71. *The Distribution of Grouped Moments in Large Samples.*

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1. We divide the whole interval $(-\infty, \infty)$ into subintervals of length δ, which we denote I_a, $a=\dots-1, 0, 1, 2, \dots$. Let I_0 contain the origin and the distance of the origin and the center of I_0 be t. Thus we can write $I_a=\left(\left(a-\frac{1}{2}\right)\delta+t, \left(a+\frac{1}{2}\right)\delta+t\right)$. Now consider a sample of size n from a certain population and let the number of individuals of the sample which fall into I_a be n_a. For this grouping, consider the sample moment of the r-th order.

$$(1) \qquad {}_\delta M_r=\sum_{a=-\infty}^{\infty}\frac{n_a}{n}(a\delta+t)^r .$$

We assume that the population variable has the finite $2r$-th moment and let its probability density be $f(x)$. Then the probability that an individual falls into I_a is

$$p_a=\int_{I_a} f(x)dx .$$

The mean value of the random variable ${}_\delta M_r$ is

$$(2) \qquad {}_\delta\mu'_r=E({}_\delta M_r)=\sum_{a=-\infty}^{\infty} p_a(a\delta+t)^r \text{[1]} .$$

Then under suitable conditions, we have

$$(3) \qquad {}_\delta\mu'_1 \fallingdotseq \mu'_1, \quad {}_\delta\mu'_2 \fallingdotseq \mu'_2+\frac{\delta^2}{12}, \quad {}_\delta\mu'_3 \fallingdotseq \mu_3+\delta^2\cdot\frac{\mu'_1}{4},$$

$${}_\delta\mu'_4 \fallingdotseq \mu'_4+\delta\cdot\frac{\mu'_2}{2}+\frac{\delta^4}{80}, \dots$$

where μ'_r is the r-th moment of the population variable[2]. The relation (3) is known as Sheppard's correction.

The object of this paper is to discuss the sampling error of ${}_\delta M_r$ in the large sample or in other words, the limit distribution of the variable ${}_\delta M_r$ as $n\to\infty$.

2. Let X $(\dots, X_{-1}, X_0, X_1, X_2, \dots)$ be a point in a space of infinite dimensions Ω and X_a take either 0 or 1. Let the probability that X_a takes 1 be p_a. In the space we define the probability such that the probability that X takes a point of the enumerable set $\{x^{(a)}\}$ $(a=\dots, -1, 0, 1, 2, \dots)$ is p_a and the probability that X is a point of a set which does not contain a point of $\{x^{(a)}\}$ is 0, where

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1) For the meaning of the mean value, we shall clarify it in the following lines

2) S. S. Wilks, Statistical inferences. Princeton Lecture, 1937.

$x^{(a)}$ denotes the point such that the α-th component is 1 and other components are all 0.

Now let $\{X^{(k)}\}$, $X^{(k)}=(\dots, X_{-1}^{(k)}, X_0^{(k)}, X_1^{(k)}, \dots)$, $(k=1, 2, \dots, n)$ be a sample of size n that is $X^{(1)}, X^{(2)}, \dots, X^{(n)}$ is independent variables of same distribution as in the population. Then we can write

$$n_a=\sum_{k=1}^{n} X_a^{(k)}$$

and hence

$$_\delta M_r=\sum_{a=-\infty}^{\infty}\frac{n_a}{n}(a\delta+t)^r=\frac{1}{n}\sum_{k=1}^{n}\sum_{a=-\infty}^{\infty}X_a^{(k)}(a\delta+t)^r.$$

Let the inner summation be

$$Y_k=\sum_{a=-\infty}^{\infty}X_a^{(k)}(a\delta+t)^r, \tag{4}$$

which is a well defined variable except in the set of probability 0 and is a function of $X^{(k)}$ which will be denoted as $g(X^{(k)})$.

Then by definition

$$\begin{aligned} E(Y_k)&=\int_\Omega g(X^{(k)})p(d\Omega) \\ &=\sum_{a=-\infty}^{\infty}p_a(a\delta+t)^r, \end{aligned} \tag{5}$$

and

$$\begin{aligned} E(Y_k^2)&=\int_\Omega g^2(X^{(k)})p(d\Omega) \\ &=\sum_{a=-\infty}^{\infty}p_a(a\delta+t)^{2r}, \end{aligned} \tag{6}$$

which is finite, for

$$\begin{aligned} \sum_{a=-\infty}^{\infty}p_a(a\delta+t)^{2r}&=\sum_{a=-\infty}^{\infty}(a\delta+t)^r\int_{I_a}f(x)dx \\ &\leqq\sum_{a=-\infty}^{\infty}\int_{I_a}x^{2r}f(x)dx\cdot\left|\frac{a\delta+t}{t+\left(a-\frac{1}{2}\right)\delta}\right|^{2r} \\ &\leqq c\int_{-\infty}^{\infty}f(x)\cdot x^{2r}dx. \end{aligned}$$

Thus
$$E(_\delta M_r)=\sum_{a=-\infty}^{\infty}p_a(a\delta+t)^r$$

and
$$\begin{aligned} E\left\{\left(_\delta M_r-E(_\delta M_r)\right)^2\right\}&=\frac{1}{n}\sum_{k=1}^{n}E\left\{\left(Y_k-E(Y_k)\right)^2\right\} \\ &=\frac{1}{n}\left[E(_\delta M_{2r})-\{E(_\delta M_r)\}^2\right]. \end{aligned}$$

Hence the variable $\sqrt{n}\,\left(_\delta M_r-E(_\delta M_r)\right)$ has the mean 0 and the variance $E(_\delta M_{2r})-\{E(_\delta M_r)\}^2$. Therefore we get, by Laplace's theorem, the following theorem.

Theorem. Suppose that the population variable has finite 2r-th moment and let the sample moment of the r-th order formed from a sample of size n of the population be ${}_\delta M_r$. *Then*

$$\sqrt{n}\left({}_\delta M_r - E({}_\delta M_r)\right)$$

converges in distribution to the normal law with mean 0 *and the variance* $E({}_\delta M_{2r}) - \left(E({}_\delta M_r)\right)^2$.

Especially we get that, for large n, ${}_\delta M_1$ is almost normal with the mean μ_1' and the variance $\frac{1}{n}\left(\mu_2 + \frac{\delta^2}{12}\right)$, and ${}_\delta M_2$ is almost normal with the mean $\left(\mu_2' + \frac{\delta^2}{12}\right)$ and the variance $\frac{1}{n}\left(\mu_4' + \frac{\delta^2}{2}\mu_2' + \frac{\delta^4}{80} - \left(\mu_2' + \frac{\delta^2}{12}\right)^2\right)$.