102. *Notes on Fourier Series (I): Riemann Sum.*

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1. Let $f(x)$ be a periodic function with period 1 and let us write

$$f_k(x)=\frac{1}{k}\sum_{\nu=0}^{k-1}f\left(x+\frac{\nu}{k}\right). \tag{1}$$

If $f(x)$ is integrable in the Riemann sense, then

$$\lim_{k\to\infty} f_k(x)=\int_0^1 f(t)dt. \tag{2}$$

Jessen[1] has shown that if $f(x)$ is integrable (in the Lebesgue sense), then

$$\lim_{n\to\infty} f_{2^n}(x)=\int_0^1 f(t)dt$$

for almost all x. Ursell[2] has shown that (2) is not necessarily true for integrable function $f(x)$ for almost all x, and (2) holds almost everywhere when $f(x)$ is positive decreasing and of squarely integrable in (0,1).

The object of the present paper is to prove the following theorem.

Theorem. Let $f(x)$ be integrable and

$$f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos 2\pi nx+b_n\sin 2\pi nx). \tag{3}$$

If $a_n\sqrt{\log n}$ and $b_n\sqrt{\log n}$ are Fourier coefficients of an integrable function, then (2) holds almost everywhere.

For the validity of (2) almost everywhere $f(x)$ can be discontinuous in a null set, for the condition of the theorem depends on the Fourier coefficients of $f(x)$ only. The condition of the theorem is satisfied when

$$\sum_{n=2}^{\infty}(a_n^2+b_n^2)\log n<\infty.$$

In this case, by the Riesz-Fischer theorem $a_n\sqrt{\log n}$ and $b_n\sqrt{\log n}$ are Fourier coefficients of squarely integrable function and then of integrable function.

2. Let us write

$$c_0=\frac{1}{2}a_0;\qquad c_n=\frac{1}{2}(a_n-ib_n),\qquad c_{-n}=\bar{c}_n\qquad (n>1),$$

1) Jessen, Annals of Math., **34** (1934).
2) Ursell, Journ. of the London Math. Soc., **12** (1937).

then (3) becomes

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}.$$

By (1)

$$f_k(x) \sim \frac{1}{k}\sum_{\nu=0}^{k-1}\{\sum_{n=-\infty}^{\infty} c_n e^{2\pi i \frac{\nu n}{k}} e^{2\pi i n x}\}$$

$$\sim \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}\left\{\frac{1}{k}\sum_{\nu=0}^{k-1} e^{2\pi i \frac{\nu n}{k}}\right\} \sim \sum_{n=-\infty}^{\infty} c_{kn} e^{2\pi i k n x},$$

that is

$$f_k(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_{nk}\cos 2\pi k n x + b_{nk}\sin 2\pi k n x). \tag{4}$$

Without loss of generality we can suppose that $a_0=0$. Thence we have to prove that

$$\lim_{k\to\infty} f_k(x) = 0$$

almost everywhere.

3. By the W. H. Young theorem

$$\frac{a_k}{2} + \sum_{n=1}^{\infty}\frac{\cos 2\pi n x}{\sqrt{\log (kn)}} \qquad (k>1)$$

is a Fourier series of a non-negative integrable function, which we denote by $\varphi_k(x)$, where a_k is taken such that

$$a_k, \quad \frac{1}{\sqrt{\log k}}, \quad \frac{1}{\sqrt{\log 2k}}$$

is a convex sequence and $a_k \to 0$ as $k \to \infty$.

By the condition of the theorem there is an integrable function $g(x)$ such that

$$g(x) \sim \sum_{n=2}^{\infty}(a_n \cos nx + b_n \sin nx)\sqrt{\log n}.$$

Since

$$\varphi_k(kt) \sim \frac{a_k}{2} + \sum_{n=1}^{\infty}\frac{\cos 2\pi k n t}{\sqrt{\log (kn)}},$$

we have

$$\int_0^1 \varphi_k(kt) g(t-x) dt \sim \sum_{n=1}^{\infty} a_{kn}\cos 2\pi k n x + b_{kn}\sin 2\pi k n x).$$

By (4) we have

$$f_k(x) = \int_0^1 \varphi_k(kt) g(t-x) dt$$

almost everywhere. Therefore it is sufficient to prove that

$$\lim_{k\to\infty}\int_0^1 \varphi_k(kt) g(t-x) dt = 0 \tag{5}$$

almost everywhere

4. If $g(t)$ is bounded, then there is an M such that $|g(x)| \leqq M$. In this case

$$\left|\int_0^1 \varphi_k(kt)g(t-x)dt\right| \leqq \int_0^1 \varphi_k(kt)\,|g(t-x)|\,dt \leqq M\int_0^1 \varphi_k(kt)dt$$

$$= \frac{M}{k}\int_0^k \varphi_k(t)dt = M\int_0^1 \varphi_k(t)dt = a_k \to 0\,, \quad \text{as} \quad k\to\infty\,.$$

Thus (5) is proved.

In the general case, let us put

$$E_n = \underset{t}{E}(|g(t)|>n) \qquad (n=1, 2, \ldots\ldots)\,,$$

then $mE_n \to 0$ as $n\to\infty$.

$$\int^1 \left|\int_{E_n} \varphi_k(kt)g(t-x)dt\right| dx \leqq \int_0^1 dx \int_{E_n} \varphi_k\big(k(t+x)\big)\,|g(t)|\,dt$$

$$\leqq \int_{E_n} |g(t)|\,dt \int_0^1 \varphi_k\big(k(t+x)\big)dx = a_k\int_{E_n} |g(t)|\,dt \to 0\,, \quad \text{as} \quad n\to\infty\,.$$

Hence there is a subsequence $\{E_{n_\nu}\}$ of $\{E_n\}$ such that

$$\lim_{\nu\to\infty}\int_{E_{n_\nu}} \varphi_k(kt)g(t-x)dt = 0$$

almost everywhere for all k.

For any positive ε there is an m such that

$$\left|\int_{E_{n_m}} \varphi_k(kt)g(t-x)dt\right| < \varepsilon$$

almost everywhere. We have

$$\int_0^1 \varphi_k(kt)\,g(t-x)dt = \int_{E_m} \varphi_k(kt)\,g(t-x)dt + \int_{CE_m} \varphi_k(kt)g(t-x)dt\,,$$

where CE denotes the complementary set of E. The second term of the right hand side tends to zero as $k\to\infty$, as was proved. Thus

$$\overline{\lim_{k\to\infty}}\left|\int_0^1 \varphi_k(kt)g(t-x)dt\right| \leqq \varepsilon$$

almost everywhere. Since ε is arbitrary, the theorem is proved.