60. An Extension of Fokker-Planck's Equation.

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Let the possible states of a stochastic system be represented by the points $x=(x_1, \ldots, x_n)$ of the n-dimensional Riemannian space R. We denote by $P(s, x, t, E)$, $s \leq t$, the transition probability that the state x at the time moment s is transferred into the Borel set $E \subseteq R$ at the later time moment t. The function P will satisfy the probability conditions

(1) $$P(s, x, t, E) \geq 0, \quad P(s, x, t, R)=1,$$

(2) $$P(s, x, s, E)=1 \text{ or} =0 \text{ according as } x \in E \text{ or } x \bar{\in} E,$$

and the Chapman-Smoluchouski's equation

(3) $$P(s, x, t, E)=\int_R P(s, x, u, dz)P(u, z, t, E), \quad s \leq u \leq t.$$

Let $C(R)$ be the Banach space of real-valued bounded continuous functions $f(x)$ on R with the norm $\|f\|=\sup|f(x)|$. We assume that

(4) $$(U_{st}f)(x)=\int_R P(s, x, t, dy)f(y)$$

defines a system of linear operators $\{U_{st}\}$ on $C(R)$ in $C(R)$. Then

(5) $(U_{st}f)(x)$ is non-negative with $f(x)$ and $\|U_{st}\|=1$,

(6) $$U_{ss}=I \text{ (the identity)}, \quad U_{su}U_{ut}f=U_{st}f.$$

In the special case of the temporal homogeneity

(7) $$U_{su}=T_{u-s},$$

the strong continuity in t of T_t implies the strong differentiability of T_tf for those f which are strongly dense in $C(R)$[1]:

(8) $$\frac{dT_tf}{dt}=\text{strong}\lim_{\Delta\downarrow 0}\frac{T_{t+\Delta}-T_t}{\Delta}f=AT_tf=T_tAf, \quad Af=\left(\frac{dT_tf}{dt}\right)_{t=0}.$$

In the general case, a formal extenssion of the above equation will be

(9) $$\frac{\partial U_{st}f}{\partial s}=-A_sU_{st}f.$$

It may be called as Fokker-Planck's equation corresponding to the stochastic process $P(s, x, t, E)$

The purpose of the present note is to give a possible form of the un-

1) E. Hille: Functional Analysis and Semi-groups, New York (1948). K. Yosida: On the differentiability and the representation of one-parameter semi-group of linear operators, Journal of the Math. Soc. of Japan, Vol. 1. No. 1 (1948).

bounded operator As as an extension of the form given by A. Kolmogoroff[1] and W. Feller.[2] It has a certain connection with the infinitely divisible law of P. Lévy,[3] and it reads as follows.

Theorem. Let there exists a sequence $\{m\}$ *of positive integers such that*

$$(A_s f)(x) = a \text{ finite } \lim_{m\to\infty} m\Big[\int_R P(s, x, s+m^{-1}, dy)f(y) - f(x)\Big] \tag{10}$$

exists if $f(x)$ *and its 1st and 2nd-derivatives are bounded and continuous in* R,

$$\lim_{a\to\infty} m \int_{d(x,y)\geqq a} P(s, x, s+m^{-1}, dy) = 0 \tag{11}$$

uniformly in m, ($d(x, y)$ = *the geodesic distance of* x *and* y).

Then we have

$$(A_s f)(x) = \sum_{j=1}^{n} a_j(s, x)\frac{\partial f}{\partial x_j} + \sum_{j,k=1}^{n} b_{jk}(s, x)\frac{\partial^2 f}{\partial x_j \partial x_k} \tag{12}$$

$$+\lim_{\varepsilon\downarrow 0}\int_{d(x,y)\geqq\varepsilon}\Big\{f(y)-f(x)-\frac{\rho(y, x)}{1+d(y, x)^2}\sum_{j=1}^{n}(y_j-x_j)\frac{\partial f}{\partial x_j}\Big\}\frac{1+d(y, x)^2}{d(y, x)^2}G(s, x, dy)$$

where i) $G(s, x, E)$ *is a countably additive non-negative set function in* E *and* $G(s, x, R) < \infty$, *ii)* $\rho(x, y)$ *is continuous in* (x, y) *such that* $\rho(x, y)$ *is 1 or 0 according as* $d(x, y) \leqq \delta/2$ *or* $\geqq \delta(\delta > 0)$, *iii) the quadratic form* $\sum_{j,k=1}^{n} b_{jk}(s, x)\xi_j\xi_k$ *is non-negative definite.*

Proof. From (10) and (11) we see that

$$G_m(s, x, E) = m\int_E \frac{d(y, x)^2}{1+d(y, x)^2} P(s, x, s+m^{-1}, dy) \tag{13}$$

satisfies

$$\lim_{a\to\infty}\int_{d(x,y)\leqq a} G_m(s, x, dy) = 0 \quad \text{uniformly in } m, \tag{14}$$

(15) $G_m(s, x, E)$ is uniformly bounded in E and in m.

Hence, for any fixed (s, x), there exists a subsequence $\{m'\}$ such that, if

$$g(x) \in C(R),$$

1) Math. Ann., *104* (1931) and *108* (1933).

2) Math. Ann., *113* (1936).

3) See K. Yosida: An operator-theoretical treatment of temporally homogeneous Markoff process, to appear in the Journal of the Math. Soc. of Japan. A formula analogus to (13) below was also obtained by K. Itô in connection with his theory of stochastic differential equations, to appear soon elsewhere. P. Lévy: Théorie de l'addition des variable aléatoires, Paris (1937), Chapitre 7.

(16) a finite $\lim_{m'\to\infty}\int_R g(y)G_{m'}(s, x, dy)$ exists and $=\int_R g(y)G(s, x, dy)$ with $G(s, x, E)$ satisfying the above i).

Now

(17) $$m\Big[\int_R P(s, x, s+m^{-1}, dy)f(y)-f(x)\Big]$$
$$=\int_R\Big\{\Big[f(y)-f(x)-\frac{\rho(x, y)}{1+d(y, x)^2}\sum_{j=1}^{n}(y_j-x_j)\frac{\partial f}{\partial x_j}\Big]\frac{1+d(y, x)^2}{d(y, x)^2}\Big\}G_m(s, x, dy)$$
$$+\int_R\frac{\rho(x, y)}{d(y, x)^2}\sum_{j=1}^{n}(y_j-x_j)\frac{f}{\partial x_j}G_m(s, x, dy)).$$

We have, for sufficiently small $d(y, x)$,
$$\{\quad\}=\sum_{j=1}^{n}(y_j-x_j)\frac{\partial f}{\partial x_j}+\sum_{k,j=1}^{n}(y_j-x_j)(y_k-x_k)\Big(\frac{\partial^2 f}{\partial X_j\partial X_k}\Big)\frac{1+d(y, x)^2}{d(y, x)^2}$$
where $X_j=x_j+\theta(y_j-x_j)$, $0<\theta<1$. Thus $\{\quad\}$ is bounded and continuous in y. Hence, by (16) the first term on the right side of (17) tends, as $m'\to\infty$, to $\int_R\{\quad\}G(s, x, dy)$. Therefore, by (10),

(18) a finite $\lim\int_R\frac{\rho(y, x)}{d(y, x)^2}\sum_{j=1}^{n}(y_j-x_j)\frac{\partial f}{\partial x_j}G_{m'}(s, x, dy)=\sum_{j=1}^{n}a_j(s, x)\frac{\partial f}{\partial x_j}$

exists and hence we have (12), by taking

(19) $$b_{jk}(s, x)=\lim_{\varepsilon\to 0}\lim_{m'\to\infty}\int_{d(y;x)\leq\varepsilon}m'(y_j-x_j)(y_k-x_k)P(s, x, s+m'^{-1}, dy).$$