26. *Fundamental Theory of Toothed Gearing (IV).*

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We have developed the general theory of profile curves in the preceding reports from (I) to (III).[1] Now we shall give its several applications to practical curves.

§ 1. Profile curves of cycloidal system.

Take a circle with radius a_γ as a rolling curve K_γ. However, in this case, as a pitch curve K we may not necessarily take a circle. Suppose that K_γ (and accordingly K) is oriented as a_γ is positive, that is, the direction of K_γ is positive, if the center O of the circle K_γ always exists on the left side to the direction. From the two points at which the straight line connecting the center O_γ of K_γ with a drawing point C invariably connected with K_γ intersects the perimeter of K_γ we choose the nearer one to C, denoting it by P_0 and adopt P_0 as origin. And denote by s the length of arc measured from the origin to an arbitrary point P on K_γ. Denote by r the signed length of the segment PC and by θ the angle between the straight line PC and the tangent to K_γ at P, where $sgn(\theta) = sgn(r)$.

If we find the relation $r=f(s)$ between r and s and the relation $r=g(\theta)$ between r and θ, they are respectively the equations of the profile curve F drawn by the drawing point C and of the path of contact Γ corresponding to F.

Now from the triangle $O_\gamma PC$ we have

$$PC^2 = O_\gamma C^2 + O_\gamma P^2 - 2O_\gamma C \cdot O_\gamma P \cos C\hat{O}_\gamma P$$

and then denoting by e the length of the spgment P_0C

$$r^2 = e^2 + 4a_\gamma(a_\gamma - e)\sin^2\frac{s}{2a_\gamma}$$

Hence, when $e>0$

$$(1)_1 \qquad r = f(s) = \sqrt{e^2 + 4a_\gamma(a_\gamma - e)\sin^2\frac{s}{2a_\gamma}}$$

and when $e<0$

$$(1)_2 \qquad r = f(s) = \begin{cases} -\sqrt{e^2 + 4a_\gamma(a_\gamma - e)\sin^2\dfrac{s}{2a_\gamma}}, & \text{where } |s| \leqq a_\gamma \cos^{-1}\left(\dfrac{a_\gamma}{a_\gamma - e}\right) \\ \sqrt{e^2 + 4a_\gamma(a_\gamma - e)\sin^2\dfrac{s}{2a_\gamma}}, & \text{where } |s| \geqq a_\gamma \cos^{-1}\left(\dfrac{a_\gamma}{a_\gamma - e}\right) \end{cases}$$

In particular, when $e=0$, that is, the drawing point C exists on the perimeter of K_γ,

1) This Proceedings, Vol. 25 (1949). No. 2.

(1)$_3$ $$r=f(s)=2a_\gamma \sin\frac{|s|}{2a_\gamma}.$$

Next, from the same triangle $O_\gamma PC$ we have

$$PC^2=O_\gamma C^2-O_\gamma P^2+2PC\cdot O_\gamma P \cos O_\gamma \hat{P} C,$$

that is,

(2) $$r^2-2a_\gamma \sin\theta\cdot r+e(2a_\gamma-e)=0,$$

The curve denoted by (2), namely, the path of constact Γ is a circular arc with the point O_γ as its center and $a_\gamma-e$ as its radius. This is the fact that we can again immediately derive from the characteristic property of path of contact which we have explained in the report (II) § 4, for the evolute of the circle K_γ is reduced to its center O_γ. From (2) we have

when $e>0$

(2)$_1$ $$r=g(\theta)=a_\gamma \sin\theta\pm\sqrt{a_\gamma^2\sin^2\theta-e(2a_\gamma-e)},$$

and when $e>0$

(2)$_2$ $$r=g(\theta)=\begin{cases} a_\gamma\sin\theta+\sqrt{a_\gamma^2\sin^2\theta-e(2a_\gamma-e)}, & \text{where } \theta\geqq 0, \\ -a_\gamma\sin\theta-\sqrt{a_\gamma^2\sin^2\theta-e(2a_\gamma-e)}, & \text{where } \theta\leqq 0. \end{cases}$$

In particular, when $e=0$, that is, the drawing point C exists on the perimeter of K_γ,

(2)$_3$ $$r=g(\theta)=2a_\gamma \sin\theta, \text{ where } \theta\geqq 0.$$

(2)$_3$ is the equation of the rolling curve K_γ itself.

Next, let the natural equations of the pitch curves K_1 and K_2 be $a_1=a_1(s)$, $a_2=a_2(s)$ respectively, then by Equation (13) in the report (III) we have the specific slidings σ_1 and σ_2 of the profile curves F and F as follows :

(3) $$\sigma_1=\sigma_1(s)=\frac{\dfrac{1}{a_1(s)}-\dfrac{1}{a_2(s)}}{\dfrac{1}{a_\gamma}-\dfrac{1}{a_2(s)}},\quad \sigma_2=\sigma_2(s)=\frac{\dfrac{1}{a_2(s)}-\dfrac{1}{a_1(s)}}{\dfrac{1}{a_\gamma}-\dfrac{1}{a_2(s)}}.$$

The values of σ_1 and σ_2 are independent of the position of the given drawing point C.

From (3) it follows :

When the rolling curve K_γ is a circle and moreover both of the pitch curves K_1 and K_2 are circles, then both of the specific slidings σ_1 and σ_2 become constant. Conversely, suppose that both of the pitch curves K_1 and K_2 are circles. If the profile curves F_1 and F_2 have constant specific slidings σ_1 and σ_2, then the rolling curve K_γ corresponding to F_1 and F_2 is necessarily a circle.[2)]

§ 2. Circular and straight profile curves.

Take a circle with radius a as a pitch curve K and settle a

2) T. Kubota : Geometry of Gears (Japanese), (1947), p. 112.

circular arc F with a point M as its center and m as its radius at K as a profile curve. When the point M exists at the inside of the circle K, we can adopt an arbitrary arc of the circle F as a profile curve. When M exists at the outside of K, we can adopt a part of the arc between the two tangents drawn M to K as a profile curve. When M exists on the perimeter of K, we can not adopt any arc of F as a profile curve (making one-point contact).

Let the circle K be oriented as the radius a is positive, and take the nearer point P_0 to M as origin from the two points at which the straight line connecting the center O of K with the center M of F intersects the perimeter of K. Denote by e the length of the segment P_0M.

Now we may conclude that the arc F is the parallel profile curve with the distance m to the point M. In this case, the direction of F is necessarily determined and accordingly the sign of m. Hence, from Equation (1), we have immediately the equation of F by Equation (3) in the report (II) as follows :

When $e>0$

$$(4)_1 \qquad r=f(s)=\pm m+\sqrt{e^2+4a(a-e)\sin^2\frac{s}{2a}}\ ,$$

and when $e>0$

$$(4)_2 \qquad r=f(s)=\begin{cases} -m-\sqrt{e^2+4a(a-e)\sin\frac{s}{2a}}\ , & \text{where } |s|\leqq a\cos^{-1}\left(\frac{a}{a-e}\right), \\ -m+\sqrt{e^2+4a(a-e)\sin\frac{s}{2a}}\ , & \text{where } |s|\geqq a\cos^{-1}\left(\frac{a}{a-e}\right). \end{cases}$$

The path of contact Γ of F is the conchoid curve, having the distance m, of the circular arc with the point O as the center and $a-e$ as the radius. And the equation of Γ is derived from (2) by Equation (11) in the report (II) :

When $e>0$

$$(5)_1 \qquad r=g(\theta)=\pm m+a\sin\theta\pm\sqrt{a^2\sin^2\theta-e(2a-e)},$$

and when $e>0$

$$(5)_2 \qquad r=g(\theta)=\begin{cases} -m+a\sin\theta+\sqrt{a^2\sin^2\theta-e(2a-e)}, & \text{where } \theta\geqq 0, \\ -m-a\sin\theta-\sqrt{a^2\sin^2\theta-e(2a-e)}, & \text{where } \theta\leqq 0. \end{cases}$$

Now, if $|e|$ is sufficiently large, then trasforming the first equation of $(4)_2$, we have

$$r=f(s)=-m+e\sqrt{1+4\frac{a(a-e)}{e^2}\sin^2\frac{s}{2a}}$$

$$=-m+e-a\left(1-\cos\frac{s}{a}\right)+\frac{1}{e}\left[2a^2\sin^2\frac{s}{2a}-2\frac{a^2(a-e)^2}{e^3}\sin^4\frac{s}{2a}+-\cdots\right],$$

that is,

(6) $$r=f(s)=-b+a\cos\frac{s}{a}+\frac{1}{e}\left[2a^2\sin^2\frac{s}{2a}-+\cdots\cdots\right],$$

where $b=m-e+a$ denotes the distance from the point O to the circle F. In (6), if $m\to-\infty$ and accordingly $e\to-\infty$ then the arc F becomes a part of the straight line with the distance b from O and its equation is given by

(7) $$r=f(s)=a\cos\frac{s}{a}-b.$$

The path of contact Γ of this straight profile curve F is derived by making $m\to-\infty$ in (5), or from (7) using the relation $\frac{s}{a}=sgn(\theta)\frac{\pi}{2}-\theta$:

(8) $$r=g(\theta)=a\,|\sin\theta|-b.$$

If we take one of the points of intersection of F and K as origin, from (7) we have the following equation (9) by substituting $s+a$ $a\cos^{-1}\frac{b}{a}$ or $s-a\cos^{-1}\frac{b}{a}$ into (7) in place of s :

(9) $$r=f(s)=a\cos\left(\frac{s}{a}\pm\cos^{-1}\frac{b}{a}\right)-b$$

or

(10) $$r=f(s)=a\sin\frac{s}{a}\sin\theta_0-b\left(1-\cos\frac{s}{a}\right),$$

where θ_0 denotes the angle between F and K. If, at this time, $a\to\infty$, then we have

(11) $$r=f(s)=s\sin\theta_0.$$

§ 3 Involute profile curves.

There exist two involutes drawn out from an arbitrary point I on a circle. When we regard these two involutes together as a curve, the point I is a cusp of this curve. If needed, we shall call the two involutes the branch cuves of this composed curve. Equation (11) in § 2 is the equation of the straight line F which intersects a straight line K at the angle of intersection θ_0, when we take K as a pitch curve and F as a profile curve. If we take a circle O_1 with radius a_1 as a pitch curve corresponding to K, then the profile curve corresponding to the straight line F is a (composed) involute F_1 of the circle which has the radius $|a_1\sin\theta_0|$ and concentric with the pitch circle O_1. In this case, we shall give the notice that, depending on the length of the straight profile curve F, we should take a part of one of the two branches of F_1 as the profile curve corresponding to F or a part of F_1 extending the two branches. If we take another circle O_2 with radius a_2 as a pitch curve, then the profile curve F_2 corresponding to F is a (composed) involute of the circle which has the radius $|a_2\sin\theta_0|$ and concentric with O_2. By the generalized Camus' theorem in the report (I), the two (composed) involutes F_1 and F_2 become a pair of profile curves, when we take the circles O_1 and O_2 as a pair of pitch curves. And the path of contact Γ is, in this case, the straight line perpendicular to the straight line F :

$$(12) \qquad r=g(\theta) : \quad \theta=\theta_0+sgn(\theta_0)sgn(s)\frac{\pi}{2} .$$

Next, the equation of the rolling curve K_γ is given by the above equation (11) and Equation (8) in the report (II) or by the above equation (12) and Equation (5) in the report (II) as follows :

$$(13) \qquad a_\gamma=a_\gamma(s)=\frac{r}{\cos\theta_0}=s\tan\theta_0$$

In accordance with Equation (4) in the report (III), the velocity of sliding of the point of contact C of F_1 and F_2 is given by

$$(14) \qquad v_p=\pm\left(\frac{1}{a_1}-\frac{1}{a_2}\right)\sin\theta_0\frac{ds}{dt}s.$$

In particular, when the pitch circles O_1 and O_2 rotate with constant angular velocities, the accelerations of sliding of the profile curves have the components by Equation (6) in the report (II) :

$$(15) \qquad w_t=\pm\left(\frac{1}{a_1}-\frac{1}{a_2}\right)\left(\frac{ds}{dt}\right)^2\sin\theta_0$$

and

$$(16)_1 \qquad w_{n1}=\left(\frac{1}{a_1}-\frac{1}{a_2}\right)^2\left(\frac{ds}{dt}\right)^2\frac{r^2}{a_2\cos\theta_0-r}$$

or

$$(16)_2 \qquad w_{n2}=\left(\frac{1}{a_2}-\frac{1}{a_1}\right)^2\left(\frac{ds}{dt}\right)^2\frac{r^2}{a_1\cos\theta_0-r}.$$

From (15) it follows that the profile curves slide one along the other with constant tangential acceleration.

Furthermore, by Equation (14) in the report (III), the specific slidings of F_1 and F_2 are respectively given by

$$(17) \qquad \sigma_1=\frac{\dfrac{1}{a_1}-\dfrac{1}{a_2}}{\dfrac{\cos\theta_0}{r}-\dfrac{1}{a_1}}, \quad \sigma_2=\frac{\dfrac{1}{a_2}-\dfrac{1}{a_1}}{\dfrac{\cos\theta_0}{r}-\dfrac{1}{a_2}}.$$

Now denote by C_2 the point which is on F_2 and corresponds to the cusp I_1 of F_1, the starting point on the base circle of the two branch curves of F_1 and by C_1 the point which is on F_1 and corresponds to the cusp I_2 of F_2. When the point of contact C of F_1 and F_2 runs on F_1 from the point I_1 to the point C_1, the point

C runs on the branch curve of F_2 on which the point C_2 exists from C_2 to the point I_2. Furthermore, if C continues to run on F_1, then on F_2 C runs on another branch curve of F_2 starting from I_2.

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