104. Analytical Characterization of Displacements in General Poincaré Space.

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In recent papers M. Sugawara has constructed a theory of automorphic functions of higher dimensions, as a generalization of Poincaré's theory[1]. He has considered the space $\mathfrak{A}_{(n)}$, whose points are symmetric matrices of order n with the property $E^{(n)}-\bar{Z}'Z>0$, and defined the displacements in $\mathfrak{A}_{(n)}$ as follows: Let $U=\begin{pmatrix}U_1U_2\\U_3U_4\end{pmatrix}$ be a matrix of order $2n$ satisfying the conditions $U'JU=J$, $U'S\bar{U}=S$, where $J=\begin{pmatrix}0 & E^{(n)}\\-E^{(n)} & 0\end{pmatrix}$, $S=\begin{pmatrix}E^{(n)} & 0\\0 & -E^{(n)}\end{pmatrix}$. Then the transformation $W=(U_1Z+U_2)(U_3Z+U_4)^{-1}$ is called a displacement in $\mathfrak{A}_{(n)}$. In the classical case $n=1$, as is well known, the transformations of the type described above exhaust *all* the one-to-one analytic transformations which map $\mathfrak{A}_{(n)}$ into itsfelf. Then arises the problem: Does this fact remain true in our general case? In what follows this problem will be discussed for the spaces $\mathfrak{A}_{(n)}$ and $\mathfrak{A}_{(n,m)}$[2]. The answer is affirmative except for $\mathfrak{A}_{(n,n)}$. As in the classical case we are led to this result by an analogue to *Schwarz's lemma* in higher dimensions.

1. The set of all matrices of type (n, m) shall be denoted by $\mathfrak{R}_{(n,m)}$.

Theorem 1. If a mapping f of $\mathfrak{R}_{(n,m)}$ into itself satisfies the conditions: (1) $f(\alpha A+\beta B)=\alpha f(A)+\beta f(B)$, *($\alpha, \beta$ being complex numbers)* (2) *according as the rank of Z is 1 or 2, the rank of the image $f(Z)$ is 1 or $\geqq 2$, then the mapping f can be written in the following form: $f(Z)=AZB$, when $n\neq m$; $f(Z)=AZB$ or $AZ'B$, when $n=m$. Here A and B are non-singular constant matrices of orders n and m respectively.*

Proof. We shall denote the matrix units by $E_{\alpha\beta}$: the (α, β)-element of $E_{\alpha\beta}$ is equal to 1 and the other elements are all zeroes. For brevity let us call that a matrix A has the form (a) or (b), according as A can be written in the form $A=\sum_{\alpha=1}^{n}a_{\alpha 1}E_{\alpha 1}$ or $A=\sum_{\beta=1}^{m}a_{1\beta}E_{1\beta}$, where $a_{\alpha 1}, a_{1\beta}$ are numbers. Now, by the condition (2), there exist non-singular matrices A_1 and B_1 (of orders n and m) such that $A_1f(E_{11})B_1=E_{11}$. Then $A_1f(E_{i1})B_1 (i>1)$ has the form (a) or (b). For, if we put $A_1f(E_{i1})B_1=\sum_{\alpha,\beta}c_{\alpha\beta}E_{\alpha\beta}$ for a fixed i, we have, by the condition (2), $c_{11}c_{\alpha\beta}-c_{\alpha 1}c_{1\beta}=0$

1) M. Sugawara, Über eine allgemeine Theorie der Fuchsschen Gruppen und Theta-Reihen, Ann. Math., **41** (1940), 488–494. On the general zetafuchsian functions, Proc. **16** (1940), 367–372. A generalization of Poincaré-space, Proc. **16** (1940), 373–377, to be cited as [S—3].

2) K. Morita, A remark on the theory of general fuchsian groups, Proc. **17** (1941), 233–237, to be cited as [M—1].

for $\alpha>1$, $\beta>1$. Applying the condition (2) to the matrix $E_{11}+E_{i1}$, we get $(c_{11}+1)c_{\alpha\beta}-c_{\alpha 1}c_{1\beta}=0$. Hence $c_{\alpha\beta}=0$ for $\alpha>1$, $\beta>1$. Noting again the rank of $A_1f(E_{i1})B_1$ we know that $A_1f(E_{i1})B_1$ must be of the form (a) or (b). This reasoning holds equally for matrices $A_1f(E_{1j})B_1$ $(j>1)$.

Next we will show that, if $A_1f(E_{21})B_1$ has the form (a), then the forms of $A_1f(E_{i1})B_1$ and $A_1f(E_{1j})B_1 (i>1,\ j>1)$ must be (a) and (b) respectively. Suppose that $A_1f(E_{i1})B_1$ were of the form (b) for some $i>2$. Then we could put $A_1f(E_{i1})B_1=\sum_{\beta=1}^{m}c_{1\beta}^{(i1)}E_{1\beta}$. Among the numbers $c_{1\beta}^{(i1)}$ there would exist a number $c_{1\beta_0}^{(i1)}\neq 0 (\beta_0>1)$. (Otherwise, the rank of $A_1f(c_{11}^{(i1)}E_{11}-E_{i1})B_1$ is equal to zero.) Similarly there exists a number $c_{\alpha_0 1}^{(21)}\neq 0 (\alpha_0>1)$, where $A_1f(E_{21})B_1=\sum_{\alpha=1}^{n}c_{\alpha 1}^{(21)}E_{\alpha 1}$. Then the rank of $A_1f(E_{21}+E_{i1})B_1$ would be equal to 2, which contradicts the condition (2). On the other hand, $A_1f(E_{1j})B_1$ are clearly of the form (b). Therefore the only possible cases are the following.

(1st case) The matrices $A_1f(E_{i1})B_1$ $(i>1)$ are all of the form (a) and $A_1f(E_{1j})B_1$ $(j>1)$ are all of the form (b).

(2nd case) The matrices $A_1f(E_{i1})B_1$ $(i>1)$ are all of the form (b) and $A_1f(E_{1j})B_1$ $(j>1)$ are all of the form (a).

(*1st case*). Let us put $A_1f(E_{i1})B_1=\sum_{\alpha=1}^{n}c_{\alpha 1}^{(i1)}E_{\alpha 1}$, $A_1f(E_{1j})B_1=\sum_{\beta=1}^{m}c_{1\beta}^{(1j)}E_{1\beta}$. By the conditions (1) and (2) we know that there exist non-singular matrices A_2 and B_2 of orders n and m satisfying the conditions: $(c_{11}^{(11)}c_{21}^{(11)}\dots c_{n1}^{(11)})A_2'=(10\dots 0)$, $(c_{11}^{(21)}c_{21}^{(21)}\dots c_{n1}^{(21)})A_2'=(010\dots 0)$, ..., $(c_{11}^{(n1)}c_{21}^{(n1)}\dots c_{n1}^{(n1)})A_2'=(0\dots 01)$; $(c_{11}^{(11)}c_{12}^{(11)}\dots c_{1m}^{(11)})B_2=(10\dots 0)$, $(c_{11}^{(12)}c_{12}^{(12)}\dots c_{1m}^{(12)})B_2=(010\dots 0)$, ..., $(c_{11}^{(1m)}c_{12}^{(1m)}\dots c_{1m}^{(1m)})B_2=(0\dots 01)$. Since $A_1f(E_{11})B_1=E_{11}$, we have $A_2A_1f(E_{i1})B_1B_2=E_{i1}$, $A_2A_1f(E_{1j})B_1B_2=E_{1j}$.

Now let us put $f_1(Z)=A_2A_1f(Z)B_1B_2$. Then the above result shows that $f_1(E_{i1})=E_{i1}$ and $f_1(E_{1j})=E_{1j}$. However, we can further prove that $f_1(E_{ij})=E_{ij}$ for all i, j. For fixed $i>1$ and $j>1$ we put $f_1(E_{ij})=\sum_{\alpha,\beta}c_{\alpha\beta}E_{\alpha\beta}$. First we will show that $c_{\alpha\beta}=0$ for all α, β such that $\alpha>1$, $\beta>1$, $(\alpha,\beta)\neq(i,j)$. If $\alpha>1$, $\alpha\neq i$, we get $c_{i1}c_{\alpha\beta}-c_{\alpha 1}c_{i\beta}=0$. Since the rank of $f_1(E_{i1}+E_{ij})$ is equal to 1, $(1+c_{i1})c_{\alpha\beta}-c_{\alpha 1}c_{i\beta}=0$. Hence $c_{\alpha\beta}=0$. In case $\beta>1$, $\beta\neq j$, we have $c_{\alpha\beta}=0$ similarly. Next we will show that $c_{ij}\neq 0$. If $c_{ij}=0$, $f_1(E_{ij})$ would be of the form (a) or (b), and hence the rank of $f_1(E_{11}+E_{ij})$ would be less than 2, contrary to the condition (2). Therefore $c_{ij}\neq 0$, and consequently $f_1(E_{ij})=c_{11}E_{11}+c_{i1}E_{i1}+c_{1j}E_{1j}+c_{ij}E_{ij}$. Now it is easily seen that $c_{11}=c_{i1}=c_{1j}=0$. By considering the rank of $f_1(E_{11}+E_{i1}+E_{1j}+E_{ij})$ we have $c_{ij}=1$. Thus we have proved that $f_1(E_{ij})=E_{ij}$ for all i and j. Accordingly $f_1(Z)=Z$ for any $Z\in\mathfrak{R}_{(n,m)}$, that is, $f(Z)=A_1^{-1}A_2^{-1}ZB_2^{-1}B_1^{-1}$.

(*2nd case*). If $n\neq m$, this case is clearly impossible. If $n=m$, this case is reduced to the 1st case by considering the transposed matrices of $A_1f(E_{i1})B_1$ and $A_1f(E_{1j})B_1$.

Thus Theorem 1 is completely proved.

Remark. As an immediate corollary to this theorem we can mention a theorem of I. Schur[3].

Theorem 2. If a mapping f of $\mathfrak{R}_{(n,m)}$ into itself satisfies the conditions: (1) $f(\alpha A+\beta B)=\alpha f(A)+\beta f(B)$, *($\alpha$, β being numbers)* (2) $\|f(Z)\|=\|Z\|$[4], *then the mapping f is of the form $f(Z)=UZV$ when $n\neq m$. In case $n=m$, $f(Z)=UZV$ or $f(Z)=UZ'V$. Here U and V are constant unitary matrices of orders n and m respectively.*

Proof. Let us assume that $n\geqq m$ (the other case being treated similarly) and put $\varphi(\lambda;Z)=\det.(\lambda E^{(m)}-\bar{Z}'Z)$, $\psi(\lambda;Z)=\varphi(\lambda;f(Z))$. Then, by the condition (2), $\varphi(\lambda;Z)$ and $\psi(\lambda;Z)$ have at least one root in common for each $Z\in\mathfrak{R}_{(n,m)}$. Now we put $z_{\alpha\beta}=x_{\alpha\beta}+iy_{\alpha\beta}$, $i=\sqrt{-1}$, $Z=(z_{\alpha\beta})$. Then $\varphi(\lambda;Z)$ and $\psi(\lambda;Z)$ can be regarded as polynomials with coefficients in the ring $K[x_{11},x_{21},\dots,x_{nm},y_{11},\dots,y_{nm}]=K[x,y]$ (To make this point clear we write $\varphi(\lambda:x,y)$ etc.), where $x_{\alpha\beta}$ and $y_{\alpha\beta}$ are considered as independent indeterminates and K means the field of all complex numbers. If we construct the resultant $R(\varphi,\psi)$ of $\varphi(\lambda;x,y)$ and $\psi(\lambda;x,y)$, $R(\varphi,\psi)$ is the zero element as an element of $K[x,y]$, since $R(\varphi,\psi)$ vanishes, if $x_{\alpha\beta}$ and $y_{\alpha\beta}$ take real values. Therefore $\varphi(\lambda;x,y)$ and $\psi(\lambda;x,y)$, regarded as elements of $K(x,y)[\lambda]$, have a common factor. However the polynomial $\varphi(\lambda;x,y)$ is irreducible. Suppose that it is reducible: $\varphi(\lambda;x,y)=g(\lambda;x,y)h(\lambda;x,y)$. Here we can assume by a well-known theorem that $g(\lambda;x,y)$ and $h(\lambda;x,y)$ belong to $K[x,y][\lambda]$. Now let us put $x_{\alpha\beta}=0$ for $\alpha>m$ and $y_{\alpha\beta}=0$ for all α,β. Then we have $\varphi(0;X)=(-1)^m\big(\det.(x_{\alpha\beta})\big)^2$, where $\varphi(\lambda;X)$ means the polynomial obtained by this substitution and $X=(x_{\alpha\beta})$ $(1\leqq\alpha,\ \beta\leqq m)$. Therefore $(-1)^m\big(\det.(x_{\alpha\beta})\big)^2=g(0;X)h(0;X)$. Since $\det.(x_{\alpha\beta})$ is irreducible in $K[x_{11},x_{21},\dots,x_{mm}]$ ($x_{\alpha\beta}$ are indeterminates), we have either $g(0;X)=(-1)^m\omega\cdot\det.(x_{\alpha\beta})$, $h(0;X)=\omega^{-1}\cdot\det.(x_{\alpha\beta})$, or $g(0;X)=(-1)^m\omega$, $h(0;X)=\omega^{-1}\big(\det.(x_{\alpha\beta})\big)^2$ (ω being a number). But, as is easily shown, both cases are impossible. Hence $\varphi(\lambda;x,y)$ is irreducible, and accordingly we have $\varphi(\lambda;x,y)=\psi(\lambda;x,y)$. Therefore hermitian matrices $\bar{Z}'Z$ and $\overline{f(Z)}'f(Z)$ are equivalent. In particular, the rank of $f(Z)$ is equal to that of Z. Hence, by Theorem 1, there exist non-singular constant matrices A and B such that $f(Z)=AZB$ (or $AZ'B$, when $n=m$. The treatment of this case we omit in the following.) To these A and B we can choose unitary matrices U_1, U_2, V_1 and V_2 so that $U_1AU_2=A_1$ and $V_2BV_1=B_1$ are positive diagonal matrices. Then we have $\|Z\|=\|f(Z)\|=\|A_1U_2^{-1}ZV_2^{-1}B_1\|$ and consequently $\|Z\|$ $(=\|U_2ZV_2\|)=\|A_1ZB_1\|$. From the last relation, we know that A_1 and B_1 are scalar matrices. Hence we get $f(Z)=U_1^{-1}U_2^{-1}ZV_2^{-1}V_1^{-1}$. This completes the proof of the theorem.

2. Now we are in a situation to prove Schwarz's lemma in higher dimensions. $\mathfrak{A}_{(n,m)}$ denotes the set of all matrices Z such that $\|Z\|<1$

3) I. Schur, Sitzungsber. preuss. Akad. Wiss. 1925, 454–463. Satz II. As is shown there, the condition of his theorem is satisfied for $r=1,2$, if it is satisfied for some $r>2$. Hence our theorem is applicable.

4) Contrary to our previous notation, $\|Z\|$ here means the norm of a matrix $Z:\|Z\|=\text{l.u.b.}\ \|Z\mathfrak{x}\|/\|\mathfrak{x}\|$, where $\mathfrak{x}$ runs over all m-dimensional vectors.

and $Z \in \mathfrak{R}_{(n,m)}$. By an analytic mapping f of $\mathfrak{A}_{(n,m)}$ into itself we mean that each (matrix-) element of $f(Z)$ is a regular function of complex variables $z_{11}, z_{21}, \cdots, z_{nm}$ in the domain $\|Z\| < 1$ $\big(Z = (z_{\alpha\beta})\big)$.

Theorem 3. Let f be an analytic mapping of $\mathfrak{A}_{(n,m)}$ into itself, which fixes the zero point. Then it holds that $\|f(Z)\| \leqq \|Z\|$. If the equality holds at every point in a neighbourhood of one point Z_0 in $\mathfrak{A}_{(n,m)}$, then f is of the form $f(Z) = UZV$ when $n \neq m$. In case $n = m$, we have either $f(Z) = UZV$ or $f(Z) = UZ'V$. Here U and V are constant unitary matrices of orders n and m respectively[5].

Proof of the first part. Take an arbitrary point A in $\mathfrak{A}_{(n,m)}$, and put $\|A\| = \alpha$. Then there exist two unitary matrices U_0 and V_0 such that $U_0 f(A) V_0 = B$ is of the form $B = \sum_j \beta_j E_{jj}$. Let us put $f_1(Z) = U_0 f(Z) V_0$. If we denote the elements of $f_1(u\alpha^{-1}A)$ by $\varphi_{ij}(u)$ $\Big(f_1(u\alpha^{-1}A) = \big(\varphi_{ij}(u)\big)\Big)$, then $\varphi_{ij}(u)$ are regular functions of a complex variable u in the domain $|u| < 1$. Moreover, $\varphi_{ij}(0) = 0$ and $|\varphi_{ij}(u)| < 1$ for $|u| < 1$, since $|\varphi_{ij}(u)| \leqq \|f_1(u\alpha^{-1}A)\|$. Hence, by Schwarz's lemma (in the case of one variable), we have $|\varphi_{ij}(u)| \leqq |u|$. Therefore $|\varphi_{ij}(\alpha)| \leqq \alpha$, that is, $|\beta_j| \leqq \alpha$ for $1 \leqq j \leqq n$, $1 \leqq j \leqq m$. Hence it follows that $\|f(A)\| = \|B\| \leqq \alpha$, which completes the proof.

Proof of the second part. In the proof of Theorem 2, we have shown that $\varphi(\lambda; x, y)$ is an irreducible polynomial[5a]. Hence $\varphi(\lambda; x, y) = 0$ defines an algebraic function $\Phi(x, y)$ of complex variables $x_{11}, x_{21}, \cdots, x_{nm}, y_{11}, \cdots, y_{nm}$. If (x^0, y^0) $\big(Z_0 = (z^0_{\alpha\beta}),\ z^0_{\alpha\beta} = x^0_{\alpha\beta} + iy^0_{\alpha\beta}\big)$ is a branch point of $\Phi(x, y)$, we can find a regular point Z_1 in its neighbourhood. Therefore we can assume that the point (x^0, y^0) is not a branch point. Since, by the assumption, a suitable branch of $\Phi(x, y)$ satisfies the equation $\psi(\lambda; x, y) = 0$ in a neighbourhood of the point (x^0, y^0), we can conclude $\varphi(\lambda; x, y) = \psi(\lambda; x, y)$ by analytic continuations. If the variables $x_{\alpha\beta}$ and $y_{\alpha\beta}$ assume real values, we have $Sp\bar{Z}'Z = Sp\overline{f(Z)}'f(Z)$ for $Z \in \mathfrak{A}_{(n,m)}$. If $Sp\bar{Z}'Z < 1$, we can regard Z as a point of $\mathfrak{A}_{(nm,1)}$. Hence the above relation shows that f is an analytic mapping of $\mathfrak{A}_{(nm,1)}$ into itself and satisfies the condition $\|f(Z)\| = \|Z\|$ for $Z \in \mathfrak{A}_{(nm,1)}$. Therefore, if we can prove that any analytic mapping g of $\mathfrak{A}_{(p,1)}$ into itself with the property $\|g(Z)\| = \|Z\|$ for $Z \in \mathfrak{A}_{(p,1)}$ is linear, we know the linearity of the given mapping f and consequently the theorem follows from Theorem 2. Now, let us put $Z = \begin{pmatrix} z_1 \\ \vdots \\ z_p \end{pmatrix}$, $g(Z) = \begin{pmatrix} g_1(z_1, \cdots, z_p) \\ \cdots\cdots\cdots\cdots \\ g_p(z_1, \cdots, z_p) \end{pmatrix}$. Since the domain $\mathfrak{A}_{(p,1)}$ is what H. Cartan calls "domaine cerclé," we have the following expansion[6]: $g_k(z_1, \cdots, z_p) = \sum_{l=1}^{\infty} P_{kl}(z_1, \cdots, z_p)$ $(k = 1, 2, \cdots, p)$. Here $P_{kl}(z_1, \cdots, z_p)$ are homogenious polynomials of degree l in $z_1, \cdots, z_p$, and the series is absolutely and uniformly convergent in the neigh-

5) This is also proved by M. Sugawara. See the foregoing paper of M. Sugawara: On the general Schwarzian lemma.

5a) Here we assume that $n \geqq m$.

6) H. Cartan, Jour. de math. pures et appl. (9), **10** (1931), 1–114.

bourhood of any point in $\mathfrak{A}_{(p,1)}$. Take an arbitrary point Z_0 in $\mathfrak{A}_{(p,1)}$ and put in the above expression $z_\nu = r z_\nu^0 e^{i\theta}$ (z_ν^0 being the components of Z_0). Then, by integrating with respect to θ from 0 to 2π, we have $\sum_{k=1}^{p} |z_k^0|^2 r^2 = \sum_{l=1}^{\infty} (\sum_{k=1}^{p} |P_{kl}(z_1^0, \dots, z_p^0)|^2) r^{2l}$, since, by the assumption, $\sum_{k=1}^{p} |g_k(r z_1^0 e^{i\theta}, \dots, r z_p^0 e^{i\theta})|^2 = \sum_{k=1}^{p} |z_k^0|^2 r^2$. Therefore $P_{kl}(z_1^0, \dots, z_p^0) = 0$ for all $l \neq 1$, that is, $g_k(z_1, \dots, z_p)$ $(k=1, 2, \dots, p)$ are linear homogenious functions of $z_1, \dots, z_p$. This completes the proof.

Remark. In order to remove the assumption $f(0)=0$ in the above theorem, we have only to make use of the metric ρ^* and the group $\mathfrak{B}_{(n,m)}$ defined previously[7].

3. The problem proposed in the introduction is solved by the following

Theorem 4. Any one-to-one analytic mapping f of $\mathfrak{A}_{(n,m)}$ on itself belongs to $\mathfrak{B}_{(n,m)}$[7], when $n \neq m$. In case $n=m$ f belongs to the transformation group generated by the mapping $\varphi_0(Z)=Z'$ and the elements of $\mathfrak{B}_{(n,n)}$.

Proof. If the zero point is fixed by f, then we have $\|f(Z)\| = \|Z\|$ and hence the theorem follows from Theorem 3.

4. Theorem 4 can also be obtained as follows. Since $\mathfrak{A}_{(n,m)}$ is a "domaine cerclé" in the sense of H. Cartan, a one-to-one analytic mapping f of $\mathfrak{A}_{(n,m)}$ on itself such that $f(0)=0$ is linear[8]. Hence $\|f(Z)\| = \|Z\|$, so that $f(Z) = UZV$ or $UZ'V$ by Theorem 2.

5. In this paragraph we are concerned with the space $\mathfrak{A}_{(n)}$.

Theorem 2′. If a linear mapping f of $\mathfrak{S}_{(n)}$ into itself ($\mathfrak{S}_{(n)}$ being the set of all symmetric matrices of order n) satisfies the condition $\|f(Z)\| = \|Z\|$, then f is of the form $f(Z) = UZU'$, where U is a constant unitary matrix.

Theorem 3′. Let f be an analytic mapping of $\mathfrak{A}_{(n)}$ into itself which fixes the zero point. Then it holds that $\|f(Z)\| \leqq \|Z\|$. If the equality holds at every point in a neighbourhood of one point Z_0 in $\mathfrak{A}_{(n)}$, then f is of the form $f(Z) = UZU'$, where U is a constant unitary matrix.

Theorem 4′. Any one-to-one analytic mapping f of $\mathfrak{A}_{(n)}$ on itself is of the type mentioned in the introduction: $f(Z) = (U_1 Z + U_2)(U_3 Z + U_4)^{-1}$, $U'JU = J$, $U'S\bar{U} = S$.

The proofs of these theorems can be done by the same method as in the case of $\mathfrak{A}_{(n,m)}$. For this purpose it is sufficient to prove Theorem 2′.

Proof of Theorem 2′. If we consider the elements $x_{\alpha\beta}$ of a symmetric matrix X as independent indeterminates, its determinant is irreducible. Hence, by proceeding analogously as in Theorem 2, it is shown that $\overline{f(Z)}' f(Z)$ and $\bar{Z}'Z$ are equivalent for any $Z \in \mathfrak{S}_{(n)}$. In particular we have $|\det. f(Z)| = |\det. Z|$. Since $\det. Z$ and $\det. f(Z)$ are regular functions of complex variables $z_{11}, \dots, z_{nn}$ $\left(Z = (z_{\alpha\beta})\right)$, there

7) Cf. [S−3] and [M−1].

8) H. Cartan, loc. cit. Théorème VI.

exists a number ω such that det. $f(Z)=\omega$ det. Z. From this, by a theorem of G. Frobenius[9], we have $f(Z)=AZA'$, where A is a non-singular constant matrix. By the similar reasoning as in Theorem 2, we know that A is a unitary matrix. This completes the proof.

In conclusion I wish to express my hearty thanks to Prof. M. Sugawara for his many valuable remarks.

9) G. Frobenius, Sitzungsber. preuss. Akad. Wiss. 1897, 994-1015. Satz III, §7.