102. On Vector Lattice with a Unit, II.

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§ 1. *Introduction and the theorems.* In a preceding note[1] one of the authors gave a representation of the vector lattice with a unit to obtain an algebraic proof of Kakutani-Krein's lattice-theoretic characterisation[2] of the space of continuous functions on a bicompact Hausdorff space. The purpose of the present note is to extend the result and to show that there exists a close analogy between the structures of the vector lattice and the algebras as in the case of the normed ring and the algebras[3].

A vector lattice E is a partially ordered real linear space, some of whose elements f are "non-negative" (written $f \geqq 0$) and in which[4]

(V 1): If $f \geqq 0$ and $\alpha \geqq 0$, then $\alpha f \geqq 0$.

(V 2): If $f \geqq 0$ and $-f \geqq 0$, then $f=0$.

(V 3): If $f \geqq 0$ and $g \geqq 0$, then $f+g \geqq 0$.

(V 4): E is a lattice by the semi-order relation $f \geqq g$ $(f-g \geqq 0)$.

In this note we further assume the existence of a "unit" $I>0$ satisfying

(V 5): For any $f \in E$ there exists $\alpha>0$ such that $-\alpha I \leqq f \leqq \alpha I$.

An element $f \in E$ is called "nilpotent" if $n|f|<I (n=1, 2, \ldots)$. The set R of all the nilpotent elements f is called the "radical" of E. Surely R constitutes a linear subspace of E. Moreover it is easy to see that R is an "ideal" of E, viz. $f \in R$ and $|g| \leqq |f|$ imply $g \in R$. Here we put as usual $|f|=f^+-f^-$, $f^+=f \vee 0=\sup(f, 0)$, $f^-=f \wedge 0=\inf(f, 0)$.

Let N be a linear subspace of E. Then the linear congruence $a \equiv b$ (mod. N) is also a lattice-congruence:

$$a \equiv b,\ a' \equiv b' \ (\text{mod. } N) \text{ implies } ab \equiv a'b' \ (\text{mod. } N),$$

if and only if N is an ideal of E[5]. An ideal N is called "non-trivial" if $N \neq 0, E$. A non-trivial ideal N is called "maximal" if it is contained in no other ideal $\neq E$. Denote by $\mathfrak{N}$ the set of all the maximal ideals N of E. The residual class E/N of E mod. any ideal $N \in \mathfrak{N}$ is "simple", viz. E/N does not contain non-trivial ideals. It is proved

1) K. Yosida: Proc. **17** (1941), 121–124. Cf. also M. H. Stone: Proc. Nat. Acad. Sci. **27** (1941), 83–87, and H. Nakano: Proc. **17** (1941), 311–317.

2) S. Kakutani: Proc. **16** (1940), 63–67. M. and S. Krein: C. R. URSS, **27** (1940), 427–430.

3) I. Gelfand: Rec. Math. **9** (1941), 1–24. We here express our hearty thanks to Tadasi Nakayama for his discussions during the preparation of the present note. He also obtained another proof of the theorem 1 below by considering the embedding of "lattice-groups" in a direct product of linearly ordered lattice-groups. See his paper shortly to appear in these Proceedings.

4) Small roman letters and small greek letters respectively denote elements $\in E$ and real numbers. We write $f>0$ if $f \geqq 0$ and $f \neq 0$.

5) Garrett Birkhoff: Lattice Theory, New York (1940), 109.

below that simple vector lattice with a unit is linear-lattice-isomorphic to the vector lattice of real numbers, the non-negative elements and the unit being represented by non-negative numbers and the number 1. We denote by $f(N)$ the real number which corresponds to $f \in E$ by the linear-lattice-homomorphism $E \rightarrow E/N$, $N \in \mathfrak{N}$.

After these preliminaries we may state our

Theorem 1. The radical R *coincides with the intersection ideal* $\bigwedge_{N \in \mathfrak{N}} N$.

The vector lattice $\bar{E}=E/R$ is again a vector lattice with a unit $\bar{I}$. By the theorem 1 the intersection ideal $\bigwedge_{\bar{N} \in \bar{\mathfrak{N}}} \bar{N}$ of all the maximal ideals $\bar{N}$ of E is the zero ideal and hence $\bar{E}$ does not contain nilpotent element $\neq 0$. Thus $\bar{E}$ satisfies the "Archimedean axiom":

$$\text{(V 6)}: \quad \text{order-limit}_{n \to \infty} \frac{1}{n} |\bar{f}| = 0 \quad \text{for all} \quad \bar{f} \in \bar{E}.$$

Therefore, by the result of the preceding note, we may add a precision to the theorem 1:

Theorem 2. By the correspondence $\bar{f} \rightarrow \bar{f}(\bar{N})$, $\bar{E}$ *is linear-lattice-isomorphically mapped on the vector lattice* $F(\bar{\mathfrak{N}})$ *of real-valued bounded functions on* $\bar{N}$ *such that* i) $\bar{f} \rightarrow \bar{f}(\bar{N})$, ii) $\bar{I}(\bar{N}) \pm 1$ *on* $\bar{\mathfrak{N}}$ *and* iii) $F(\bar{\mathfrak{N}})$ *is dense in the set of all the real-valued continuous functions* $c(\bar{N})$ *on* $\bar{\mathfrak{N}}$ *by the "norm"* $\|c\|=\sup_{\bar{N}} |c(\bar{N})|$. *Here the topology in* $\bar{\mathfrak{N}}$ *is defined by calling open the set of all the points* $\bar{N} \in \bar{\mathfrak{N}}$ *which satisfy* $|\bar{f}_i(\bar{N}) - \bar{f}_i(\bar{N}_0)| < \varepsilon_i$, $i=1, 2, \dots, n$, *where* $\bar{N}_0$, $\bar{N} \in \bar{\mathfrak{N}}$, $\varepsilon_i > 0$, n *and* $\bar{f}_i(-\bar{I} \leqq \bar{f}_i \leqq \bar{I})$ *are arbitrary.*

The theorems 1 and 2 show the analogy to a fundamental theorem in the theory of algebras, viz. the theorem stating that the residual class of an algebra mod. its maximal nilpotent ideal is a direct sum of total matric algebras.

§ 2. *The proof of the theorem 1* may be obtained by the following four lemmas.

Lemma 1. Let E be a simple vector lattice with a unit I, then we must have $E=\{\alpha I\}$, $-\infty < \alpha < \infty$.

Proof. E does not contain a nilpotent element $f=0$, for otherwise E would contain the non-trivial ideal $N_0=\mathfrak{E}_g(|g| \leqq \eta |f|, \ \eta < \infty)$. Hence E satisfies the Archimedean axiom (V 6). Let $E \ni f \neq \gamma I$ for any γ. Let $\alpha = \inf \alpha'$, $\alpha' I \geqq f$, $\beta = \sup \beta'$, $\beta' I \leqq f$, then $\beta I \leqq f \leqq \alpha I$ and $\alpha > \beta$. Hence $(f-\delta I)^+ \neq 0$, $(f-\delta I)^- \neq 0$ for $\beta < \delta < \alpha$. Then the set $N_0 = \mathfrak{E}_g(|g| \leqq \eta (f-\delta I)^+, \ \eta < \infty)$ is a non-trivial ideal, contrary to the hypothesis.

Lemma 2. For any non-trivial ideal N_0 there exists a maximal ideal $N \supset N_0$.

Proof. Let $N_0 \subset N_1 \subset N_2 \subset \cdots \subset N_\eta \subset \cdots$, $\eta < \omega$, be a transfinite sequence of non-trivial ideals. If ω is a limit ordinal, define $f \equiv g$

(mod. N_ω) to mean $f\equiv g$ (mod. N_η) for some $\eta<\omega$. That N_ω is a non-trivial ideal follows from the fact that $I\not\equiv 0$ (mod. N_η), $\eta<\omega$. This process defines a transfinite sequence of linear-lattice-congruence relations on E, each more inclusive than the last. Hence it cannot continue indefinitely. Therefore we would obtain the demanded maximal ideal $N\supset N_0$.

Lemma 3. We have $R\leqq\bigwedge_{N\in\mathfrak{N}} N$.

Proof. Let $f>0$ and $nf<I$ $(n=1,2,\dots)$, then for any $N\in\mathfrak{N}$ we have $n\cdot f(N)\leqq I(N)=1$ $(n=1,2,\dots)$ and hence $f(N)=0$, that is, $f\in N$.

Lemma 4. We have $R\geqq\bigwedge_{N\in\mathfrak{N}} N$.

Proof. Let $f>0$ be not nilpotent, then we have to show that there exists an ideal $N\in\mathfrak{N}$ such that $f\bar{\in}N$. This may be proved as follows.

Let $n\cdot f\not\leqq I$. Such an integer $n\geqq 1$ surely exists, since f is not nilpotent. We may assume that $n\cdot f\not\geqq I$, since otherwise $f\bar{\in}N$ for any $N\in\mathfrak{N}$. Thus $p=I-(n\cdot f)\wedge I>0$. For any positive integer m we do not have $m\cdot p\geqq I$. If otherwise we would have $\frac{1}{m}I\leqq I-(n\cdot f)\wedge I$ and hence

$$n\cdot f\wedge I=n\cdot f\wedge\Big(1-\frac{1}{m}\Big)I, \tag{1}$$

which implies

$$n\cdot f\leqq\Big(1-\frac{1}{m}\Big)I, \tag{2}$$

contrary to $n\cdot f\not\leqq I$. Thus the set $N_0=\mathop{\mathfrak{E}}_{g}(|g|\leqq\eta p,\eta<\infty)$ is a non-trivial ideal and hence there exists a maximal ideal $N\ni N_0$, by the lemma 2. Then $O=p(N)=1-\big(n\cdot f(N)\big)\wedge 1$, and thus $f(N)>0$, that is, $f\bar{\in}N$.

The deduction of (2) from (1). From (1) we have

$$\Big(n\cdot f-\Big(1-\frac{1}{m}\Big)I\Big)\wedge\frac{1}{m}I=\Big(n\cdot f-\Big(1-\frac{1}{m}\Big)I\Big)\wedge 0\leqq 0\,,$$

and hence, by the distributivity of the vector lattice,

$$0=\Big\{\Big(n\cdot f-\Big(1-\frac{1}{m}\Big)I\Big)\wedge\frac{1}{m}I\Big\}\vee 0=\Big(n\cdot f-\Big(1-\frac{1}{m}\Big)I\Big)^+\wedge\frac{1}{m}I\,.$$

Thus $\Big(n\cdot f-\Big(1-\frac{1}{m}\Big)I\Big)^+\wedge I=0$. Put $b=\Big(n\cdot f-\Big(1-\frac{1}{m}\Big)I\Big)^+$ and assume that $b>0$. By (V 5) we have $b<\alpha I$ with $\alpha>1$. Then $0<b=b\wedge\alpha I$, and hence $0<\frac{b}{\alpha}\wedge I\leqq b\wedge I$, contrary to $b\wedge I=0$. Thus $b=0$, which is equivalent to (2).

§3. *An example due to T. Nakayama.* The following example shows that the existence of the unit is important for the theorem 1. Consider linear functions $\alpha x+\beta$ with an indeterminate symbol x. We put $\alpha x+\beta\geqq\gamma x+\delta$ if, and only if, 1) $\alpha>\delta$ or 2) $\alpha=\gamma$ and $\beta\geqq\delta$. Then the totality of the vectors $f=(\alpha_1 x+\beta_1, \alpha_2 x+\beta_2, \dots)$ forms a vector lattice by componentwise addition and componentwise order relation. Now, consider the sublattice E consisting of those f such that almost all α_i are zero. This vector lattice possesses no unit. Further, if we call an element g nilpotent when $n|g|<f$ $(n=1, 2, \dots)$ for a certain $f>0$, then g is nilpotent in E if and only if all its α_i vanish and almost all its β_i vanish. On the other hand, the intersection of all the maximal ideals in E contains the totality of those f such that all its α_i are zero. This last property may be proved by the fact that a simple vector lattice is linear-lattice-isomorphic to real numbers (proof similar as in the lemma 1). In fact, let $c=(\gamma_1, \gamma_2, \dots)$ with all $\gamma_i \geqq 0$ be $\bar{\in}$ a maximal ideal N, then, since E/N is isomorphic to real numbers, we have $(2x, 0, 0, \dots) \equiv \delta c$ (mod. N). Hence $(x, 0, 0 \dots) \in N$ and similarly $(0, x, 0, 0, \dots)$, $(0, 0, x, 0, 0, \dots)$, $\dots \in N$. Thus if only a finite number of $\alpha_i x+\beta_i \neq 0$, then $(\alpha_1 x+\beta_1, \alpha_2 x+\beta_2, \dots) \in N$. Let M denote the totality of such elements. N/M is a maximal ideal of E/M. Since $n\gamma_i < i\gamma_i$ for almost all i, we have $nc < c_1 = (\gamma_1, 2\gamma_2, 3\gamma_3, \dots)$ (mod. M). Thus c (mod. M) is contained in any maximal ideal of E/M and hence c (mod. M) $\in N/M$. Therefore $c \in N$, contrary to the assumption.