101. *Concircular Geometry III. Theory of Curves.*

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In the two recent papers "Concircular Geometry I, and II[1]," we have considered concircular transformations $\bar{g}_{\mu\nu}=\rho^2 g_{\mu\nu}$ of a Riemannian metric $ds^2=g_{\mu\nu}du^\mu du^\nu$, that is to say, conformal transformations $\bar{g}_{\mu\nu}=\rho^2 g_{\mu\nu}$ with the function ρ satisfying

$$\rho_{\mu\nu}\equiv\frac{\partial\rho_\mu}{\partial u^\nu}-\rho_\lambda\left\{{}^{\lambda}_{\mu\nu}\right\}-\rho_\mu\rho_\nu+\frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta g_{\mu\nu}=\phi g_{\mu\nu}\,,$$

where ρ_μ denotes $\partial\log\rho/\partial u^\mu$ and $\left\{{}^{\lambda}_{\mu\nu}\right\}$ the three-index symbols of Christoffel formed with $g_{\mu\nu}$, and we have discussed the integrability conditions of these partial differential equations.

The purpose of the present note is to develop the theory of curves in the concircular geometry.

§ 1. *Frenet formulae.* Let us consider a curve $u^\lambda(s)$ in a Riemann space, s being the curve length measured from a fixed point on the curve, and form the vector

$$V^\lambda=\frac{\delta^3u^\lambda}{\delta s^3}+\frac{\delta u^\lambda}{\delta s}g_{\mu\nu}\frac{\delta^2u^\mu}{\delta s^2}\frac{\delta^2u^\nu}{\delta s^2} \tag{1.1}$$

where $\dfrac{\delta}{\delta s}$ denotes the covariant differentiation along the curve.

If we effect a conformal transformation of the metric

$$\bar{g}_{\mu\nu}=\rho^2 g_{\mu\nu}\,, \tag{1.2}$$

the vector V^λ will be transformed into

$$\bar{V}^\lambda=\frac{1}{\rho^3}\left[V^\lambda+\frac{\delta u^\lambda}{\delta s}\rho_{\mu\nu}\frac{\delta u^\mu}{\delta s}\frac{\delta u^\nu}{\delta s}-g^{\lambda\alpha}\rho_{\alpha\nu}\frac{\delta u^\nu}{\delta s}\right]. \tag{1.3}$$

Hence, if the conformal transformation (1.2) is a concircular one, that is to say, if the function ρ satisfies

$$\rho_{\mu\nu}=\phi g_{\mu\nu}\,, \tag{1.4}$$

the equations (1.3) become

$$\bar{V}^\lambda=\frac{1}{\rho^3}V^\lambda, \tag{1.5}$$

which shows that the direction defined by the vector V^λ is invariant under a concircular transformation.

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1) K. Yano: Concircular Geometry I, Proc. **16** (1940), 195–200, and Concircular Geometry II, Proc. **16** (1940), 354–360.

Putting

$$\begin{cases} \dfrac{\delta u^\lambda}{\delta s} = \underset{1}{\eta}^\lambda, \\ V^\lambda = \overset{1}{k}\underset{2}{\eta}^\lambda, \end{cases} \tag{1.6}$$

where

$$(\overset{1}{k})^2 = g_{\mu\nu} V^\mu V^\nu,$$

we can see that

$$g_{\mu\nu}\underset{1}{\eta}^\mu \underset{1}{\eta}^\nu = 1, \qquad g_{\mu\nu}\underset{1}{\eta}^\mu \underset{2}{\eta}^\nu = 0 \quad \text{and} \quad g_{\mu\nu}\underset{2}{\eta}^\mu \underset{2}{\eta}^\nu = 1, \tag{1.7}$$

and that the law of transformations of $\underset{1}{\eta}^\lambda$ and $\underset{2}{\eta}^\lambda$ under a concircular transformation (1.2) is given by

$$\underset{1}{\bar{\eta}}^\lambda = \frac{1}{\rho}\underset{1}{\eta}^\lambda \quad \text{and} \quad \underset{2}{\bar{\eta}}^\lambda = \frac{1}{\rho}\underset{2}{\eta}^\lambda, \tag{1.8}$$

respectively. The vector $\underset{2}{\eta}^\lambda$ being transformed by (1.8), the covariant derivative $\frac{\delta}{\delta s}\underset{2}{\eta}^\lambda$ of $\underset{2}{\eta}^\lambda$ along the curve is transformed by the following equations

$$\frac{\delta}{\delta \bar{s}}\underset{2}{\bar{\eta}}^\lambda = \frac{1}{\rho^2}\left[\frac{\delta}{\delta s}\underset{2}{\eta}^\lambda + \underset{1}{\eta}^\lambda \rho_\nu \underset{2}{\eta}^\nu\right], \tag{1.9}$$

from which we have

$$\underset{1}{\bar{\eta}}_\nu \frac{\delta}{\delta \bar{s}}\underset{2}{\bar{\eta}}^\nu = \frac{1}{\rho}\left[\underset{1}{\eta}_\nu \frac{\delta}{\delta s}\underset{2}{\eta}^\nu + \rho_\nu \underset{2}{\eta}^\nu\right]$$

or multiplying by $\underset{1}{\bar{\eta}}^\lambda = \frac{1}{\rho}\underset{1}{\eta}^\lambda$,

$$\underset{1}{\bar{\eta}}^\lambda \underset{1}{\bar{\eta}}_\nu \frac{\delta}{\delta \bar{s}}\underset{2}{\bar{\eta}}^\nu = \frac{1}{\rho^2}\left[\underset{1}{\eta}^\lambda \underset{1}{\eta}_\nu \frac{\delta}{\delta s}\underset{2}{\eta}^\nu + \underset{1}{\eta}^\lambda \rho_\nu \underset{2}{\eta}^\nu\right]. \tag{1.10}$$

Subtracting (1.10) from (1.9), we get the equations

$$\frac{\delta}{\delta \bar{s}}\underset{2}{\bar{\eta}}^\lambda - \underset{1}{\bar{\eta}}^\lambda \underset{1}{\bar{\eta}}_\nu \frac{\delta}{\delta \bar{s}}\underset{2}{\bar{\eta}}^\nu = \frac{1}{\rho^2}\left[\frac{\delta}{\delta s}\underset{2}{\eta}^\lambda - \underset{1}{\eta}^\lambda \underset{1}{\eta}_\nu \frac{\delta}{\delta s}\underset{2}{\eta}^\nu\right], \tag{1.11}$$

which show that the direction defined by the vector

$$\frac{D}{Ds}\underset{2}{\eta}^\lambda = \frac{\delta}{\delta s}\underset{2}{\eta}^\lambda - \underset{1}{\eta}^k \underset{1}{\eta}_\nu \frac{\delta}{\delta s}\underset{2}{\eta}^\nu \tag{1.12}$$

is invariant under a concircular transformation.

From equations (1.7) and (1.12), we have

$$g_{\mu\nu}\underset{1}{\eta}^\mu \frac{D}{Ds}\underset{2}{\eta}^\nu = 0, \qquad g_{\mu\nu}\underset{2}{\eta}^\mu \frac{D}{Ds}\underset{2}{\eta}^\nu = 0.$$

Thus we see that the concircularly invariant direction given by the vector $\frac{D}{Ds}\underset{2}{\eta}^\lambda$ is orthogonal to the both of concircularly invariant directions given by the vectors $\underset{1}{\eta}^\lambda$ and $\underset{2}{\eta}^\lambda$. Thus putting

$$\frac{D}{Ds}\underset{2}{\eta}^\lambda=\overset{2}{k}\underset{3}{\eta}^\lambda, \tag{1.13}$$

where

$$(\overset{2}{k})^2=g_{\mu\nu}\left(\frac{D}{Ds}\underset{2}{\eta}^\mu\right)\left(\frac{D}{Ds}\underset{2}{\eta}^\nu\right), \tag{1.14}$$

we have

$$g_{\mu\nu}\underset{a}{\eta}^\mu\underset{b}{\eta}^\nu=\delta_{ab} \qquad (a, b=1, 2, 3). \tag{1.15}$$

Under a concircular transformation, the vectors $\underset{1}{\eta}^\lambda$, $\underset{2}{\eta}^\lambda$ and $\underset{3}{\eta}^\lambda$ being transformed by

$$\underset{1}{\bar\eta}^\lambda=\frac{1}{\rho}\underset{1}{\eta}^\lambda, \qquad \underset{2}{\bar\eta}^\lambda=\frac{1}{\rho}\underset{2}{\eta}^\lambda \quad\text{and}\quad \underset{3}{\bar\eta}^\lambda=\frac{1}{\rho}\underset{3}{\eta}^\lambda, \tag{1.16}$$

respectively, we can see by the same process as used above that the vector defined by

$$\frac{D}{Ds}\underset{3}{\eta}^\lambda=\frac{\delta}{\delta s}\underset{3}{\eta}^\lambda-\underset{1}{\eta}^\lambda\underset{1}{\eta}_\nu\frac{\delta}{\delta s}\underset{3}{\eta}^\nu \tag{1.17}$$

is transformed as follows

$$\frac{D}{D\bar s}\underset{3}{\bar\eta}^\lambda=\frac{1}{\rho^2}\frac{D}{Ds}\underset{3}{\eta}^\lambda, \tag{1.18}$$

and that the vector $\frac{D}{Ds}\underset{3}{\eta}^\lambda$ satisfies the equations

$$g_{\mu\nu}\underset{1}{\eta}^\mu\frac{D}{Ds}\underset{3}{\eta}^\nu=0, \qquad \overset{2}{k}+g_{\mu\nu}\underset{2}{\eta}^\mu\underset{3}{\eta}^\nu=0, \qquad g_{\mu\nu}\underset{3}{\eta}^\mu\frac{D}{Ds}\underset{3}{\eta}^\nu=0,$$

or

$$\begin{cases} g_{\mu\nu}\underset{1}{\eta}^\mu\left(\overset{2}{k}\underset{2}{\eta}^\nu+\frac{D}{Ds}\underset{3}{\eta}^\nu\right)=0, \qquad g_{\mu\nu}\underset{2}{\eta}^\mu\left(\overset{2}{k}\underset{2}{\eta}^\nu+\frac{D}{Ds}\underset{3}{\eta}^\nu\right)=0, \\ g_{\mu\nu}\underset{3}{\eta}^\mu\left(\overset{2}{k}\underset{2}{\eta}^\nu+\frac{D}{Ds}\underset{3}{\eta}^\nu\right)=0, \end{cases} \tag{1.19}$$

which show that the concircularly invariant direction given by the vector $\overset{2}{k}\underset{2}{\eta}^\lambda+\frac{D}{Ds}\underset{3}{\eta}^\lambda$ is orthogonal to the three concircularly invariant directions given by $\underset{1}{\eta}^\lambda$, $\underset{2}{\eta}^\lambda$ and $\underset{3}{\eta}^\lambda$. Thus putting

$$\overset{2}{k}\underset{2}{\eta}^\lambda+\frac{D}{Ds}\underset{3}{\eta}^\lambda=\overset{3}{k}\underset{4}{\eta}^\lambda$$

or

$$\frac{D}{Ds}\underset{3}{\eta}^\lambda=-\overset{2}{k}\underset{2}{\eta}^\lambda+\overset{3}{k}\underset{4}{\eta}^\lambda, \tag{1.20}$$

where

$$(\overset{3}{k})^2=g_{\mu\nu}\left(\overset{2}{k}\underset{2}{\eta}^\mu+\frac{D}{Ds}\underset{3}{\eta}^\mu\right)\left(\overset{2}{k}\underset{2}{\eta}^\nu+\frac{D}{Ds}\underset{3}{\eta}^\nu\right),$$

we have

$$g_{\mu\nu}\underset{a}{\eta}^{\mu}\underset{b}{\eta}^{\nu}=\delta_{ab} \qquad (a, b=1, 2, 3, 4). \tag{1.21}$$

Furthering this process, finally we obtain the following equations

$$\left\{\begin{aligned} &\underset{1}{\eta}^{\lambda}=\frac{\delta u^{\lambda}}{\delta s}, \\ &\underset{2}{\eta}^{\lambda}=\frac{1}{\overset{1}{k}}V^{\lambda}, \\ &\frac{D}{Ds}\underset{2}{\eta}^{\lambda}=\overset{2}{k}\underset{3}{\eta}^{\lambda}, \\ &\frac{D}{Ds}\underset{a}{\eta}^{\lambda}=-\overset{a-1}{k}\underset{a-1}{\eta}^{\lambda}+\overset{a}{k}\underset{a+1}{\eta}^{\lambda} \qquad (a=3, 4, \dots, n, \quad \overset{n}{k}=0). \end{aligned}\right. \tag{1.22}$$

These are the Frenet formulae in concircular geometry.[1)]

§ 2. *Geodesic circles on hypersurfaces.*

Let

$$u^{\lambda}=u^{\lambda}(u^{\dot{1}}, u^{\dot{2}}, \dots, u^{\dot{n}-\dot{1}}) \tag{2.1}$$

be the equations of a hypersurface V_{n-1} in our Riemannian space, u^{i} $(i, j, k, \dots=\dot{1}, \dot{2}, \dots, \dot{n}-\dot{1})$ being parameters for V_{n-1}. Then the fundamental tensor g_{jk} and the Christoffel symbols $\{{}^{i}_{jk}\}$ of V_{n-1} are respectively given by

$$g_{jk}=B_{j}^{\cdot\mu}B_{k}^{\cdot\nu}g_{\mu\nu} \tag{2.2}$$

and

$$\{{}^{i}_{jk}\}=B^{i}_{\cdot\lambda}(B_{j}^{\cdot\mu}B_{k}^{\cdot\nu}\{{}^{\lambda}_{\mu\nu}\}+B_{j,k}^{\cdot\lambda}) \tag{2.3}$$

where

$$B_{j}^{\cdot\mu}=\frac{\partial u^{\mu}}{\partial u^{j}}, \qquad B^{i}_{\cdot\lambda}=g^{ij}g_{\lambda\mu}B_{j}^{\cdot\mu} \quad \text{and} \quad B_{j,k}^{\cdot\lambda}=\frac{\partial B_{j}^{\cdot\lambda}}{\partial u^{k}}. \tag{2.4}$$

The Euler-Schouten curvature tensor of V_{n-1} in V_{n} being defined by

$$H_{jk}^{\cdot\cdot\lambda}=B_{j,k}^{\cdot\lambda}+B_{j}^{\cdot\mu}B_{k}^{\cdot\nu}\{{}^{\lambda}_{\mu\nu}\}-B_{i}^{\cdot\lambda}\{{}^{i}_{jk}\}, \tag{2.5}$$

it is easily seen that the tensor defined by

$$M_{jk}^{\cdot\cdot\lambda}=H_{jk}^{\cdot\cdot\lambda}-\frac{1}{n-1}H^{a\cdot\lambda}_{\cdot a}g_{jk}, \tag{2.6}$$

where

$$H^{i\cdot\lambda}_{\cdot k}=g^{ij}H_{jk}^{\cdot\cdot\lambda}, \tag{2.7}$$

is concircularly invariant.

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1) The method of obtaining these conformal formulae was already indicated in K. Yano, Sur la connexion de Weyl-Hlavatý et ses applications à la géométrie conforme, Proc. Physico-Math. Soc. Japan **22** (1940), 595–621.

Denoting by B^λ the unit vector normal to the hypersurface V_{n-1}, we can put

$$H_{jk}^{\cdot\cdot\lambda}=H_{jk}B^\lambda \quad\text{and}\quad M_{jk}^{\cdot\cdot\lambda}=M_{jk}B^\lambda, \tag{2.8}$$

$H_{jk}^{\cdot\cdot\lambda}$ and $M_{jk}^{\cdot\cdot\lambda}$ being orthogonal to the hypersurface, if we regard them as vectors in V_n with respect to the index λ. Then the equations of Weingarten may be written as

$$B^\lambda_{;j}=-B_i^{\cdot\lambda}H^{i}_{\cdot j}, \tag{2.9}$$

where

$$H^{i}_{\cdot j}=g^{ia}H_{aj}, \tag{2.10}$$

and the semi-colon denotes the covariant derivative.

We shall now consider a curve $u^i(s)$ on this hypersurface. This is also regarded as defining a curve $u^\lambda(s)$ in V_n. Then differentiating $u^\lambda(s)$ along the curve we have

$$\frac{\delta u^\lambda}{\delta s}=B_i^{\cdot\lambda}\frac{\delta u^i}{\delta s}, \tag{2.11}$$

$$\frac{\delta^2 u^\lambda}{\delta s^2}=B_i^{\cdot\lambda}\frac{\delta^2 u^i}{\delta s^2}+H_{jk}^{\cdot\cdot\lambda}\frac{\delta u^j}{\delta s}\frac{\delta u^k}{\delta s}, \tag{2.12}$$

$$\frac{\delta^3 u^\lambda}{\delta s^3}=B_i^{\cdot\lambda}\frac{\delta^3 u^i}{\delta s^3}+3H_{jk}^{\cdot\cdot\lambda}\frac{\delta^2 u^j}{\delta s^2}\frac{\delta u^k}{\delta s}+H_{jk;h}^{\cdot\cdot\lambda}\frac{\delta u^j}{\delta s}\frac{\delta u^k}{\delta s}\frac{\delta u^h}{\delta s}. \tag{2.13}$$

From these equations, we obtain

$$\begin{aligned}\frac{\delta^3 u^\lambda}{\delta s^3}+\frac{\delta u^\lambda}{\delta s}g_{\mu\nu}\frac{\delta^2 u^\mu}{\delta s^2}\frac{\delta^2 u^\nu}{\delta s^2}&=B_i^{\cdot\lambda}\left(\frac{\delta^3 u^i}{\delta s^3}+\frac{\delta u^i}{\delta s}g_{jk}\frac{\delta^2 u^j}{\delta s^2}\frac{\delta^2 u^k}{\delta s^2}\right)\\&+3H_{jk}^{\cdot\cdot\lambda}\frac{\delta^2 u^j}{\delta s^2}\frac{\delta u^k}{\delta s}+H_{jk;h}^{\cdot\cdot\lambda}\frac{\delta u^j}{\delta s}\frac{\delta u^k}{\delta s}\frac{\delta u^h}{\delta s}\\&+B_i^{\cdot\lambda}H_{jk}H_{hl}\frac{\delta u^i}{\delta s}\frac{\delta u^j}{\delta s}\frac{\delta u^k}{\delta s}\frac{\delta u^h}{\delta s}\frac{\delta u^l}{\delta s}.\end{aligned}$$

Substituting, in these equations, the following relations

$$H_{jk}^{\cdot\cdot\lambda}=M_{jk}^{\cdot\cdot\lambda}+\frac{1}{n-1}H^{a\cdot\lambda}_{\cdot a}g_{jk},$$

$$\begin{aligned}H_{jk;h}^{\cdot\cdot\lambda}&=\left(M_{jk}B^\lambda+\frac{1}{n-1}H^{a}_{\cdot a}B^\lambda g_{jk}\right)_{;h}\\&=\left(M_{jk;h}+\frac{1}{n-1}H^{a}_{\cdot a;h}g_{jk}\right)B^\lambda-\left(M_{jk}+\frac{1}{n-1}H^{a}_{\cdot a}g_{jk}\right)B_i^{\cdot\lambda}H^{i}_{\cdot h}\\&=\left(M_{jk;h}+\frac{1}{n-1}H^{a}_{\cdot a;h}g_{jk}\right)B^\lambda\\&\qquad-B_i^{\cdot\lambda}\left(M_{jk}+\frac{1}{n-1}H^{a}_{\cdot a}g_{jk}\right)\left(M^{i}_{\cdot h}+\frac{1}{n-1}H^{b}_{\cdot b}\delta^i_h\right),\end{aligned}$$

where

$$M^{i}_{\cdot h}=g^{ij}M_{jh},$$

we find

$$(2.14)\qquad \frac{\delta^3 u^\lambda}{\delta s^3}+\frac{\delta u^\lambda}{\delta s}g_{\mu\nu}\frac{\delta^2 u^\mu}{\delta s^2}\frac{\delta^2 u^\nu}{\delta s^2}=B_i^{\cdot\lambda}\left(\frac{\delta^3 u^i}{\delta s^3}+\frac{\delta u^i}{\delta s}g_{jk}\frac{\delta^2 u^j}{\delta s^2}\frac{\delta^2 u^k}{\delta s^2}\right)$$

$$+3M_{jk}^{\cdot\cdot\lambda}\frac{\delta^2 u^j}{\delta s^2}\frac{\delta^2 u^k}{\delta s^2}-B_i^{\cdot\lambda}\Big[M_{jk}M^i_{\cdot h}\frac{\delta u^j}{\delta s}\frac{\delta u^k}{\delta s}\frac{\delta u^h}{ds}$$

$$+\frac{1}{n-1}H^a_{\cdot a}\left(M^i_{\cdot h}\frac{\delta u^h}{\delta s}-\frac{\delta u^i}{\delta s}M_{jk}\frac{\delta u^j}{\delta s}\frac{\delta u^k}{\delta s}\right)$$

$$-\frac{\delta u^i}{\delta s}M_{jk}M_{hl}\frac{\delta u^j}{\delta s}\frac{\delta u^k}{\delta s}\frac{\delta u^h}{\delta s}\frac{\delta u^l}{\delta s}\Big]$$

$$+B^\lambda M_{jk;h}\frac{\delta u^j}{\delta s}\frac{\delta u^k}{\delta s}\frac{\delta u^h}{\delta s}+\frac{1}{n-1}B^\lambda H^a_{\cdot a;h}\frac{\delta u^h}{\delta s}.$$

Suppose now that any geodesic circles of the hypersurface V_{n-1} can also be regarded as a geodesic circle of the enveloping space V_n, then wo have

$$(2.15)\qquad 3M_{jk}^{\cdot\cdot\lambda}\frac{\delta^2 u^j}{\delta s^2}\frac{\delta u^k}{\delta s}-B_i^{\cdot\lambda}\Big[M_{jk}M^i_{\cdot h}\frac{\delta u^j}{\delta s}\frac{\delta u^k}{\delta s}\frac{\delta u^h}{\delta s}$$

$$+\frac{1}{n-1}H^a_{\cdot a}\left(M^i_{\cdot h}\frac{\delta u^h}{\delta s}-\frac{\delta u^i}{\delta s}M_{jk}\frac{\delta u^j}{\delta s}\frac{\delta u^k}{\delta s}\right)$$

$$-\frac{\delta u^i}{\delta s}M_{jk}M_{hl}\frac{\delta u^j}{\delta s}\frac{\delta u^k}{\delta s}\frac{\delta u^h}{\delta s}\frac{\delta u^l}{\delta s}\Big]$$

$$+B^\lambda M_{jk;h}\frac{\delta u^j}{\delta s}\frac{\delta u^k}{\delta s}\frac{\delta u^h}{\delta s}$$

$$+\frac{1}{n-1}B^\lambda H^a_{\cdot a;h}\frac{\delta u^h}{\delta s}=0$$

for any $\dfrac{\delta^2 u^i}{\delta s^2}$ and $\dfrac{\delta u^i}{\delta s}$ arbitrary except the condition

$$g_{jk}\frac{\delta^2 u^j}{\delta s^2}\frac{\delta u^k}{\delta s}=0.$$

From the equation (2.15) we have

$$M_{jk}^{\cdot\cdot\lambda}=\alpha^\lambda g_{jk},$$

from which we conclude

$$(2.16)\qquad M_{jk}^{\cdot\cdot\lambda}=0$$

because of the identity $g^{jk}M_{jk}^{\cdot\cdot\lambda}=0$.

Substituting (2.16) in (2.15), we have

$$(2.17)\qquad H^a_{\cdot a;h}=0.$$

Thus we have the

Theorem. *If any geodesic circle of a hypersurface V_{n-1} can be regarded as a geodesic circle of the enveloping space V_n, then the hyper-*

surface is totally umbilical and the mean curvature is constant on the hypersurface.

Remark. The property that the mean curvature of a totally umbilical hypersurface is constant is not a conformal one, but is a concircular one.

For, under a concircular transformation (1.2), the Euler-Schouten tensor $H_{jk}^{\cdot\cdot\lambda}$ being transformed by

$$\bar{H}_{jk}^{\cdot\cdot\lambda}=H_{jk}^{\cdot\cdot\lambda}-g_{jk}\rho_a B^a B^\lambda ,$$

we have

$$\rho\bar{H}^{a}_{\cdot a}=H^{a}_{\cdot a}-(n-1)\rho_a B^a . \tag{2.18}$$

Differentiating this equation covariantly, we have

$$\rho\rho_j\bar{H}^{a}_{\cdot a}+\rho\bar{H}^{a}_{\cdot a;j}=H^{a}_{\cdot a;j}-(n-1)\rho_{a;\beta}B^a B_j^{\cdot\beta}+(n-1)\rho_a B_i^{\cdot a}H^{i}_{\cdot j} \tag{2.19}$$

where

$$\rho_j=\rho_a B_j^{\cdot a}=\frac{\partial \log \rho}{\partial u^j} .$$

Substituting

$$\rho_{a;\beta}=\psi g_{a\beta}+\rho_a\rho_\beta$$

and (2.18) in (2.19), we find

$$\rho\bar{H}^{a}_{\cdot a;j}=H^{a}_{\cdot a;j}+(n-1)\rho_i M^{i}_{\cdot j}$$

or

$$\frac{\rho}{n-1}\bar{H}^{a}_{\cdot a;j}=\frac{1}{n-1}H^{a}_{\cdot a;j}+\rho_i M^{i}_{\cdot j} . \tag{2.20}$$

The equations (2.20) show that the property that the mean curvature of a totally umbilical hypersurface is constant is a concircular one.