1}^{\infty}l(a_r,b_r)+\sum_{r=1}^{\infty}\left(\overline{f(a_r),f(b_r)}\right)=l(A)+\sum_{r=1}^{\infty}\left(\overline{f(a_r),f(b_r)}\right)$ as n tends to infinity.

Theorem 1. Let A be a generalized arc of a rectifiable Jordan curve C, whose extreme values are τ_0, τ_1. Suppose that A is contained in a circle of radius $S(S>0)$, and that to each point p of A, we can associate two half-lines pq, pq' with the properties:

1) $\overline{T}$ is the set of all points of T together with its limiting points.

2) $l(A)$ is the linear measure of $\overline{A}$ on C.

(1) *These half straight lines pq, pq' make a given angle θ* $(0<\theta<\pi)$.

(2) *The angle qpq' has no common points with the arc $\mathfrak{A}(\tau_0, \tau_1)$ of C except p.*

(3) *pq, pq' are situated in a given side of the curve C.*

Then the length of A does not exceed a fixed number $l_1=l_1(\theta, S)$ depending only on θ and S, namely $l(A)\leqq l_1$. Moreover we can set

$$l_1(\theta, S)=2S\left\{\left[\frac{4\pi}{\theta}\right]+1\right\}\Big/\sin\frac{\theta}{4}.^{1)}$$

Proof. Let p be a point of the generalized arc A. We have two half-lines pq, pq' making the angle θ at p, which have no common points with the arc $\mathfrak{A}(\tau_0, \tau_1)$, and are situated in the prescribed side of the curve C.

We now consider the unit circle: $x^2+y^2=1$ on the plane of C, and draw two radius ou, ou'[2] which are parallel to pq, pq' respectively in the same sense. We shall denote by $I(p)$ the arc uu' of the unit circle, and by $\frac{1}{n}I(p)$ $(n=2, 3, \ldots\ldots)$ the sub-arc of $I(p)$, whose middle point coincides with that of $I(p)$ and whose length is $\frac{\theta}{n}$.

Let $q_\nu(\nu=0, 1, 2, \ldots\ldots, \mu-1)$ be μ points on the unit circle, whose coordinates in polar form are $\left(1, \frac{2\pi\nu}{\mu}\right)$. If we set $\mu=\mu(\theta)=\left[\frac{4\pi}{\theta}\right]+1$, then for any p, $\frac{1}{2}I(p)$ will contain at least one of these μ points. Denote by A_ν the set of all points p of A for which $q_\nu\in\frac{1}{2}I(p)$. A_ν is a generalized arc of C. Let τ_0^ν, τ_1^ν be the extreme values of A_ν, and divide the interval $[\tau_0^\nu, \tau_1^\nu]$ into n sub-intervals by the points of T_ν[3]: $t_0<t_1<t_2<\cdots<t_n$. Let us evaluate the sum $\sum_{m=1}^{n}\overline{(p_{m-1}, p_m)}$; $p_m=f(t_m)$.

As the rotation does not alter the mutual positions of the curve and segments, we can suppose, without loss of generality, that $q_r=(0, 1)$. Then the arc $\mathfrak{A}(\tau_0^\nu, \tau_1^\nu)$ satisfies the following condition (γ): It has no points in common (except p_m) with the half-line $p_mp'_m$ issued vertically from p_m in the positive direction of the y-axis. Hence, the generalized arc A_ν has no more than one point contained in the same straight line which is parallel to the y-axis. Let (x_m, y_m) be the coordinates of the point p_m. The correspondence between x_m and p_m is *biuniform*. Moreover, these half-lines $p_mp'_m$ must be situated in the same side of the curve C.

Let $p_{n_1}, p_{n_2}, p_{n_3}, p_{n_4}$ be any four of the points p_m. We suppose that $n_1<n_2<n_3<n_4$. Then their abscissae $x_{n_1}, x_{n_2}, x_{n_3}, x_{n_4}$ give a per-

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1) $[x]$ is the greatest integer which does not exceed x.

2) We denote by o the origin of the plane.

3) T_ν is the set of all t such as $f(t)\in A_\nu$.

mutation of their mutual positions. For example, to the case: $x_{n_2} < x_{n_3} < x_{n_1} < x_{n_4}$ we associate the permutation (2 3 1 4). There are 24 such permutations. In the cases corresponding to the permutations (1 4 2 3), (4 2 3 1), (2 3 1 4), (3 1 4 2) and their inverses, the arc $\mathfrak{A}(\tau_0^\nu, \tau_1^\nu)$ should have common points with some of vertical half-lines $p_{n_1}p'_{n_1}$, $p_{n_2}p'_{n_2}$, $p_{n_3}p'_{n_3}$, $p_{n_4}p'_{n_4}$, which contradicts the condition (γ). In the cases corresponding to (1 3 4 2), (3 4 2 1), (4 2 1 3), (2 1 3 4) and their inverses, the arc $\mathfrak{A}(\tau_0^\nu, \tau_1^\nu)$, which satisfies the condition (γ), should have the vertical half-lines $p_{n_2}p'_{n_2}$ and $p_{n_3}p'_{n_3}$ in the opposite side of the curve, which contradicts the property (3). Consequently only the cases corresponding to the 8 permutations: (1 2 3 4), (2 3 4 1), (3 4 1 2), (4 1 2 3) and their inverses are possible. Thus we see that, regarding the order of $x_0, x_1, x_2, \ldots\ldots, x_n$, there are only two possible cases: viz. there exists an index m_0 (which may be 0) such that

$$(1) \quad x_{m_0} < x_{m_0+1} < \cdots\cdots < x_n < x_0 < x_1 < x_2 < \cdots\cdots < x_{m_0-1}$$

or

$$(2) \quad x_{m_0} > x_{m_0+1} > \cdots\cdots > x_n > x_0 > x_1 > x_2 > \cdots\cdots > x_{m_0-1}.$$

Draw two half-lines $p_m p''_m$, $p_m p'''_m$ from p_m, making the angles $\pm\frac{\theta}{4}$ with the vertical lines $p_m p'_m$. Then all the points $p_{m'} (m \neq m')$ are situated below the angle $p''_m p_m p'''_m$, and every segment (p_{m-1}, p_m) $(m = 1, 2, \ldots\ldots, n)$, except (p_{m_0-1}, p_{m_0}), is one side of a triangle whose other two sides are on the adjacent sides of the angles constructed above.

Each of these two sides of the triangle has constant direction. Hence it is easily seen that the length $\sum_{m=1}^{n}(p_{m-1}, p_m)$ does not exceed $\frac{2S}{\sin\frac{\theta}{4}} + (\overline{p_{m_0-1}, p_{m_0}})$. But, we can choice the suitable values $t_0 < t_1 < t_2 < \cdots\cdots < t_n$ of T_ν and make n tend to infinity, so that the sum $\sum_{m=1}^{n}(p_{m-1}, p_m)$ converges to $l(A_\nu) + \sum_{\nu=1}^{\infty} l\big(\overline{f(a_r), f(b_r)}\big)$. Moreover we can suppose that in this case the values t_{m_0-1} and t_{m_0} tend to two fixed values v, v' respectively such that the interval (v, v') is one of the contiguous intervals (a_r, b_r) belonging to the complement of $\overline{T}_\nu$. Thus we have $l(A_\nu) \leqq l = 2S \Big/ \sin\frac{\theta}{4}$.

Since $A = \sum_{\nu=0}^{\mu-1} A_\nu$ and consequently $l(A) \leqq \sum_{\nu=0}^{\mu-1} l(A_\nu)$, $l(A)$ can not exceed the number $\mu\cdot l = 2S\left\{\left[\frac{4\pi}{\theta}\right]+1\right\} \Big/ \sin\frac{\theta}{4}$. Q. E. D.

Theorem 2. Let C be a rectifiable Jordan curve contained in a circle of radius $R (R > 0)$. Suppose that to every point p of C, we can associate a circle with center p and radius $r (r > 0)$, and two diameters qpq_1, $q'pq'_1$ of the circle making the angle θ $(0 < \theta < \pi)$, which satisfy the following condition (α): the curve C has no common points with two sectorial domains bounded by qpq_1, $q'pq'_1$ and two arcs qq', $q_1q'_1$ of the circumference.

Then the length of the curve C does not exceed a number $l_0 = l_0(\theta, r, R)$ depending only on θ, r and R, namely $l(C) \leqq l_0$.

Proof. Divide the whole plane into squares by straight lines which are parallel respectively to the x- and y-axis and pass through the points $\left(m\frac{2r}{\nu(\theta)}, 0\right)$ and $\left(0, m\frac{2r}{\nu(\theta)}\right)$, where $\nu(\theta)$ is a function of θ, and $m=0, \pm 1, \pm 2, \ldots\ldots$ We have thus a net covering the whole plane, whose meshes are squares of side $\frac{2r}{\nu(\theta)}$. All squares which have common points with C (in the interior or on the sides) can be contained in a square whose side has length $2\left(R+\frac{2r}{\nu(\theta)}\right)$. Hence, the number of former squares is not greater than

$$\lambda=\lambda(\theta, r, R)=\left[\left(2+\frac{R\nu(\theta)}{r}\right)^2\right].$$

Denote by Q_μ, $\mu=1, 2, 3, \ldots\ldots, \lambda'$; $\lambda' \leqq \lambda$, the closed square which has common points with the curve C, and put $C_\mu=Q_\mu \cdot C$. C_μ are generalized arcs of C.

Let μ be fixed, and let (a, b) be the center of the square Q_μ. Describe a circle S_μ with center (a, b) and radius $\frac{r}{2}$. To each point p of C_μ, we can associate a circle $S(p)$ with center p and two diameters qpq_1, $q'pq'_1$ making the angle θ, which satisfy the condition (α). The circle S_μ is contained in the interior of $S(p)$ and the point p lies within S_μ. Consequently the domains between two diameters qpq_1, $q'pq'_1$ cut off, from the circumference of S_μ, two distinct arcs which we denote by $I(p)$ and $I'(p)$. The lengths of these arcs are greater than $2r\left(\frac{1}{2}-\frac{\sqrt{2}}{\nu(\theta)}\right)\sin\frac{\theta}{2}$, and this is greater than $\frac{r}{2}\sin\frac{\theta}{2}$, if $\nu(\theta)\geqq 8$.

We say that two points p_1, p_2 of C are *connected in* S_μ, if the arc $\mathfrak{A}(t_1, t_2)$, $p_1=f(t_1)$, $p_2=f(t_2)$ of C has no points on the outside of the circle S_μ. We can denote it by $p_1 \sim p_2$ in S_μ. If $p_1 \sim p_2$ in S_μ and $p_2 \sim p_3$ in S_μ, then we have also $p_1 \sim p_3$ in S_μ. Thus the points of C_μ are classified, collecting the points connected each other in S_μ in the same class A^μ.

Let us see now that there exist at most $N(\theta)=\left[16\pi : \sin\frac{\theta}{2}\right]$ classes A^μ. To prove this, we first confirm that, if we take one point p_ν from each of these classes, then any three of $\frac{1}{4}I(p_\nu)$ has no point in common. For, if $\frac{1}{4}I(p_\nu)$ and $\frac{1}{4}I(p_{\nu'})$ have a point in common, then the common part $I(p_\nu)\cdot I(p_{\nu'})$ will contain an arc of length $\frac{3}{4}\frac{r\sin\frac{\theta}{2}}{2}$. And hence, if we set $\nu(\theta)>2\sqrt{2} : \sin\left(\frac{3}{16}\sin\frac{\theta}{2}\right)$ or $\left[24 : \sin\frac{\theta}{2}\right]+1$, then $I'(p_\nu)$ and $I'(p_{\nu'})$ will also have common points. Now each class considered above, determines an arc $\mathfrak{A}(\tau, \tau')$ of C, where τ, τ' are the extreme values of t such as $f(t) \sim p$ in S_μ, p being a point of the class. Let $\mathfrak{A}(\tau_\nu, \tau'_\nu)$ and $\mathfrak{A}(\tau_{\nu'}, \tau'_{\nu'})$ be the arcs determined by the classes of $p_\nu, p_{\nu'}$. We can suppose that $\tau_\nu < \tau_{\nu'}$. Then the curve C goes from the point p_ν to the outside of the circle S_μ and then comes to $p_{\nu'}$. And, afterwards C can not have a point without S_μ. Thus we see that no three of $\frac{1}{4} I(p_\nu)$ can have points in common. From this result, it follows immediately that $\frac{1}{2} N(\theta) \cdot \frac{1}{4} \frac{r \sin \frac{\theta}{2}}{2} \leqq \pi r$ or that $N(\theta) \leqq \left[16\pi : \sin \frac{\theta}{2}\right]$.

Thus we have at most $\lambda \cdot N(\theta)$ generalized arcs (classes) A_ν $\left(\nu = 1, 2, \ldots\ldots, n,\ n \leqq \lambda N(\theta)\right)$, which satisfy the following condition (β): to each point p of A_ν, we can associate two straight lines qpq_1, $q'pq'_1$ making the angle θ at p, and whose two opposite angles have no points in common with the arc $\mathfrak{A}(\tau, \tau')$ of the curve C. Thus, by Theorem 1, we have $l(A_\nu) \leqq l_1\left(\theta, \frac{r}{2}\right)$.

On the other hand, since $C = \sum_{\nu=1}^{n} A_\nu$, we get $l(C) \leqq \sum_{\nu=1}^{n} l(A_\nu) \leqq \lambda N(\theta) l_1$.

Q. E. D.