48. *On the Theory of Conformal Transformations between two Rheonomic Spaces.*

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Introduction. Defining the conformal transformations in a rheonomic space of A. WUNDHEILER[1] we state in the present paper the conformal invariants and introduce four special rheonomic spaces (rheonomic space without stretching, sub-rheonomic space, quasi-rheonomic space, rheonomic flat space). By the help of these invariants we find also the conditions for that a rheonomic space be conformal to one of these spaces and the conformal properties of sub-space.

§1. **The conformal parameters of connection.** Let V_n be an n-dimensional rheonomic space whose fundamental differential form is given by

$$ds^2 = a_{ij}\, dx^i\, dx^j + 2a_i\, dx^i\, dt + A\, dt^2 \tag{1.1}$$

and whose parameters of connection

$$\Gamma^k_{ij} = \frac{1}{2} a^{kh}(\partial_i a_{jh} + \partial_j a_{hi} - \partial_h a_{ij}),$$

$$\Gamma^k_i = \frac{1}{2} a^{kh}(\partial_t a_{ih} + \partial_i a_h - \partial_h a_i).$$

We consider the case that points of the domains $D, \bar{D}$ of the two rheonomic spaces, $V_n, \bar{V}_n$, are in one-to-one correspondence to each other in such a way that the following relation holds good:

$$d\bar{s} = \sigma ds, \tag{1.2}$$

that is,

$$\bar{a}_{ij} = \sigma^2 a_{ij}, \quad \bar{a}_i = \sigma^2 a_i, \quad \bar{A} = \sigma^2 A,$$

where σ may be a function of x^i, t. In this case we say that the correspondence between the two spaces is conformal in the domains $D, \bar{D}$ and that the transformation from one space to the other is a

1) Cf. A. WUNDHEILER, Rheonome Geometrie. Absolute Mechanik. Prace Mathematyczno-Fizycyne, **40**. (1932), pp. 97–142.

conformal transformation. Then the parameters of connections Γ^k_{ij}, Γ^k_i in V_n and $\bar\Gamma^k_{ij}, \bar\Gamma^k_i$ in $\bar V_n$ are related by

$$\bar\Gamma^k_{ij} = \Gamma^k_{ij} + \delta^k_i \sigma_j + \delta^k_j \sigma_i - \sigma^k a_{ij}, \tag{1.3}$$
$$\bar\Gamma^k_i = \Gamma^k_i + \delta^k_i \sigma_t - a^k \sigma_i, \quad -a_i \sigma^k$$

where $\sigma_j = \partial_j \log\sigma$, $\sigma_t = \partial_t \log\sigma$, $\sigma^k = a^{kj}\sigma_j$, $a^k = a^{kj} a_j$. We obtain after a contraction in (1.3)

$$\sigma_j = \frac{1}{n}(\bar\Gamma^g_{gj} - \Gamma^g_{gj}), \quad \sigma_t = \frac{1}{n}(\bar\Gamma^g_g - \Gamma^g_g). \tag{1.4}$$

Putting (1.4) in (1.3) we have two conformal invariants

$$K^k_{ij} = \Gamma^k_{ij} - \frac{1}{n}\delta^k_i \Gamma^g_{gj} - \frac{1}{n}\delta^k_j \Gamma^g_{gi} + \frac{1}{n} a^{kh} \Gamma^g_{gh} a_{ij}, \tag{1.5}$$

$$K^k_j = \Gamma^k_j - \frac{1}{n} a^k \Gamma^g_{gj} - \frac{1}{n}\delta^k_j \Gamma^g_g + \frac{1}{n} a^{kh} \Gamma^g_{gh} a_j. \tag{1.6}$$

which are called the *rheonomic conformal parameters of connection*. (1.5) is the same form as the T. Y. Thomas's conformal parameters of connection in Riemannian geometry. Corresponding to $\Gamma^{*k}_i = \Gamma^k_i - a^j \Gamma^k_{ij}$ which appears in the expression of the covariant differential of a strong vector[1)]

$$\delta v^k = dv^k + \Gamma^k_{ij}\,\delta x^j v^i + \Gamma^{*k}_i v^i\, dt,$$

where exists the invariant

$$K^{*k}_i = K^k_i - a^h K^k_{ih}$$

which is essential in our theory. In consequence of the changes of Γ^k_{ij}, Γ^k_i, a_{ij}, a_i, Γ^g_{gj}, Γ^g_g under rheonomic transformations $x' = x'(x, t)$:

$$\Gamma'^k_{ij} = \frac{\partial x'^k}{\partial x^m}\left(\frac{\partial x^h}{\partial x'^j}\frac{\partial x^g}{\partial x'^i}\Gamma^m_{hg} + \frac{\partial^2 x^m}{\partial x'^j \partial x'^i}\right),$$

$$\Gamma'^h_i = \frac{\partial x'^k}{\partial x^m}\left(\frac{\partial x^h}{\partial x'^i}\frac{\partial x^g}{\partial t'}\Gamma^m_{hg} + \frac{\partial x^h}{\partial x^h}\Gamma^m_h + \frac{\partial^2 x^m}{\partial x'^j \partial t'}\right),$$

$$a'_{ik} = \frac{\partial x^l}{\partial t'^i}\frac{\partial x^m}{\partial x'^k} a_{lm}, \quad a'_i = \frac{\partial x^l}{\partial x'^i}\frac{\partial x^m}{\partial t'} a_{lm} + \frac{\partial x^l}{\partial x'^i} a_l, \tag{1.7}$$

$$\Gamma'^g_{gj} = \frac{\partial x^h}{\partial x'^j}\Gamma^l_{lh} + \partial'_i \log\Delta,$$

1) A. WUNDHEILER named it "Stark Vektor".

$$\Gamma_g'^g = \frac{\partial x^h}{\partial t'}\Gamma_{lh}^l + \Gamma_{lh}^l + \partial_t' \log \Delta, \quad \Delta = \left|\frac{\partial x^i}{\partial x'^k}\right|$$

resp.

these invariants are transformed as follows:

$$K_{hl}'^g = \frac{\partial x'^g}{\partial x^k}\left(\frac{\partial x^i}{\partial x'^h}\frac{\partial x^j}{\partial x'^l}K_{ij}^k + \frac{\partial^2 x^k}{\partial x'^h \partial x'^l}\right) - \frac{2}{n}\delta^g_{(h}\psi'_{l)} + \frac{1}{n}a'^{gm}\psi'_m a'_{hl},^{1)}$$

$$K_h'^g = \frac{\partial x'^g}{\partial x^k}\left(\frac{\partial x^i}{\partial x'^h}\frac{\partial t^j}{\partial t'}K_{ij}^k + \frac{\partial x^i}{\partial x'^h}K_i^k + \frac{\partial^2 x^k}{\partial x'^h \partial t'}\right)$$

$$-\frac{1}{n}\psi_h' a'^k - \frac{1}{n}\psi_t'\delta_h^g + \frac{1}{n}\psi_m' a'^{gm} a_h'$$

where $\psi_i' = \partial_i' \log \Delta$, $\psi_t' = \partial_t' \log \Delta$. Consequently we can find easily the transformed formulas of $K_i^{*\prime k} = K_i'^k - a'^h K_{ij}'^k$.

§ 2. **The conformal stretch tensor.** Rheonomic spaces being conformal to that without stretching. Under conformal transformations the stretch-tensor

(2.1) $$W_{ij} = \frac{1}{2}(\partial_t a_{ij} - a_{j/i} - a_{i/j})^{2)}$$

varies in the rule

(2.2) $$\overline{W}_{ij} = \sigma^2\{W_{ij} + a_{ij}(\sigma_t - a^h\sigma_h)\},$$

by the help of

$$\bar{a}_{j/i} = \sigma^2(a_{j/i} + 2a_{[j}\sigma_{i]} + a^h\sigma_h a_{ij}).^{3)}$$

Multiplying $\bar{a}_{ij}$ and summing up with respect to i, j, (2.2) gives us

(2.3) $$\sigma_t - a^h\sigma_h = \frac{1}{n}(\overline{W} - W), \quad \text{where} \quad W = W_{ij}a^{ij},$$

Substituting (2.3), (2.2) goes into

(2.4) $$\overline{W}_{ij} - \frac{1}{n}\bar{a}_{ij}\overline{W} = \sigma^2\left(W_{ij} - \frac{1}{n}a_{ij}W\right).$$

Since

$$\overline{T} = \bar{A} - \bar{a}_i\bar{a}^i = \sigma^2 A - \sigma^2 a_i a^i = \sigma^2 T,$$

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1) $\delta^g_{(h}\psi'_{l)} = \frac{1}{2}(\delta^g_h\psi'_l + \delta^g_l\psi'_h)$.

2) This is nothing but WUNDHEILER'S "Dehnungstensor" and $a_{i/j} = \partial_j a_i - \Gamma^k_{ij}a_k$.

3) $a_{[j}\sigma_{i]} = \frac{1}{2}(a_j\sigma_i - a_i\sigma_j)$.

we have

$$\frac{1}{\bar{T}}\left(\bar{W}_{ij}-\frac{1}{n}\bar{a}_{ij}\bar{W}\right)=\frac{1}{T}\left(W_{ij}-\frac{1}{n}a_{ij}W\right). \tag{2.5}$$

This conformal invariant is a strong tensor, which is called the *conformal stretch tensor* and denoted by Ω_{ij}.

From $\bar{W}_{ij}=0$, it follows $\Omega_{ij}=0$. Inversely if $\Omega_{ij}=0$, we consider the differential equation

$$-\frac{1}{n}W=\sigma_t-a^h\,\sigma_h, \tag{2.6}$$

whose solution exists always and this solution σ makes $\bar{W}_{ij}$ equal to zero. Consequently

Theorem 1. *The vanishing of the conformal stretch tensor is the necessary and sufficient condition that a rheonomic space be conformal to that without stretching.*

§ 3. **Rheonomic flat space. Sub-rheonomic space and quasi-rh onomic space.** In a rheonomic space the curvature is defined by

$$(\bar{\delta}\delta-\delta\bar{\delta})\,u^i=R^i_{jkl}(x,t)\,u^j\,\bar{\delta}^k_x\,\delta^l_x+R^i_{jk}(x,t)\,u^j\,(\bar{\delta}^k_x\,dt-\delta^k_x\,\bar{d}t) \tag{3.1}$$

where u^i is a strong vector and δ, $\bar{\delta}$ are two arbitrary dispacements. In the case $dt=0$, $\bar{d}t=0$, (3.1) has the form

$$(\delta\bar{\delta}-\delta\bar{\delta})\,u^i=R^i_{jkl}(x,t)\,u^j\,\bar{d}x^k\,dx^l.$$

When $R^i_{jkl}(x,t)$ is equal to zero, the virtual space is a flat space. We shall call the rheonomic space with $R^i_{jkl}=0$ the *rheonomic flat space*. For the dispacements $\delta x^k=0$, $\bar{d}t=0$, (3.1) is reduced into

$$(\bar{\delta}\delta-\delta\bar{\delta})\,u^i=R^i_{jk}(x,t)\,u^j\,dx^h\,dt$$

which vanishes for $R^i_{jk}(x,t)=0$. In the Canal[1] space the last equation means the flatness of any surfaces whose 2-direction contains the direction of trajectory. The rheonomic space with $R^i_{jk}=0$ will be named the *sub-rheonomic space*.

The definitions of R^i_{hk}, W_{ij}, lead us to the important relation

$$R^k_{hi}=a^{kl}\,(W_{il/h}-W_{ih/l}). \tag{3.2}$$

Hence we have

1) A WUNDHEILER; the previous paper. § 9.

Theorem 2. *R^k_{hi} can be represented by derivatives of the stretch-tensor W_{ij} and a_{ij}.*

Let us call the rheonomic space with $W_{ij/k}=0$ the *quasi-rheonomic space,* then we obtain the following theorem

Theorem 3. *A sub-rheonomic space is a particular quasi-rheonomic space, and a quasi-rheonomic space is a particular rheonomic space without stretching.*

§ 4 The condition that a rheonomic space be conformal to one of the other three special ones.

1. By using the same method as that in Riemannian geometry[1], we can conclude that

Theorem 4. *The necessary and sufficient condition that a V_n for $n>2$ be mapped conformally on a rheonomic flat space is that the conformal rheonomic curvature tensor*

$$C^k_{hij}=R^k_{hij}-\frac{2}{n-2}(R_{h[i}\delta^k_{j]}+a_{k[i}R^k_{j]})+\frac{2R}{(n-1)(n-2)}a_{k[i}\delta^k_{j]}$$

be a zero tensor when $n=3$ and when $n>3$ that

$$C_{hij}=(n-3)C^k_{hij/k}$$

be a zero tensor.

2. Under the conformal transformations $W_{ij/k}$ varies in the rule

$$\overline{W}_{ij/k}=\sigma^2[W_{ij/k}+a_{ij}T_{1k}-2W_{k(j}\sigma_{i)}+2W_{l(j}\sigma^l a_{i)k}] \tag{4.1}$$

where $T_{1k}=(\sigma_t-a^h\sigma_h)_{1k}$. Multiplying by $\bar{a}_{ij}$ and summing with respect to i,j, we obtain from (4.1)

$$T_{1k}=\frac{1}{n}(\bar{a}_{ij}\overline{W}_{ij/k}-a^{ij}W_{ij/k}). \tag{4.2}$$

Substituting (4.2) in (4.1), we have

$$\begin{aligned}\overline{W}^*_{ij/k}&=\sigma^2[W^*_{ij/k}-2W_{k(j}\sigma_{i)}+2W_{l(j}\sigma^l a_{i)k}]\\&=\sigma^2[W^*_{ij/k}-2W^*_{k(j}\sigma_{i)}+2W^*_{l(j}\sigma^l a_{i)k}],\quad\text{where}\end{aligned} \tag{4.3}$$

$$W^*_{ij}=W_{ij}-\frac{1}{n}a_{ij}W.$$

Multiplying by $\bar{a}^{jk}$ and summing with respect to j,k, the last equation gives rise to

1) Cf. L. P. EISENHART; Riemannian Geometry p. 89.

$$(4.4)\qquad \bar{a}^{jk}\,W^*_{ij/k} = a^{jk}\,W^*_{ij/k} + n\,W^*_{li}\,\sigma^l.$$

a) When the rank of (W^*_{ij}) is n, it follows from (4.4)

$$(4.5)\qquad \sigma^k = \frac{1}{n}\,\overline{W}^{*\,l}_{il/}\,\overline{W}^{*/ik}\,\sigma^2 - \frac{1}{n}\,W^{*\,l}_{il/}\,W^{*/ik},$$

$W^{*/jl}$ being determined from $W^*_{li}\,W^{*/jl} = \delta^j_i$. From (2.3) we have

$$(4.6)\qquad \sigma_t = a^h\,\sigma_h + \frac{1}{n}(\overline{W} - W).$$

Substituting (4.5) in (4.3) we get the conformal invariant

$$\Pi^i_{j/k} = a^{ir}\,(W^*_{rj/k} + \frac{2}{n}\,W^*_{k(j}\,W^{*/}{}_{r)}{}^l\,W^{*\ m}_{lm/} - \frac{2}{n}\,W^{*\,l}_{l(j/}\,a_{r)k})$$

Now we put $\overline{W}_{ij/k} = 0$ in (4.1), (4.2), (4.3), (4.4) and (4.5), and denote them (4.1′), (4.2′), (4.3′), (4.4′) and (4.5′) resp.. For that (4.5′) satisfy (4.1′) it is necessary that $\Pi^i_{j/k} = 0$. The conditions of integrability of (4.5′) and (4.6′) are

$$(4.7')\qquad \begin{cases} W^{*\,l}_{il/[h}\,W^{*/i}_{k]} + W^{*\,l}_{il/}\,W^{*/}{}_{[k}{}^{i}{}_{/h]} = 0, \\ W^{*\,l}_{il/}\,W^{*/i}_{k}\,\partial_h\,a^k + W^{*/i}_{k}\,(a^k\,\partial_h\,W^{*\,l}_{il/} - \partial_t\,W^{*\,l}_{il/}\,\delta^k_h) \\ \qquad + W_{/h} + W^{*\,l}_{il/}\,(a^k\,\partial_h\,W^{*/i}_{k} - \partial_t\,W^{*/}{}_k{}^i) = 0. \end{cases}$$

Hence we have the theorem

Theorem 5. *The necessary and sufficient condition that a rheonomic space be mapped conformally on a quasi-rheonomic space that* $\Pi^i_{j/k} = 0$ *and* (4.7′) *de satisfied.*

b) When the rank of (W^*_{ij}) is $< n, \neq 0$, from (4.3′), (4.4′)

$$(4.8')\qquad 0 = W^+_{ij/k} - \frac{2}{n}\,a_{k/i}\,W^{*\,l}_{j)l/} - 2\,W^*_{k(j}\,a_{i)l}\,\sigma^l.$$

Now we shall denote the rank of the matrix $(W_{k(j}\,a_{i)1}, W_{k(j}\,a_{i)2}, \ldots, W_{k(j}\,a_{i)n})$ with α. When $\alpha = n$, we can find the condition in the same way as the previous *a*). When $\alpha < n$, the differential equations (4.8′) have their solution always, hence it requires no condition. When $\alpha > n$, there exists no conformal transformatiou which satisfies (4.8′).

3. Under conformal transformations R^k_{hi} varies in the rule

$$(4.9)\qquad \overline{R}^k_{hi} = \bar{a}^{kl}\,(\overline{W}_{il/h} - \overline{W}_{ih/l})$$

$$= R^k_{hi} + 2a^{kl} a_{i[l} T_{/h]} - 2a^{kl} W_{i[h} \sigma_{l]} + 2 W_{m[l} \sigma^m a_{h]i} a^{kl}.$$

$$= R^k_{hi} + 2a^{kl} a_{i[l} T_{/h]} - 2a^{kl} W^*_{i[h} \sigma_{l]} + 2 W^*_{m[l} \sigma^m a_{h]i} a^{kl}.$$

Putfing $k = i$ in (4.9) and summing with respect to k, we have

$$T_{/h} = \frac{1}{n-1} \{\bar{R}^k_{hk} - R^k_{hk}) - n\, \sigma^m W^*_{mh}\}. \tag{4 10}$$

Substituting (4.10) in (4.9) we get

$$\begin{aligned} \bar{R}^k_{hi} - \frac{\delta^k_i}{n-1} \bar{R}^k_{hk} + \frac{1}{n-1} \bar{a}_{ih} \bar{a}^{kl} \bar{R}^j_{lj} &= R^k_{hi} - \frac{\delta^k_i}{n-1} R^k_{hl} \\ &+ \frac{1}{n-1} a_{ih} a^{kl} R^j_{lj} + \sigma^m \Big(\frac{1}{n-1} \delta^k_i W^*_{mh} - \frac{1}{n-1} a_{ih} a^{kl} W^*_{ml} \\ &- \delta^k_m W^*_{ih} + a^{kl} W^*_{il} a_{hm} \Big). \end{aligned} \tag{4.11}$$

When $\bar{R}^k_{hi} = 0$, the left hand side in (4.11) is equal to zero. The same discussion as that in § 2. gives the condition that a rheonomic space be mapped conformally on a sub-rheonomic space.

§ 5. **Conformal properties of sub-space.** Let the sub-space V_m in a rheonomic space V_n be defined by

$$u^a = u^a(x^i, t) \qquad a = 1, \ldots, m, \tag{5.1}$$

then the metric functions of V_m

$$a_{\alpha\beta} = b^i_\alpha b^j_\beta a_{ij}, \quad a_\alpha = b^i_\alpha b^j_t a_{ij} + b^k_\alpha a_k, \quad \text{where} \quad b^i_\alpha = \frac{\partial x^i}{\partial u^\alpha},$$

are transformed as follows

$$\bar{a}_{\alpha\beta} = \sigma^2 a_{\alpha\beta}, \qquad \bar{a}_\alpha = \sigma^2 a_\alpha \tag{5.2}$$

by a conformal transformation. In use of (5.2) $\Gamma^\alpha_{\beta\gamma}$, Γ^α_β vary into

$$\begin{aligned} \bar{\Gamma}^\alpha_{\beta\gamma} &= \Gamma^\alpha_{\beta\gamma} + \delta^\alpha_\beta \sigma_\gamma + \delta^\alpha_\gamma \sigma_\beta - \sigma^\alpha a_{\beta\gamma}, \\ \bar{\Gamma}^\alpha_\beta &= \Gamma^\alpha_\beta + \delta^\alpha_\beta \sigma_t + a^\alpha \sigma_\beta - a_\beta \sigma^\alpha, \end{aligned} \tag{5.3}$$

where $\sigma_\alpha = b^i_\alpha \sigma_i$. Let D_α be D-symbol and $D_t b^i_\alpha = b^k_\alpha \nabla_t b^i_k = b^k_\alpha (b^i_{k/t} + a^j b^i_{k/j})$, then the Euler-Schouten's tensors

$$H^i_{\alpha\beta} = D_\alpha b^i_\beta, \qquad H^i_\alpha = D_t b^i_\alpha$$

are transformed in the rule

$$\bar{H}^i_{\alpha\beta} = \partial_\beta b^i_\alpha + b^k_\alpha b^h_\beta \bar{\Gamma}^i_{kh} - b^i_\gamma \bar{\Gamma}^\gamma_{\alpha\beta} = H^i_{\alpha\beta} - a^i_{\alpha\beta}(a^{ij} - b^i_\gamma b^j_\delta g^{\gamma\delta}) \sigma_j, \tag{5.4}$$

(5.5) $\overline{H}^i_\alpha = b^k_\alpha \nabla_t b^i_k = H^i_\alpha$

by a conformal transformation.

The conformal invariant produced from (5.4) is

$$M^i_{\alpha\beta} = H^i_{\alpha\beta} - \frac{1}{m} a_{\alpha\beta}\, a^{\gamma\delta} H^i_{\gamma\delta},$$

whereas the relation $M^i_{\alpha\beta} = 0$ is also invariant conformally, that is, in the terms of geometry

Theorem 6. *Under conformal transformations the umbilic points of a rheonomic sub-space is invariant conformally and consequently a total umbilic surface is also same.*

Since H^i_α is a conformal invariant, the relation $H^i_\alpha = 0$ is invariant conformally. From that follows

Theorem 7. *The property that b^i_α is parallel along* t-*curve remain unaltered under conformal transformations.*