70. On the flat conformal differential geometry II.

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§ 2. *Canonical fundamental equations of flat conformal geometry.*

In the preceding Chapter,[1] we have established the so-called fundamental differential equations for the flat conformal geometry, and discussed the transformation law of the coefficients of the fundamental equations under change of factors and under coordinate transformations respectively. We have also studied the integrability conditions of the fundamental differential equations.

In the present Chapter, we shall introduce a canonical form of the fundamental equations and discuss the transformation law of the coefficients of the canonical fundamental equations under coordinate transformations, the factor being fixed in this case. We shall, in the last Paragraph of the present Chapter, also study the integrability conditions of the canonical fundamental differential equations.

1. *Canonical fundamental equations.*[2]

We have seen that, the fundamental equations established for a repère $[A_0, A_\lambda, A_\infty]$ being

$$(2.1)\qquad \begin{cases} \dfrac{\partial A_0}{\partial \xi^\nu} = A_\nu , \\ \dfrac{\partial A_\mu}{\partial \xi^\nu} = \Pi^0_{\mu\nu} A_0 + \Pi^\lambda_{\mu\nu} A_\lambda + \Pi^\infty_{\mu\nu} A_\infty , \\ \dfrac{\partial A_\infty}{\partial \xi^\nu} = \Pi^\lambda_{\infty\nu} A_\lambda , \end{cases}$$

if we perform a change of factor

$$(2.3)\qquad {}^*A_0 = \phi A_0 ,$$

the repère $[A_0, A_\lambda, A_\infty]$ will be transformed into another repère $[{}^*A_0, {}^*A_\lambda, {}^*A_\infty]$ following the formulae

$$(2.3)\qquad \begin{cases} {}^*A_0 = \phi A_0 , \\ {}^*A_\mu = \phi(\phi_\mu A_0 + A_\mu) , \\ {}^*A_\infty = \dfrac{1}{\phi}\left(\dfrac{1}{2} \phi^\lambda \phi_\lambda A_0 + \phi^\lambda A_\lambda + A_\infty \right) ; \end{cases}$$

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1) K. Yano: On the flat conformal differential geometry I. Proc., **21** (1945) 419.

2) T. Y. Thomas: On conformal geometry. Proc. Nat. Acad. Sci. U.S.A., **12** (1926), 352-359.

where

$$\phi_\mu=\frac{\partial \log \phi}{\partial \xi^\mu} \qquad \text{and} \qquad \phi^\lambda=g^{\lambda\mu}\phi_\mu .$$

To obtain a canonical repère, we impose the condition that the determinant formed with the components of the fundamental tensor $*A_\mu \cdot *A_\nu = G_{\mu\nu}$ is equal to the unity, from which we have

$$|G_{\mu\nu}|=|\phi^2 g_{\mu\nu}|=\phi^{2n}g=1,$$

or

$$\phi=g^{-\frac{1}{2n}}, \tag{2.4}$$

g denoting the determinant formed with $g_{\mu\nu}$. Substituting the value (2.4) of ϕ into the equations (2.3), we find

$$\left\{\begin{aligned} &*A_0=g^{-\frac{1}{2n}}A_0,\\ &*A_\mu=g^{-\frac{1}{2n}}\left(\frac{\partial \log g^{-\frac{1}{2n}}}{\partial \xi^\mu}A_0+A_\mu\right),\\ &*A_\infty=g^{\frac{1}{2n}}\left(\frac{1}{2}g^{\mu\nu}\frac{\partial \log g^{-\frac{1}{2n}}}{\partial \xi^\mu}\frac{\partial \log g^{-\frac{1}{2n}}}{\partial \xi^\nu}\right.\\ &A_0 \qquad +g^{\mu\nu}\left.\frac{\partial \log g^{-\frac{1}{2n}}}{\partial \xi^\mu}A_\nu+A_\infty\right). \end{aligned}\right. \tag{2.5}$$

We shall call the repère $[*A_0, *A_\lambda, *A_\infty]$ the canonical repère or the repère naturel. Such repères are characterized by the condition

$$|G_{\mu\nu}|=|*A_\mu\cdot *A_\nu|=1. \tag{2.6}$$

The canonical repère $[*A_0, *A_\lambda, *A_\infty]$ being defined at every point of the conformal space C_n, we can establish, by a process analogous to that used in Chapter 1, 1°, the canonical fundamental equations

$$\left\{\begin{aligned} &\frac{\partial *A_0}{\partial \xi^\lambda}= \qquad\qquad *A_\lambda,\\ &\frac{\partial *A_\mu}{\partial \xi^\nu}=*\Pi^0_{\mu\nu}*A_0+*\Pi^\lambda_{\mu\nu}*A_\lambda+*\Pi^\infty_{\mu\nu}*A_\infty,\\ &\frac{\partial *A_\infty}{\partial \xi^\nu}= \qquad *\Pi^\lambda_{\infty\nu}*A_\lambda \end{aligned}\right. \tag{2.7}$$

with respect to the canonical repère $[*A_0, *A_\lambda, *A_\infty]$, where

$$\left\{\begin{aligned} &*\Pi^\lambda_{\mu\nu}=K^\lambda_{\mu\nu}=\frac{1}{2}G^{\lambda\alpha}\left(\frac{\partial G_{\alpha\mu}}{\partial \xi^\nu}+\frac{\partial G_{\alpha\nu}}{\partial \xi^\mu}-\frac{\partial G_{\mu\nu}}{\partial \xi^\alpha}\right),\\ &*\Pi^\infty_{\mu\nu}=G_{\mu\nu}, \quad *\Pi^\lambda_{\infty\nu}=G^{\lambda\mu}*\Pi^0_{\mu\nu}, \end{aligned}\right. \tag{2.8}$$

$*\Pi^0_{\mu\nu}$ being determined in Paragraph 3 of the present Chapter.

It will be noticed that the coefficients $*\Pi^\lambda_{\mu\nu}=K^\lambda_{\mu\nu}$, that is, Christoffel symbols formed with $G_{\mu\nu}$, satisfy the conditions

(2.9) $$*\Pi^{\lambda}_{\lambda\nu}=K^{\lambda}_{\lambda\nu}=0\,.$$

2°. *Transformation law of coefficients of the canonical fundamental equations.*

The factor being fixed for the canonical repère, we have only to consider the coordinate transformations

(2.10) $$\bar{\xi}^{\lambda}=\bar{\xi}^{\lambda}(\xi^1,\xi^2,\ldots\ldots,\xi^n)\,.$$

Under this coordinate transformation, the repère $[A_0, A_\lambda, A_\infty]$ is transformed into another repère $[\bar{A}_0, \bar{A}_\lambda, \bar{A}_\infty]$ following the formulae

$$\bar{A}_0=A_0,\quad \bar{A}_\lambda=\frac{\partial\xi^\alpha}{\partial\bar{\xi}^\lambda}A_\alpha,\quad \bar{A}_\infty=A_\infty,$$

and $g_{\mu\nu}=A_\mu\cdot A_\nu$ into $\bar{g}_{\mu\nu}=\bar{A}_\mu\cdot\bar{A}_\nu$ following

$$\bar{g}_{\mu\nu}=\frac{\partial\xi^\beta}{\partial\bar{\xi}^\mu}\frac{\partial\xi^\gamma}{\partial\bar{\xi}^\nu}g_{\beta\gamma},$$

from which $\bar{g}=\Delta^2 g$, and consequently

$$\bar{g}^{-\frac{1}{2n}}=\Delta^{-\frac{1}{n}}g^{-\frac{1}{2n}},$$

Δ being the Jacobian of the coordinate transformation.

Substituting these equations into (2.5) written for the barred quantities, we find

$$(2.11)\quad\begin{cases}*\bar{A}_0=\Delta^{-\frac{1}{n}}*A_0\,,\\[6pt]
*\bar{A}_\lambda=\Delta^{-\frac{1}{n}}\left(\dfrac{\partial\log\Delta^{-\frac{1}{n}}}{\partial\bar{\xi}^\lambda}*A_0+\dfrac{\partial\xi^\alpha}{\partial\bar{\xi}^\lambda}*A_\alpha\right),\\[6pt]
*\bar{A}_\infty=\Delta^{-\frac{1}{n}}\left(\dfrac{1}{2}\bar{G}^{\mu\nu}\dfrac{\partial\log\Delta^{-\frac{1}{n}}}{\partial\bar{\xi}^\mu}\dfrac{\partial\log\Delta^{-\frac{1}{n}}}{\partial\bar{\xi}^\nu}*A_0+\right.\\[6pt]
\qquad\qquad\left.\bar{G}^{\mu\nu}\dfrac{\partial\log\Delta^{-\frac{1}{n}}}{\partial\bar{\xi}^\mu}\dfrac{\partial\xi^\gamma}{\partial\bar{\xi}^\nu}*A_\gamma+\Delta^{\frac{2}{n}}*A_\infty\right),\end{cases}$$

which give the transformation law of the canonical repère $[*A_0, *A_\lambda, *A_\infty]$ under a coordinate transformation (2.10), where

$$\bar{G}_{\mu\nu}=\Delta^{-\frac{2}{n}}\frac{\partial\xi^\beta}{\partial\bar{\xi}^\mu}\frac{\partial\xi^\gamma}{\partial\bar{\xi}^\nu}G_{\beta\gamma}\ \text{ and }\ \bar{G}^{\mu\nu}=\Delta^{\frac{2}{n}}\frac{\partial\bar{\xi}^\mu}{\partial\xi^\beta}\frac{\partial\bar{\xi}^\nu}{\partial\xi^\gamma}G^{\beta\gamma}\,.$$

Substituting the equation (2.11) into the canonical fundamental equations

$$(2.12)\quad\begin{cases}\dfrac{\partial*\bar{A}_0}{\partial\bar{\xi}^\lambda}= *\bar{A}_\lambda\,,\\[6pt]
\dfrac{\partial*\bar{A}_\mu}{\partial\bar{\xi}^\nu}=*\bar{\Pi}^0_{\mu\nu}*\bar{A}_0+*\bar{\Pi}^\lambda_{\mu\nu}*\bar{A}_\lambda+*\bar{\Pi}^\infty_{\mu\nu}*\bar{A}_\infty\,,\\[6pt]
\dfrac{\partial*\bar{A}_\infty}{\partial\bar{\xi}^\nu}= *\bar{\Pi}^\lambda_{\infty\nu}*\bar{A}_\lambda\,,\end{cases}$$

established for the canonical repère $[*\bar{A}_0, *\bar{A}_\lambda, *\bar{A}_\infty]$, and taking account of the canonical equations (2.7), we find

$$
(2.13)\quad \begin{cases}
\bar{\Pi}^0_{\mu\nu}=\dfrac{\partial\xi^\beta}{\partial\bar{\xi}^\mu}\dfrac{\partial\xi^\tau}{\partial\bar{\xi}^\nu}\Pi^0_{\beta\tau}+\left(\dfrac{\partial\bar{\psi}_\mu}{\partial\bar{\xi}^\nu}-\bar{\psi}_\lambda*\bar{\Pi}^\lambda_{\mu\nu}\right) \\
\qquad\qquad +\bar{\psi}_\mu\bar{\psi}_\nu-\dfrac{1}{2}\bar{G}^{\beta\tau}\bar{\psi}_\beta\bar{\psi}_\tau\bar{G}_{\mu\nu}\,, \\
\bar{\Pi}^\lambda_{\mu\nu}=\dfrac{\partial\bar{\xi}^\lambda}{\partial\xi^\alpha}\left(\dfrac{\partial\xi^\beta}{\partial\bar{\xi}^\mu}\dfrac{\partial\xi^\tau}{\partial\bar{\xi}^\nu}\Pi^\alpha_{\beta\tau}+\dfrac{\partial^2\xi^\alpha}{\partial\bar{\xi}^\mu\partial\bar{\xi}^\nu}\right) \\
\qquad\qquad +\delta^\lambda_\mu\bar{\psi}_\nu+\delta^\lambda_\nu\bar{\psi}_\mu-\bar{G}^{\lambda\alpha}\bar{\psi}_\alpha\bar{G}_{\mu\nu}\,, \\
\bar{\Pi}^\infty_{\mu\nu}=\Delta^{-\frac{2}{n}}\dfrac{\partial\xi^\beta}{\partial\bar{\xi}^\mu}\dfrac{\partial\xi^\tau}{\partial\bar{\xi}^\nu}\Pi^\infty_{\beta\tau}\,, \\
\bar{\Pi}^\lambda_{\infty\nu}=\Delta^{\frac{2}{n}}\dfrac{\partial\bar{\xi}^\lambda}{\partial\xi^\alpha}\dfrac{\partial\xi^\tau}{\partial\bar{\xi}^\nu}\Pi^\alpha_{\infty\tau}+\bar{G}^{\lambda\mu}\left(\dfrac{\partial\bar{\psi}_\mu}{\partial\bar{\xi}^\nu}-\bar{\psi}_\alpha*\bar{\Pi}^\alpha_{\mu\nu}\right) \\
\qquad\qquad +\bar{G}^{\lambda\mu}\bar{\psi}_\mu\bar{\psi}_\nu-\dfrac{1}{2}\bar{G}^{\beta\tau}\bar{\psi}_\beta\bar{\psi}_\tau\delta^\lambda_\nu\,,
\end{cases}
$$

where

$$
(2.14)\qquad \bar{\psi}_\lambda=\frac{\partial\log\Delta^{-\frac{1}{n}}}{\partial\bar{\xi}^\lambda}
$$

The formulae (2.13) give the transformation law of the coefficients of canonical fundamental equations.

3°. *Integrability conditions of the canonical fundamental equations.*

We shall now consider the integrability conditions of the canonical fundamental equations.

Applying the method analogous to that used in the third Paragraph of Chapter 1, we obtain the

Theorem 2: In order that $*\Pi^0_{\mu\nu}$, $*\Pi^\lambda_{\mu\nu}=K^\lambda_{\mu\nu}$, $*\Pi^\infty_{\mu\nu}=G_{\mu\nu}(|G_{\mu\nu}|=1)$ and $*\Pi^\lambda_{\infty\nu}=G^{\lambda\mu}*\Pi^0_{\mu\nu}$ *be coefficients of the canonical fundamental equations of flat conformal geometry, it is necessary and sufficient that,*

when $n=3$

$$
(2.15)\qquad *\Omega^0_{\cdot\mu\nu\omega}\equiv*\Pi^0_{\mu\nu/\omega}-*\Pi^0_{\mu\omega/\nu}=0\,,
$$

when $n>3$

$$
(2.16)\qquad *\Omega^\lambda_{\mu\nu\omega}\equiv*R^\lambda_{\cdot\mu\nu\omega}-\frac{1}{n-2}(*R_{\mu\nu}\delta^\lambda_\omega-*R_{\mu\omega}\delta^\lambda_\nu+G_{\mu\nu}*R^\lambda_{\cdot\omega}-G_{\mu\omega}*R^\lambda_{\cdot\nu})
$$
$$
+\frac{*R}{(n-1)(n-2)}(G_{\mu\nu}\delta^\lambda_\omega-G_{\mu\omega}\delta^\lambda_\nu)=0\,,
$$

$*\Pi^0_{\mu\nu}$ being necessarily given by

$$
(2.17)\qquad *\Pi^0_{\mu\nu}=-\frac{*R_{\mu\nu}}{n-2}+\frac{*RG_{\mu\nu}}{2(n-1)(n-2)}\,.
$$

In this statement, the solidus denotes the formal covariant derivative with respect to $K^\lambda_{\mu\nu}$ and

$$*R^{\lambda}_{\cdot\mu\nu\omega}=\frac{\partial K^{\lambda}_{\mu\nu}}{\partial\xi^{\omega}}-\frac{\partial K^{\lambda}_{\mu\omega}}{\partial\xi^{\nu}}+K^{\alpha}_{\mu\nu}K^{\lambda}_{\alpha\omega}-K^{\alpha}_{\mu\omega}K^{\lambda}_{\alpha\nu},$$

and

$$*R_{\mu\nu}=*R^{\lambda}_{\cdot\mu\nu\lambda},\quad *R=G^{\mu\nu}*R_{\mu\nu}.$$

The fact that these conditions are invariant under coordinate transformations may be verified as follows.

As is shown at the end of Chapter 1, the quantities $\Omega^{0}_{\cdot\mu\nu\omega}$, $\Omega^{\lambda}_{\cdot\mu\nu\omega}$ and $*\Omega^{0}_{\cdot\mu\nu\omega}$, $*\Omega^{\lambda}_{\cdot\mu\nu\omega}$ are related by the equations

$$*\Omega^{0}_{\cdot\mu\nu\omega}=\Omega^{0}_{\cdot\mu\nu\omega}-\frac{\partial\log g^{-\frac{1}{2n}}}{\partial\xi^{\lambda}}\Omega^{\lambda}_{\cdot\mu\nu\omega} \tag{2.18}$$

and

$$*\Omega^{\lambda}_{\cdot\mu\nu\omega}=\Omega^{\lambda}_{\cdot\mu\nu\omega} \tag{2.19}$$

respectively. The last equation shows that the quantities $*\Omega^{\lambda}_{\cdot\mu\nu\omega}$ are components of a tensor, which coincides with Weyl's conformal curvature tensor. The equations (2.18) shows that $*\Omega^{0}_{\cdot\mu\nu\omega}$ are not in general components of a tensor. But $\Omega^{\lambda}_{\cdot\mu\nu\omega}$ vanishing for $n=3$, this equation shows also that when $n=3$, $*\Omega^{0}_{\cdot\mu\nu\omega}$ are components of a tensor which coincides with conformal covariant of J.M. Thomas.[1]

Thus the conditions in the Theorem 2 are all stated in the form invariant under coordinate transformations.

§ 3. *Theory of curves.*

In the present Chapter, we shall study the theory of curves in the conformal space C_n. We shall establish first the Frenet formulae with respect to a projective parameter.

These formulae were already obtained by the present author. Here we shall show that we can deduce, from the Frenet formulae with respect to a projective parameter, the Frenet formulae which do not depend on the projective parameter and coincide essentially with those obtained by the present author in the study of linear connections of Weyl-Hlavatý and of its application to the conformal geometry.

We state the results for the curve in the conformally flat space, but it is evident that the results are valid also for the curves in the space with conformal connection.

1°. *The Frenet formulae with respect to a projective parameter.*

Let us consider a curve $\xi^{\lambda}=\xi^{\lambda}(t)$ in the conformal space C_n and put

$$S_{(0)}=\rho A_0, \tag{3.1}$$

(1) J. M. Thomas: Conformal invariants. Proc. Nat. Acad. Sci. U.S.A., **12** (1926), 389-393.

where ρ is a function of the parameter t which will be determined later.

Differentiating (3.1) along the curve, we have

$$\frac{d}{dt}S_{(0)}=\frac{d\rho}{dt}A_0+\rho\frac{d\xi^\lambda}{dt}A_\lambda,$$

from which

$$\left[\frac{d}{dt}S_{(0)}\right]\cdot\left[\frac{d}{dt}S_{(0)}\right]=\rho^2 g_{\mu\nu}\frac{d\xi^\mu}{dt}\frac{d\xi^\nu}{dt}.$$

The $S_{(0)}$ being a point-hypersphere on the curve,

$$S_{(1)}=\frac{d}{dt}S_{(0)} \tag{3.2}$$

is a hypersphere passing through the point $S_{(0)}$ and orthogonal to the curve. We shall choose the factor ρ in such a manner that we have $S_{(1)}\cdot S_{(1)}=1$, from which we obtain

$$\rho=\frac{dt}{ds}, \tag{3.3}$$

where s is a parameter defined by the condition

$$g_{\mu\nu}\frac{d\xi^\mu}{ds}\frac{d\xi^\nu}{ds}=1.$$

Substituting (3.3) into (3.1) and (3.2), we heve respectively

$$S_{(0)}=\dot{t}A_0 \tag{3.4}$$

and

$$S_{(1)}=\frac{\ddot{t}}{\dot{t}}A_0+\frac{d\xi^\lambda}{ds}A_\lambda, \tag{3.5}$$

where a dot denotes the differentiation with respect to the parameter s.

Differentiating the relations $S_{(0)}\cdot S_{(1)}=0$ and $S_{(1)}\cdot S_{(1)}=1$ along the curve, we find respectively

$$1+S_{(0)}\cdot\frac{d}{dt}S_{(1)}=0 \quad\text{and}\quad S_{(1)}\cdot\frac{d}{dt}S_{(1)}=0.$$

Hence

$$S_{(\infty)}=\frac{d}{dt}S_{(1)} \tag{3.6}$$

is in general a hypersphere orthogonal to the hypersphere $S_{(1)}$. We shall choose the parameter t in such a way that the hypersphere $S_{(\infty)}$ reduces to a point-hypersphere.

Substituting (3.5) into (3.6), we find

$$S_{(\infty)}=\frac{1}{\dot{t}}\left[\left(\frac{\dddot{t}}{\dot{t}}-\frac{\ddot{t}^2}{\dot{t}^2}+a^0\right)A_0+\left(a^\lambda+\frac{\ddot{t}}{\dot{t}}\frac{d\xi^\lambda}{ds}\right)A_\lambda+A_\infty\right], \tag{3.7}$$

where

$$a^0=\Pi^0_{\mu\nu}\frac{d\xi^\mu}{ds}\frac{d\xi^\nu}{ds} \quad\text{and}\quad a^\lambda=\frac{d^2\xi^\lambda}{ds^2}+\Pi^\lambda_{\mu\nu}\frac{d\xi^\mu}{ds}\frac{d\xi^\nu}{ds}.$$

Consequently, in order that $S_{(\infty)}$ be a point-hypersphere, we must have

$$\{t, s\} = \frac{1}{2} g_{\mu\nu} a^\mu a^\nu - \dot{a}^0, \tag{3.8}$$

where $\{t, s\}$ denotes the Schwarzian derivative of t with respect to s.

The equations (3.8) shows the projective character of the parameter t thus defined. We shall call t the projective parameter on the curve.

We thus attached, to every point $S_{(0)}$ of the curve, a unit hypersphere $S_{(1)}$ passing through the point $S_{(0)}$ and orthogonal to the curve and a point-hypersphere $S_{(\infty)}$ on $S_{(1)}$ satisfying

$$S_{(0)} \cdot S_{(\infty)} = -1$$

which will be easily verified.

Differentiating the relations

$$S_{(0)} \cdot S_{(\infty)} = -1, \quad S_{(1)} \cdot S_{(\infty)} = 0, \quad S_{(\infty)} \cdot S_{(\infty)} = 0$$

along the curve, we find

$$S_{(0)} \cdot \frac{d}{dt} S_{(\infty)} = 0, \quad S_{(1)} \cdot \frac{d}{dt} S_{(\infty)} = 0, \quad S_{(\infty)} \cdot \frac{d}{dt} S_{(\infty)} = 0.$$

Thus, if $\frac{d}{dt} S_{(\infty)}$ is not identically zero, it is a hypersphere passing through two points $S_{(0)}$ and $S_{(\infty)}$ and orthogonal to the hypersphere $S_{(1)}$, consequently we can put

$$\frac{d}{dt} S_{(\infty)} = \kappa_{(1)} S_{(2)}, \tag{3.9}$$

where $S_{(2)}$ is a unit hypersphere passing through $S_{(0)}$ and $S_{(\infty)}$ and orthogonal to $S_{(1)}$.

To obtain the expression of $\kappa_{(1)}$ and $S_{(2)}$, we differentiate (3.7) along the curve, then we find

$$\frac{d}{dt} S_{(\infty)} = \frac{1}{\dot{t}^2} \Big[\Big(\frac{d}{ds} \{t, s\} + \frac{da^0}{ds} + \Pi^0_{\mu\nu} a^\mu \frac{d\xi^\nu}{ds} \Big) A_0$$
$$+ \Big(\frac{da^\lambda}{ds} + \Pi^\lambda_{\mu\nu} a^\mu \frac{d\xi^\nu}{ds} + a^0 \frac{d\xi^\lambda}{ds} + \Pi^\lambda_{\infty\nu} \frac{d\xi^\nu}{ds} + 2\{t, s\} \frac{d\xi^\lambda}{ds} \Big) A_\lambda \Big]$$

or

$$\frac{d}{dt} S_{(\infty)} = \frac{1}{\dot{t}^2} \Big[\Big(g_{\mu\nu} \frac{\delta a^\mu}{ds} a^\nu + \Pi^0_{\mu\nu} a^\mu \frac{d\xi^\nu}{ds} \Big) A_0$$
$$+ \Big(\frac{\delta a^\lambda}{ds} + (g_{\mu\nu} a^\mu a^\nu - a^0) \frac{d\xi^\lambda}{ds} + \Pi^\lambda_{\infty\nu} \frac{d\xi^\nu}{ds} \Big) A_\lambda \Big] \tag{3.10}$$

by virtue of (3.8), where $\frac{\delta}{ds}$ denotes the covariant defferentiation along the curve with respect to the Christoffel symbols $\Pi^\lambda_{\mu\nu} = \{^{\lambda}_{\mu\nu}\}$, say,

$$\frac{\delta a^\lambda}{ds} = \frac{da^\lambda}{ds} + \Pi^\lambda_{\mu\nu} \frac{d\xi^\mu}{ds} \frac{d\xi^\nu}{ds}.$$

Thus if we put

$$(3.11) \qquad \frac{d}{dt} S_{(\infty)} = \frac{1}{\dot{t}^2} [v^0 A_0 + v^\lambda A_\lambda] ,$$

where

$$(3.12) \qquad \begin{cases} v^0 = g_{\mu\nu} v^\mu a^\nu , \\ v^\lambda = \dfrac{\delta a^\lambda}{ds} + (g_{\mu\nu} a^\mu a^\nu - a^0) \dfrac{d\xi^\lambda}{ds} + \Pi^\lambda_{\infty\nu} \dfrac{d\xi^\nu}{ds} , \end{cases}$$

we have

$$\kappa_{(1)} = \frac{k_{(1)}}{\dot{t}^2} \quad \text{where} \quad k^2_{(1)} = g_{\mu\nu} v^\mu v^\nu .$$

Consequently, the equations (3.9) and (3.11) give

$$(3.13) \qquad S_{(2)} = \zeta^0_{(2)} A_0 + \zeta^\lambda_{(2)} A_\lambda ,$$

where

$$\zeta^0_{(2)} = \frac{v^0}{k_{(1)}} \quad \text{and} \quad \zeta^\lambda_{(2)} = \frac{v^\lambda}{k_{(1)}} .$$

It is to be noticed that $\zeta^\lambda_{(1)} = \dfrac{d\xi^\lambda}{ds}$ and $\zeta^\lambda_{(2)}$ are unit and mutually orthogonal vectors.

If $\dfrac{d}{dt} S_{(\infty)} = 0$, we have

$$\frac{d}{dt} S_{(\infty)} = \frac{d^2}{dt^2} S_{(1)} = \frac{d^3}{dt^3} S_{(0)} = 0 ,$$

and consequently

$$S_{(0)} = [S_{(0)}]_0 + t[S_{(1)}]_0 + \frac{1}{2} t^2 [S_{(\infty)}]_0 ,$$

which shows that $S_{(0)}$ describes a circle passing through two fixed points $[S_{(0)}]_0$ and $[S_{(\infty)}]_0$ and orthogonal to the fixed hypersphere $[S_{(1)}]_0$.

For a circle, we have, from (3.10),

$$(3.14) \qquad g_{\mu\nu} \frac{\delta a^\mu}{ds} a^\nu + \Pi^0_{\mu\nu} a^\mu \frac{d\xi^\nu}{ds} = 0$$

and

$$(3.15) \qquad \frac{\delta a^\lambda}{ds} + (g_{\mu\nu} a^\mu a^\nu - a^0) \frac{d\xi^\lambda}{ds} + \Pi^\lambda_{\infty\nu} \frac{d\xi^\nu}{ds} = 0 .$$

The equations (3.14) being a consequence of (3.15), (3.15) are differential equations of a circle.[1]

Returning to the general case, we differentiate the relations

$$S_{(0)} \cdot S_{(2)} = 0 , \quad S_{(1)} \cdot S_{(2)} = 0 , \quad S_{(\infty)} \cdot S_{(2)} = 0 , \quad S_{(2)} \cdot S_{(2)} = 1$$

along the curve, then we find

1) K. Yano: Sur les circonférences généralisées dans l'espace à connexion conforme. Proc., **14** (1938), 329–332,

$$S_{(0)}\cdot\frac{d}{dt}S_{(2)}=0,\ S_{(1)}\cdot\frac{d}{dt}S_{(2)}=0,\ \kappa_{(1)}+S_{(\infty)}\cdot\frac{d}{dt}S_{(2)}=0,\ S_{(2)}\cdot\frac{d}{dt}S_{(2)}=0\,.$$

Thus the hypersphere

$$-\kappa_{(1)}S_{(0)}+\frac{d}{dt}S_{(2)}$$

passes through the points $S_{(0)}$ and $S_{(\infty)}$ and orthogonal to the hyperspheres $S_{(1)}$ and $S_{(2)}$ and consequently, we can put

$$-\kappa_{(1)}S_{(0)}+\frac{d}{dt}S_{(2)}=\kappa_{(2)}S_{(3)}\,, \tag{3.16}$$

where $S_{(3)}$ is a unit hypersphere passing through the points $S_{(0)}$ and $S_{(\infty)}$ and orthogonal to the hyperspheres $S_{(1)}$ and $S_{(2)}$.

Substituting (3.4) and (3.13) into (3.16), we find

$$\frac{1}{\dot{t}}\left[\left(-\kappa_{(1)}\dot{t}^2+\frac{d\zeta^0_{(2)}}{ds}+\Pi^0_{\mu\nu}\zeta^\mu_{(2)}\zeta^\nu_{(1)}\right)A_0 \right.$$
$$\left. +\left(\frac{\delta}{ds}\zeta^\lambda_{(2)}+\zeta^0_{(2)}\zeta^\lambda_{(1)}\right)A_\lambda\right]=\kappa_{(2)}S_{(3)}$$

by virtue of the relation

$$g_{\mu\nu}\zeta^\mu_{(2)}\zeta^\nu_{(1)}=0\,,$$

where we have put

$$\zeta^\lambda_{(1)}=\frac{d\xi^\lambda}{ds}\,.$$

Thus, we can put

$$S_{(3)}=\zeta^0_{(3)}A_0+\zeta^\lambda_{(3)}A_\lambda\,, \tag{3.7}$$

where

$$\zeta^0_{(3)}=\frac{1}{k_{(2)}}\left[-k_{(1)}+\frac{d\zeta^0_{(2)}}{ds}+\Pi^0_{\mu\nu}\zeta^\mu_{(2)}\zeta^\nu_{(1)}\right],$$
$$\zeta^\lambda_{(3)}=\frac{1}{k_{(2)}}\left[\frac{\delta}{ds}\zeta^\lambda_{(2)}+\zeta^0_{(2)}\zeta^\lambda_{(1)}\right]$$

and

$$k_{(1)}=\dot{t}^2\kappa_{(1)}\,,$$
$$k^2_{(2)}=\dot{t}^2\kappa^2_{(2)}=g_{\mu\nu}\left(\frac{\delta}{ds}\zeta^\mu_{(2)}+\zeta^0_{(2)}\zeta^\mu_{(1)}\right)\left(\frac{\delta}{ds}\zeta^\nu_{(2)}+\zeta^0_{(2)}\zeta^\nu_{(1)}\right).$$

Differentiating the relations

$$S_{(0)}\cdot S_{(3)}=0\,,\ S_{(1)}\cdot S_{(3)}=0\,,\ S_{(\infty)}\cdot S_{(3)}=0\,,\ S_{(2)}\cdot S_{(3)}=0\,,\ S_{(3)}\cdot S_{(3)}=1\,.$$

along the curve, we find

$$S_{(0)}\cdot\frac{d}{dt}S_{(3)}=0,\quad S_{(1)}\cdot\frac{d}{dt}S_{(3)}=0,\quad S_{(\infty)}\cdot\frac{d}{dt}S_{(3)}=0\,,$$
$$\kappa_{(2)}+S_{(2)}\cdot\frac{d}{dt}S_{(3)}=0,\quad S_{(3)}\cdot\frac{d}{dt}S_{(3)}=0\,.$$

Thus the hypersphere

$$\kappa_{(2)}S_{(2)}+\frac{d}{dt}S_{(3)}$$

passes through the points $S_{(0)}$ and $S_{(\infty)}$ and is orthogonal to the hyperspheres $S_{(1)}$, $S_{(2)}$ and $S_{(3)}$. Consequently, we can put

$$\kappa_{(2)}S_{(2)}+\frac{d}{dt}S_{(3)}=\kappa_{(3)}S_{(4)}\,, \tag{3.18}$$

where $S_{(4)}$ is a unit hypersphere passing through $S_{(0)}$ and $S_{(\infty)}$ and orthogonal to $S_{(1)}$, $S_{(2)}$ and $S_{(3)}$.

Substituting (3.13) and (3.17) into (3.18), we find

$$\frac{1}{\dot{t}}\left[\dot{t}\kappa_{(2)}\zeta^0_{(2)}+\frac{d}{ds}\zeta^0_{(3)}+\Pi^0_{\mu\nu}\zeta^\mu_{(3)}\zeta^\nu_{(1)}\right]A_0$$
$$+\frac{1}{\dot{t}}\left[\dot{t}\kappa_{(2)}\zeta^\lambda_{(2)}+\frac{\delta}{ds}\zeta^\lambda_{(3)}+\zeta^0_{(3)}\zeta^\lambda_{(1)}\right]A_\lambda=\kappa_{(3)}S_{(4)}$$

by virtue of the relation

$$g_{\mu\nu}\zeta^\mu_{(3)}\zeta^\nu_{(1)}=0\,.$$

Thus we can put

$$S_{(4)}=\zeta^0_{(4)}A_0+\zeta^\lambda_{(4)}A_\lambda\,, \tag{3.19}$$

where

$$\zeta^0_{(4)}=\frac{1}{k_{(3)}}\left[k_{(2)}\zeta^0_{(2)}+\frac{d}{ds}\zeta^0_{(3)}+\Pi^0_{\mu\nu}\zeta^\mu_{(3)}\zeta^\nu_{(1)}\right]$$
$$\zeta^\lambda_{(4)}=\frac{1}{k_{(3)}}\left[k_{(2)}\zeta^\lambda_{(2)}+\frac{\delta}{ds}\zeta^\lambda_{(3)}+\zeta^0_{(3)}\zeta^\lambda_{(1)}\right]$$

and

$$k^2_{(3)}=\dot{t}^2\kappa^2_{(3)}=g_{\mu\nu}\left(k_{(2)}\zeta^\mu_{(2)}+\frac{\delta}{ds}\zeta^\mu_{(3)}+\zeta^0_{(3)}\zeta^\mu_{(1)}\right)\left(k_{(2)}\zeta^\nu_{(2)}+\frac{\delta}{ds}\zeta^\nu_{(3)}+\zeta^0_{(3)}\zeta^\nu_{(1)}\right)$$

Proceeding in this manner, we arrive at the formulae

$$\left\{\begin{aligned}
&\frac{d}{dt}S_{(0)}=S_{(1)}\,,\quad \frac{d}{dt}S_{(1)}=S_{(\infty)}\,,\quad \frac{d}{dt}S_{(\infty)}=\kappa_{(1)}S_{(2)}\,,\\
&\qquad\frac{d}{dt}S_{(2)}=\kappa_{(1)}S_{(0)}+\kappa_{(2)}S_{(3)}\,,\\
&\qquad\frac{d}{dt}S_{(3)}=-\kappa_{(2)}S_{(2)}+\kappa_{(3)}S_{(4)}\,,\\
&\qquad\frac{d}{dt}S_{(4)}=-\kappa_{(3)}S_{(3)}+\kappa_{(4)}S_{(5)}\,,\\
&\qquad\cdots\cdots\cdots\\
&\qquad\frac{d}{dt}S_{(n-1)}=-\kappa_{(n-2)}S_{(n-2)}+\kappa_{(n-1)}S_{(n)}\,,\\
&\qquad\frac{d}{dt}S_{(n)}=-\kappa_{(n-1)}S_{(n-1)}\,,
\end{aligned}\right. \tag{3.20}$$

where $S_{(1)}$, $S_{(2)}$, ...; $S_{(n)}$ are n mutually orthogonal unit hyperspheres passing through two points $S_{(0)}$ and $S_{(\infty)}$, $S_{(0)}$ being the point on the curve and $S_{(1)}$ being a unit hypersphere orthogonal to the curve.

These are the Frenet formulae with respect to a projective parameter.[1]

If we put

$$
(3.21)\quad \begin{cases}
S_{(0)}=tA_0, \quad S_{(1)}=\dfrac{t}{\dot{t}}A_0+\zeta^{\lambda}_{(1)}A_\lambda, \\
S_{(\infty)}=\dfrac{1}{\dot{t}}\left[\left(\dfrac{\dddot{t}}{\dot{t}}-\dfrac{\ddot{t}^2}{\dot{t}^2}+a^0\right)A_0+\left(a^\lambda+\dfrac{\ddot{t}}{\dot{t}}\zeta^{\lambda}_{(1)}\right)A_\lambda+A_\infty\right], \\
S_{(2)}=\zeta^0_{(2)}A_0+\zeta^\lambda_{(2)}A_\lambda, \\
S_{(3)}=\zeta^0_{(3)}A_0+\zeta^\lambda_{(3)}A_\lambda, \\
\dots\dots\dots \\
S_{(n-1)}=\zeta^0_{(n-1)}A_0+\zeta_{(n-1)}A_\lambda, \\
S_{(n)}=\zeta^0_{(n)}A_0+\zeta^\lambda_{(n)}A_\lambda,
\end{cases}
$$

the quantities $\zeta^0_{(\alpha)}$ and $\zeta^\lambda_{(\alpha)}$ satisfy the relations

$$
g_{\mu\nu}\zeta^\mu_{(\alpha)}\zeta^\nu_{(\beta)}=\delta_{\alpha\beta},
$$

$$
(3.32)\quad \begin{cases}
\dfrac{d}{ds}\zeta^0_{(2)}+\Pi^0_{\mu\nu}\zeta^\mu_{(2)}\zeta^\nu_{(1)}=k_{(1)}+k_{(2)}\zeta^0_{(3)}, \\
\dfrac{d}{ds}\zeta^0_{(3)}+\Pi^0_{\mu\nu}\zeta^\mu_{(3)}\zeta^\nu_{(1)}=-k_{(2)}\zeta^0_{(2)}+k_{(3)}\zeta^0_{(4)}, \\
\dfrac{d}{ds}\zeta^0_{(4)}+\Pi^0_{\mu\nu}\zeta^\mu_{(4)}\zeta^\nu_{(1)}=-k_{(3)}\zeta^0_{(3)}+k_{(4)}\zeta^0_{(5)}, \\
\dots\dots\dots \\
\dfrac{d}{ds}\zeta^0_{(n-1)}+\Pi^0_{\mu\nu}\zeta^\mu_{(n-)}\zeta^\nu_{(1)}=-k_{(n-2)}\zeta^0_{(n-2)}+k_{(n-1)}\zeta^0_{(n)}, \\
\dfrac{d}{ds}\zeta^0_{(n)}+\Pi^0_{\mu\nu}\zeta^\mu_{(n)}\zeta^\nu_{(1)}=-k_{(n-1)}\zeta^0_{(n-1)}.
\end{cases}
$$

and

$$
(3.23)\quad \begin{cases}
\dfrac{\delta}{ds}\zeta^\lambda_{(2)}+\zeta^0_{(2)}\zeta^\lambda_{(1)}=k_{(2)}\zeta^\lambda_{(3)}, \\
\dfrac{\delta}{ds}\zeta^\lambda_{(3)}+\zeta^0_{(3)}\zeta^\lambda_{(1)}=-k_{(2)}\zeta^\lambda_{(2)}+k_{(3)}\zeta^\lambda_{(4)}, \\
\dfrac{\delta}{ds}\zeta^\lambda_{(4)}+\zeta^0_{(4)}\zeta^\lambda_{(1)}=-k_{(3)}\zeta^\lambda_{(3)}+k_{(4)}\zeta^\lambda_{(5)}, \\
\dots\dots\dots \\
\dfrac{\delta}{ds}\zeta^\lambda_{(n-1)}+\zeta^0_{(n-1)}\zeta^\lambda_{(1)}=-k_{(n-2)}\zeta^\lambda_{(n-2)}+k_{(n-1)}\zeta^\lambda_{(n)}, \\
\dfrac{\delta}{ds}\zeta^\lambda_{(n)}+\zeta^0_{(n)}\zeta^\lambda_{(1)}=-k_{(n-1)}\zeta_{(n-1)}.
\end{cases}
$$

1) K. Yano: Sur la théorie des espaces à connexion conforme. Journal of the Faculty of Science, Tokyo Imperial University, Vol. IV, Part, 1, (1939) 1–59.

The Frenet formulae (3.20) are established with respect to a projective parameter t. It is evident that the parameter t is invariant under both change of factors and transformations of coordinates. But the parameter t being defined by a Schwarzian equation, it is determined only up to a transformation

$$\bar{t}=\frac{at+b}{ct+d}, \quad (ad-bc\neq 0).$$

Consequently, the Frenet formulae (3.20) is certainly conformal but contain some thing arbitrary depending upon the choice of the parameter t.

The modification of these formulae will be discussed in the following Paragraph.

But the Frenet formulae (3.22) and (3.23) do not depend upon the projective parameter, consequently these are purely conformal Frenet formulae for the curve.

It will be easily verified that our formulae (3.23) essentially coincide with those obtained by the present author in the study of the linear connections of Weyl-Hlavatý and of its application to the conformal geometry.[1]

1) K. Yano: Sur la connexion de Weyl-Hlavatý et ses applications à la géométrie conforme. Proc. Physico-Math. Soc. Japan, 22 (1940), 595–621.