90. *The Exceptional Values of Functions with the Set of Linear Measure Zero of Essential Singularities, II.*

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1. Let D be a domain and E be a compact sub-set, of D, of linear measure zero in the sense of Carathéodory.

If $w=f(z)$ is regular in $D-E$ and has E as its essential singularities, then, *near each point* $z_0 \in E$, $f(z)$ *takes every finite value except perhaps those belonging to a set of Newtonian capacity Zero.* This result, an extension of the one obtained by M. L. Cartwright[1] was proved in our former Note with the same title as the present one, Proc. **17** (1941).

Now, according to the result obtained recently by the present author,[2] a set of Newtonian capacity zero may be the sum of enumerably infinite sets of Carathéodory's linear measure finite, so that the exceptional set stated above might be of linear measure positive.

In this note, we shall show that the intersection of the exceptional set with any straight line is of linear measure zero.

2. We shall denote by $m(E)$ Carathéodory's lineare measure[3] or the length of E, by E^e the complementary set of E, and by $\{p\,;\,P\}$ the set of all the points p with the property P.

Lemma 1. *Let F be a closed set on a rectifiable Jordan arc L*[4] *and of positive linear measure. Then there exists a point $\zeta_0 \in F$ such that*

$$\int_{F\cdot A} d\zeta \neq 0$$

for every sufficiently small arc $A(\subset L)$ containing ζ_0.

Proof. Let L be represented by the equation:

$$\zeta=\zeta(s) \qquad (0 \leqq s \leqq l)$$

with the arc length s as its parameter.

Then, at almost every point of s, $\zeta'(s)$ exists and $|\zeta'(s)|=1$.

Writing $\zeta'(s)=e^{i\varphi(s)}$, $\big(\varphi(s)\,;\text{ real}\big)$, we have

$$\int_{F\cdot A} d\zeta=\int_{M\cdot I} e^{i\varphi(s)}ds\,,$$

1) M. L. Cartwright. The exceptional values of functions with a non-enumerable set of essential singularities, Quart. J. Math., Vol. **8** (1937).

2) S. Kametani. On some properties of Hausdorff's measure and the concept of capacity in generalized potentials, Proc. **18** (1942).

3) S. Saks. Theory of the Integral (1937), p. 53.

4) We may suppose that the set F does not contain any of the end points of L. We suppose it hereafter, if necessary, without explicitly saying so.

where $M=\{s;\ \zeta(s)\in F\}$ and $I=\{s;\ \zeta(s)\in A\}$.

By Lebesgue's theory on the differentiation of the indefinite integral, we have at almost every point $s=s_0\in M$

$$\frac{1}{m(I)}\int_{M\cdot I} e^{i\varphi(s)}ds \rightarrow e^{i\varphi(s_0)},$$

as $m(I)\rightarrow 0$ where $I\ni s_0$, whence we have also

$$\left|\int_{F\cdot A} d\zeta\right| = \left|\int_{M\cdot I} e^{i\varphi(s)}ds\right| \geqq m(I)(1-\eta) \qquad (0<\eta<1),$$

if $I(\ni s_0)$ is sufficiently small. Since the are A corresponding to I contains the point $\zeta_0=\zeta(s_0)$, our lemma is proved.

Lemma 2. *Let F be a closed set on a rectifiable Jordan arc L and of positive linear measure. Then the regular function defined by the following integral:*

$$H(w)=\int_F \frac{1}{\zeta-w}\,d\zeta$$

is non-constant in F^c.

Proof. Supposing the contrary, let $H(w)$ be a constant in F^c. Then, we would have

$$\int_C H(w)dw=0$$

for any rectifiable closed Jordan curve C not meeting F.

We distinguish here two cases.

1°) If F contains continuums, then it contains an arc $A\subset L$ such that near both end points of A, we can find small arcs ($\subset L$) which do not meet F.

Therefore, we can find an arc $A'\supset A$ with both end points $\notin F$, as near to A as we may.

Since $\int_{A'F} d\zeta \rightarrow \int_A d\zeta$ when $A'\rightarrow A$, and $\int_A d\zeta=\zeta_1-\zeta_0\neq 0$, where ζ_0 and ζ_1 are both end points of A, we have, for some arc A' with both end points $\notin F$,

$$\int_{F\cdot A'} d\zeta \neq 0.$$

2°) If F does not contain any continuum, then near every point of F, we may find small arcs ($\subset L$) which do not meet F. Now choose a point ζ_0 stated in Lemma 1.

Then, for every small arc $A\ni\zeta_0$,

$$\int_{F\cdot A} d\zeta \neq 0.$$

Therefore we can find also an arc $A'\subset A$, $A'\ni\zeta_0$ with both end-points not belonging to F such that

$$\int_{F\cdot A'} d\zeta \neq 0.$$

In both cases, *there exists a closed rectifiable Jordan curve C which contains A' in its interior, except both end points, and does not meet F.*

Since F and C are compact, the distance of them is positive and the function $(\zeta-w)^{-1}$ for $\zeta\in F$ and $w\in C$ is bounded, which enables us to invert the order of integration as follows:

$$0=\int_C H(w)dw=\int_C\left\{\int_F\frac{1}{\zeta-w}\,d\zeta\right\}dw=\int_F\left\{\int_C\frac{1}{\zeta-w}\,dw\right\}d\zeta$$
$$=\int_{F\cdot A'}\left(-\frac{1}{2\pi i}\right)d\zeta=-\frac{1}{2\pi i}\int_{F\cdot A'}d\zeta\neq 0\,,$$

in which we notice $\displaystyle\int_C\frac{1}{\zeta-w}\,dw=0$ for ζ exterior to C.

Thus we arrive at a contradiction, and our lemma is proved.

Let $L:\ \zeta=\zeta(\lambda)$ $(0\leqq\lambda\leqq 1)$ be a rectifiable Jordan arc, and w any point outside L. Dividing L into a finite number of sub-arcs

$$L_\nu:\ \ \zeta=\zeta(\lambda)\quad(\lambda_{\nu-1}\leqq\lambda\leqq\lambda_\nu)$$

by a finite number of arbitrary points on L, $\zeta_\nu=\zeta(\lambda_\nu)$, $0=\lambda_0<\lambda_1<\cdots<\lambda_{m-1}<\lambda_m=1$.

We consider the angle $\zeta_{\nu-1}\hat{w}\zeta_\nu$ made by the vector $\overrightarrow{\zeta_\nu w}$ to the vector $\overrightarrow{\zeta_{\nu-1}w}$, whose value is uniquely determined by the condition that $\zeta_{\nu-1}\hat{w}\zeta$ should be a continuous function of ζ along L_ν and $\zeta_{\nu-1}\hat{w}\zeta_{\nu-1}=0$. Thus $\zeta_{\nu-1}\hat{w}\zeta_\nu$ becomes a single-valued function of w outside L_ν. Next we shall consider the sum of the angles thus determined:

$$\sum_{\nu=1}^{m}|\,\zeta_{\nu-1}\hat{w}\zeta_\nu\,|\,.$$

Let its upper bound, varying the mode of division of L, be $V(w,L)$. We shall write:

$$V(L)=\sup_w V(w,L)\,.$$

It is evident that if L is a straight line, then $V(L)$ is finite and $=\pi$, and more generally, if L is a bounded convex arc, then $V(L)$ is also finite.

Lemma 3. *Let F be a closed set of positive length on a rectifiable Jordan arc L with $V(L)<+\infty$.*

Then, there exists a function $G(w)$ with the following properties: $G(w)$ is

(1) *a uniform (single-valued), analytic function of w in F^c,*

(2) *non-constant*

and (3) *bounded.*

Proof. From the fact that F is a set $\in\mathfrak{G}_\delta$, we may find a descending sequence $\{O^{(n)}\}$ of sets which are open in L and satisfying

$$\lim_n O^{(n)}=\prod_{n=1}^{\infty}O^{(n)}=F\,,$$

Since $O^{(n)}(\supset F)$ is open in L, it consists of a sequence $\{A_\nu^{(n)}\}$ of arcs which are open in L and any two of them do not meet, where we may suppose, by Borel-Lebesgue's theorem, that $\{A_\nu^{(n)}\}$ consists of a *finite number* of such arcs.

The sum of such arcs, depending on n, will be denoted by $\alpha(n)$. Then $F\subset\alpha(n)\subset O^{(n)}$, whence

$$\lim_{n\to\infty}\alpha(n)=F. \tag{1}$$

Fixing n for a moment, let the arcs of $\alpha(n)$ be $a_1,\dots,a_m$, where a_ν $\zeta=\zeta(\lambda)$, $(\lambda_\nu^{(1)}<\lambda<\lambda_\nu^{(2)})$ and $\lambda_0^{(1)}<\lambda_0^{(2)}\leqq\lambda_1^{(1)}<\lambda_1^{(2)}\leqq\cdots\leqq\lambda_m^{(1)}<\lambda_m^{(2)}$.

At a point of the arc $\bar a_\nu$, the closure of a_ν, let us fix any element of the analytic function log $(w-\zeta)$ of the variable ζ and continuate it analytically along $\bar a_\nu$. The function thus defined along $\bar a_\nu$ will be denoted also by log $(w-\zeta)$.

Then we have evidently

$$\log(w-\zeta_\nu^{(2)})-\log(w-\zeta_\nu^{(1)})=-\int_{\bar a_\nu}\frac{d\zeta}{w-\zeta}=-\int_{a_\nu}\frac{d\zeta}{w-\zeta}, \tag{2}$$

where $\zeta_\nu^{(1)}=\zeta(z_\nu^{(1)})$ and $\zeta_\nu^{(2)}=\zeta(z_\nu^{(2)})$.

From the right-hand side of the above relation, we find that (2) *represents a uniquely determined, single-valued function of w in $\bar a_\nu^c$, which is independent of the special choice of log $(w-\zeta)$.*

It is also evident that

$$\Im\{\log(w-\zeta_\nu^{(2)})-\log(w-\zeta_\nu^{(1)})\}=\zeta_\nu^{(1)}\hat{w}\zeta_\nu^{(2)}.$$

Let us consider, depending on n now, the following function:

$$A_n(w)=\sum_{\nu=1}^{m}\{\log(w-\zeta_\nu^{(2)})-\log(w-\zeta_\nu^{(1)})\}.$$

Then we have evidently

$$|\Im A_n(w)|=\Big|\sum_{\nu=1}^{m}\Im\{\log(w-\zeta_\nu^{(2)})-\log(w-\zeta_\nu^{(1)})\}\Big|\leqq V(L). \tag{3}$$

It is also evident from (2) that

$$A_n(w)=-\int_{\alpha(n)}\frac{d\zeta}{w-\zeta}$$

We have from above and (1)

$$\lim_{n\to\infty}\Big\{-\int_{\alpha(n)}\frac{d\zeta}{w-\zeta}\Big\}=-\int_F\frac{d\zeta}{w-\zeta},$$

which will be denoted by $H(w)$. By Lemma 2, $H(w)$ is not a constant and by (3) we have in the limit

$$|\Im H(w)|\leqq V(L). \tag{4}$$

Let us consider next the following function:

$$G(w)=e^{-iH(w)}$$

From $\Re\{-iH(w)\}=\Im H(w)$ and (4), we have for $w \in F^c$

$$|G(w)|=e^{\Im H(w)} \leqq e^{V(L)},$$

from which follows the boundedness of $G(w)$.

Since $H(w)$ is a non-constant, single-valued, analytic function, so is the function $G(w)$ for $w \in F^c$, which proves our lemma completely.

3. Theorem A. *Let D be a domain and E be a compact sub-set of linear measure 0 lying in D. If $w=f(z)$ is regular in $D-E$ and has E as its essential singularities, let us denote by S the set of all the finite values which are not assumed by $f(z)$ in $D-E$: $S=\{\omega\,;\,\omega \neq f(z)$ for $z \in D-E\}$.*

Then the intersection of S with any rectifiable Jordan arc L such that $V(L)<\infty$ is of linear measure zero.

Proof. The proof is chiefly depends on Lemma 3.

Supposing the contrary, let L be an arc such that

$$m(L.S)>0 \quad \text{and} \quad V(L)<\infty.$$

As L and S are closed sets, $F=L.S$ is a closed set.

By Lemma 3, there exists a single-valued, bounded, and non-constant analytic function $G(w)$ defined for all values of F^c.

Let us now consider the following function:

$$\Phi(z)=G[f(z)].$$

Since $G(w)$ is bounded and single-valued outside F whose values are not taken by $f(z)$ in $D-E$, $\Phi(z)$ is a single-valued bounded analytic function in D except for the set E of linear measure 0. Then, by Besicovitch's theorem[1] which we have also used already in our former note, $\Phi(z)$ becomes regular analytic throughout D, if properly defined on E.

$G(w)$ being not a constant, we can find two values w' and $w'' \in F^c$, such that

$$(5) \qquad G(w') \neq G(w'').$$

But, since the set of values taken by $f(z)$ in every neighbourhood of each $z_0 \in E$ is everywhere dense by Besicovitch's theorem[1], we can find two sequence of points $\{z'_\nu\}$ and $\{z''_\nu\}$ both tending to the point z_0 and satisfying

$$f(z'_\nu) \to w' \quad \text{and} \quad f(z''_\nu) \to w'' \quad \text{as} \quad \nu \to \infty.$$

Then, we would have

$$\Phi(z_0)=\lim_{\nu\to\infty}\Phi(z'_\nu)=\lim_{\nu\to\infty}G[f(z'_\nu)]=G(w')$$

and also

$$\Phi(z_0)=\lim_{\nu\to\infty}\Phi(z''_\nu)=G(w''),$$

which is impossible on account of (5), and our theorem is proved.

1) A. S. Besicovitch. On sufficient conditions for a function to be analytic, etc Proc. London Math. Soc. (2) Vol. **32** (1931).

As a corollary of the above theorem, we have immediately:

Theorem B. *Under the same assumption of Theorem A, the intersection of S with any straight line or any convex arc is of linear measure zero.*

Theorem A may be slightly generalized by the typical argument as follows:

Theorem C. *Under the same assumption of Theorem A, let S be the set of all the finite values not assumed or assumed only a finite number of times by $f(z)$ near $z_0 \in E$, then the intersection of S with any rectifiable Jordan arc L such that $V(L) < +\infty$ is of linear measure zero.*

Proof. Noticing that E does not contain any continuum, there exists for $z_0 \in E$ a sequence of open domains $\{D_\nu\}$ with the following properties:

1° $D = D_1 \supset D_2 \supset \cdots \ni z_0$,

2° $\delta(D_\nu) \rightarrow 0$ as $\nu \rightarrow \infty$, where $\delta(D_\nu) = \sup_{z', z'' \in D_\nu} |z' - z''|$,

3° each $E_\nu = D_\nu . E$ is a closed set.

For each ν, let us put

$$S_\nu = \{\omega ; \omega \neq f(z) \text{ for } z \in D_\nu - E_\nu\}.$$

Then by 1° and 3°, $S_1 \subset S_2 \subset \cdots$

Writing $S = \sum_{\nu=1}^{\infty} S_\nu$. we find the intersection of S with any rectifiable arc L such that $V(L) < \infty$ is of linear measure 0, since, by Theorem A, we have

$$m(S.L) \leqq \sum_{\nu=1}^{\infty} m(S_\nu . L) = 0.$$

Now it is evident by 2° that any finite value $\notin S$ is assumed by $f(z)$ infinitely many times near z_0, which proves our theorem completely.