

## 55. On the Foci of Algebraic Curves.

By Asajiro ICHIDA.

Lecturer at the Waseda University.

(Comm. by T. KUBOTA, M. J. A., Sept. 13, 1949.)

1. The points of intersection of tangents drawn from the imaginary circular points at infinity to an algebraic curve of the  $n$ -th class are called the Foci of the algebraic curves. As is known there are  $n^2$  foci. Now we determine the locus of foci of algebraic curves in the pencil of algebraic curves of the  $n$ -th class, and make the extension of it in space.

2. Let us prove the dual theorem.

**Theorem 1.** *Supposing the points  $P, Q$  to be intersections of variable algebraic curve in the pencil of algebraic curves of the  $n$ -th order with the two given straight lines  $g, h$ , the straight line  $FQ$  envelops an algebraic curve of the  $(2n-1)$ -th class, which has the straight lines  $g, h$  as  $(n-1)$ -ple tangents.*

**Proof.** In proving the above theorem, let us assume the equation of the pencil of algebraic curves of the  $n$ -th order to be

$$\sum_{i+j+k=n} A_{ijk} x^i y^j z^k = 0,$$

$$A_{ijk} = a_{ijk} + \lambda b_{ijk}$$

and the straight lines  $g, h$

$$g; \quad z = 0,$$

$$h; \quad y = 0,$$

then the coordinates of the point  $P(x_1, y_1, 0)$  are given by

$$\sum_{i+j=n} A_{ij0} x^i y^j = 0,$$

and the coordinates of the point  $Q(x_1', 0, z_1')$  are given by

$$\sum_{i+k=n} A_{i0k} x^i z^k = 0.$$

Let us use line coordinates  $u, v, w$  of the line  $FQ$ , then we have

$$ux_1 + vy_1 = 0,$$

$$ux_1' + wz_1' = 0.$$

Hence

$$\sum_{i+j=n} A_{ij0} (-1)^i v^i u^j = 0,$$

$$\sum_{i+k=n} A_{i0k} (-1)^i w^i u^k = 0.$$

Eliminating  $\lambda$ , from both equations, we get

$$\left| \begin{array}{cc} \sum_{i+j=n} a_{ij0}(-1)^i v^i u^j & \sum_{i+j=n} b_{ij0}(-1)^i v^i u^j \\ \sum_{i+k=n} a_{j0k}(-1)^i w^i u^k & \sum_{i+k=n} b_{j0k}(-1)^i w^i u^k \end{array} \right| = 0. \quad (A)$$

In this expression, the coefficient of the term  $u^n$  is

$$\left| \begin{array}{cc} a_{n00}(-1)^n v^n & b_{n00}(-1)^n v^n \\ a_{n00}(-1)^n w^n & b_{n00}(-1)^n w^n \end{array} \right| \equiv 0,$$

hence the expression in the left hand side of (A) is divisible by  $u$ . That is, (A) is an equation of an algebraic curve of the  $(2n-1)$ -th class.

Furthermore (A) is of the  $n$ -th degree with respect to  $v$ , so that the line  $u=0$ ,  $w=0$  or  $y=0$  is the  $n-1$ -ple tangent of (A). And the same is true for  $w$ , so (A) has also  $z=0$  as  $(n-1)$ -ple tangent.

3. Now we examine the locus of  $R$ , which is the intersection of two lines  $P_i Q_i$ ,  $P_j Q_j$ , taken two positions of the line  $PQ$ .

Theorem 2. *The locus  $\bar{C}$  of the point  $R$  is the algebraic curve of the  $(n^2-1)$ -th order which has the intersection of two straight lines  $g$ ,  $h$  as a  $(n-1)^2$ -ple point.*

Proof. In proving the above theorem, let us count the intersections of  $\bar{C}$  and  $g$ , that is, we count the cases where the point  $R$  comes on  $g$ .

In case the point  $R$  comes on the line  $g$ , there are but two conditions as below,

- (i)  $Q_i$  are on  $g$ ,
- (ii) two  $P_i$  are coincident.

(i) In case  $Q_i$  are on  $g$ , the coordinates of  $Q_i$  must be  $(1, 0, 0)$ , so

$$A_{n00} = 0,$$

$$\lambda = -\frac{a_{n00}}{b_{n00}}.$$

Accordingly, the coordinates of  $P_i$  is  $(1, 0, 0)$ , too. That is,  $(n-1)^2$  intersections of each  $n-1$  straight lines come to the intersection of  $g$ ,  $h$ .  $\bar{C}$  has the intersection of  $g$ ,  $h$ , as  $(n-1)^2$ -ple point.

(ii) For the coincidence of two  $P_i$ ,

$$\sum_{i+j=n} A_{ij0} x^i y^j = 0$$

must have a double root, and its discriminant is

$$\begin{vmatrix}
 nA_{n00} & (n-1)A_{n-1, 1, 0} & A_{1, n-1, 0} & \dots & \dots & 0 \\
 0 & nA_{n00} & \dots & \dots & A_{1, n-1, 0} & 0 \\
 & \dots & \dots & \dots & & \\
 & \dots & nA_{n00} & \dots & A_{1, n-1, 0} & \\
 A_{n-1, 1, 0} & \dots & nA_{0n0} & \dots & 0 & \\
 & \dots & \dots & \dots & & \\
 & \dots & A_{n-1, 1, 0} & \dots & nA_{0n0} & 
 \end{vmatrix} = 0.$$

This is of the  $2(n-1)$ -th degree with respect to  $A$ , that is, with respect to  $\lambda$ , and so there are  $2(n-1)$  case of coincidence of two  $P_i$ . Accordingly the number of the intersections of  $\bar{C}$  and  $g$  is found to be

$$(n-1)^2 + 2(n-1) = n^2 - 1,$$

and  $\bar{C}$  is of the  $(n^2-1)$ -th order.

4. Dualizing the above theorem, we get the following

**Theorem 3.** *Draw tangents from the two points  $G, H$  to variable algebraic curve in the pencil of algebraic curves of the  $n$ -th class, then the locus of the intersection of these tangents is the algebraic curve of the  $(2n-1)$ -th order which has  $G, H$  as  $n-1$ -ple points. And the envelope of the straight line joining the two points of intersection is an algebraic curve of the  $(n^2-1)$ -th class which has the straight line  $GH$  as a  $(n-1)^2$ -ple tangent.*

Taking  $G, H$  as the imaginary circular points at infinity, we get

**Theorem 4.** *The locus of the foci of the algebraic curves in the pencil of algebraic curves of the  $n$ -th class is an  $(n-1)$ -ply circular algebraic curve of the  $2n-1$ -th order, and the envelope of the straight line joining the two foci is a  $(n-1)^2$ -ply parabolic algebraic curve of the  $n^2-1$ -th class.*

Taking  $G, H$  as general points at infinity, we get :

**Theorem 5.** *Draw tangents parallel to two fixed directions to the algebraic curve in the pencil of algebraic curves of the  $n$ -th class, then the locus of its points of intersection is an algebraic curve of the  $(2n-1)$ -th order, and it has  $2(n-1)$  asymptotes, of which, the  $n-1$  asymptotes are parallel to one of two directions the remaining asymptotes being parallel to another. The envelope of lines joining the two points of intersection is a  $(n-1)^2$ -ply parabolic curve of the  $n^2-1$ -th class.*

5. In case  $n=2^*$ , we put the pencil of algebraic curves

$$A_{200}u^2 + A_{020}v^2 + A_{002}w^2 + 2A_{101}vw + 2A_{101}wu + 2A_{110}uw = 0,$$

then the locus of the foci is

$$\begin{aligned} & (x^2 + y^2) \left\{ x \begin{vmatrix} a_{011} & a_{002} \\ b_{011} & b_{002} \end{vmatrix} - y \begin{vmatrix} a_{101} & a_{002} \\ b_{101} & b_{002} \end{vmatrix} \right\} \\ & + x^2 \left\{ 2 \begin{vmatrix} a_{101} & a_{011} \\ b_{101} & b_{011} \end{vmatrix} - \begin{vmatrix} a_{110} & a_{002} \\ b_{110} & b_{002} \end{vmatrix} \right\} + xy \left\{ \begin{vmatrix} a_{200} & a_{002} \\ b_{200} & b_{002} \end{vmatrix} - \begin{vmatrix} a_{020} & a_{002} \\ b_{020} & b_{002} \end{vmatrix} \right\} \\ & + y^2 \left\{ 2 \begin{vmatrix} a_{101} & a_{011} \\ b_{101} & b_{011} \end{vmatrix} + \begin{vmatrix} a_{110} & a_{002} \\ b_{110} & b_{002} \end{vmatrix} \right\} + x \left\{ 2 \begin{vmatrix} a_{110} & a_{101} \\ b_{110} & b_{101} \end{vmatrix} - \begin{vmatrix} a_{011} & a_{020} \\ b_{011} & b_{020} \end{vmatrix} + \begin{vmatrix} a_{011} & a_{200} \\ b_{011} & b_{200} \end{vmatrix} \right\} \\ & + y \left\{ -2 \begin{vmatrix} a_{110} & a_{011} \\ b_{110} & b_{011} \end{vmatrix} - \begin{vmatrix} a_{101} & a_{020} \\ b_{101} & b_{020} \end{vmatrix} + \begin{vmatrix} a_{101} & a_{200} \\ b_{101} & b_{200} \end{vmatrix} \right\} \\ & + \begin{vmatrix} a_{110} & a_{020} \\ b_{110} & b_{020} \end{vmatrix} - \begin{vmatrix} a_{110} & a_{200} \\ b_{110} & b_{200} \end{vmatrix} = 0, \end{aligned}$$

and the envelope of the axis is

$$\begin{vmatrix} (u^2 - v^2)a_{110} - uv(a_{200} - a_{020}) + uwa_{011} - vwa_{101} & a_{101}u + a_{011}v + a_{002}w \\ (u^2 - v^2)b_{110} - uv(b_{200} - b_{020}) + uwb_{011} - vwb_{101} & b_{101}u + b_{011}v + b_{002}w \end{vmatrix} = 0.$$

6. In space, we can treat the same problem in a similar manner.

**Theorem 6.** *Supposing the points  $P, Q$  to be the points of intersection of an algebraic surface of the pencil of algebraic surfaces of the  $n$ -th class and the intersecting lines  $g, h$ , the straight line  $FQ$  envelops a plane algebraic curve of the  $(2n-1)$ -th order which has the straight lines  $g, h$  as  $(n-1)$ -ple tangents.*

Consider the section by the plane through  $g, h$ , this theorem is reduced to Theorem 1.

Dualizing this we get

**Theorem 7.** *Draw two tangent planes, which are perpendicular to one of two fixed directions respectively, to an algebraic surface in the pencil of algebraic surfaces of the  $n$ -th class, their lines of intersection generate an algebraic cone of the  $2n-1$ -th order, and it has  $2(n-1)$  asymptotic planes, of which,  $n-1$  asymptotic planes are perpendicular to one of the two directions and the remaining asymptotic planes perpendicular to another.*

\* In Salmon, Conic Sections, p. 275, we find the theorem as follows.

Let the four tangents to a conic be  $\alpha, \beta, \gamma, \delta$  in trilinear coordinates, these must be connected by an identical relation

$$a\alpha + b\beta + c\gamma + d\delta = 0.$$

Then the locus of the foci is

$$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} + \frac{d}{\delta} = 0.$$

7. Theorem 8. *Supposing the points P, Q, R are the intersections of an algebraic surface in the net of algebraic surfaces of the n-th order, and the three straight lines g, h, l meeting in the same point, the plane PQR envelops an algebraic surface of the 3n-2-th class which has the straight lines g, h, l as (n-1)-ple lines.*

Proof. Let us put the equation of the net of algebraic surfaces of the n-th order as

$$\sum_{i+j+k+l=n} A_{ijkl}x^i y^j z^k w^l = 0,$$

$$A_{ijkl} = a_{ijkl} + \lambda b_{ijkl} + \mu c_{ijkl},$$

and the straight lines *g, h, l*

$$g; \quad z=0, \quad w=0,$$

$$h; \quad y=0, \quad w=0,$$

$$l; \quad y=0, \quad z=0,$$

then the points *P* ( $x_1, y_1, 0, 0$ ), *Q* ( $x_1', 0, z_1', 0$ ), *R* ( $x_1'', 0, 0, w_1''$ ) are given by

$$\sum_{i+j=n} A_{ij00}x^i y^j = 0,$$

$$\sum_{i+k=n} A_{i0k0}x^i z^k = 0,$$

$$\sum_{i+l=n} A_{i00l}x^i w^l = 0.$$

If we denote plane coordinates of the plane *PQR*, by *u, v, s, t* then we have

$$ux_1 + vy_1 = 0,$$

$$ux_1' + sz_1' = 0,$$

$$ux_1'' + tw_1'' = 0.$$

Hence

$$\sum_{i+j=n} A_{ij00}(-1)^i v^i u^j = 0,$$

$$\sum_{i+k=n} A_{i0k0}(-1)^i s^i u^k = 0,$$

$$\sum_{i+l=n} A_{i00l}(-1)^i t^i u^l = 0.$$

Eliminating  $\lambda$  and  $\mu$ , from these equations, we get

$$\left| \begin{array}{ccc} \sum_{i+j=n} a_{ij00}(-1)^i v^i u^j & \sum_{i+j=n} b_{ij00}(-1)^i v^i u^j & \sum_{i+j=n} c_{ij00}(-1)^i v^i u^j \\ \sum_{i+k=n} a_{i0k0}(-1)^i s^i u^k & \sum_{i+k=n} b_{i0k0}(-1)^i s^i u^k & \sum_{i+k=n} c_{i0k0}(-1)^i s^i u^k \\ \sum_{i+l=n} a_{i00l}(-1)^i t^i u^l & \sum_{i+l=n} b_{i00l}(-1)^i t^i u^l & \sum_{i+l=n} c_{i00l}(-1)^i t^i u^l \end{array} \right| \equiv 0. \quad (B)$$

In this expression, the coefficient of the term  $u^0$  is

$$\begin{vmatrix} a_{n000}(-1)^n v^n & b_{n000}(-1)^n v^n & c_{n000}(-1)^n v^n \\ a_{n000}(-1)^n s^n & b_{n000}(-1)^n s^n & c_{n000}(-1)^n s^n \\ a_{n000}(-1)^n t^n & b_{n000}(-1)^n t^n & c_{n000}(-1)^n t^n \end{vmatrix} \equiv 0.$$

And the coefficient of the term  $u$  is

$$\begin{vmatrix} a_{n-1, 1, 0, 0}(-1)^{n-1} v^{n-1} & b_{n000}(-1)^n v^n & c_{n000}(-1)^n v^n \\ a_{n-1, 0, 1, 0}(-1)^{n-1} s^{n-1} & b_{n000}(-1)^n s^n & c_{n000}(-1)^n s^n \\ a_{n-1, 0, 0, 1}(-1)^{n-1} t^{n-1} & b_{n000}(-1)^n t^n & c_{n000}(-1)^n t^n \end{vmatrix} \\ + \begin{vmatrix} a_{n000}(-1)^n v^n & b_{n-1, 1, 0, 0}(-1)^{n-1} v^{n-1} & c_{n000}(-1)^n v^n \\ a_{n000}(-1)^n s^n & b_{n-1, 0, 1, 0}(-1)^{n-1} s^{n-1} & c_{n000}(-1)^n s^n \\ a_{n000}(-1)^n t^n & b_{n-1, 0, 0, 1}(-1)^{n-1} t^{n-1} & c_{n000}(-1)^n t^n \end{vmatrix} \\ + \begin{vmatrix} a_{n000}(-1)^n v^n & b_{n000}(-1)^n v^n & c_{n-1, 1, 0, 0}(-1)^{n-1} v^{n-1} \\ a_{n000}(-1)^n s^n & b_{n000}(-1)^n s^n & c_{n-1, 0, 1, 0}(-1)^{n-1} s^{n-1} \\ a_{n000}(-1)^n t^n & b_{n000}(-1)^n t^n & c_{n-1, 0, 0, 1}(-1)^{n-1} t^{n-1} \end{vmatrix} \equiv 0.$$

Hence the expression in the left hand side of (B) is divisible by  $u^2$ . That is, (B) is an equation of an algebraic surface of the  $3n-2$ -th class.

Furthermore (B) is of the  $n$ -th degree with respect to each  $v, s, t$ , so the surface has  $g, h, l$  as  $n-1$ -ple lines.

Dualizing we get the following:

**Theorem 9.** *Draw three tangential planes, perpendicular to one of three fixed directions respectively, to an algebraic surface in the net of algebraic surfaces of the  $n$ th class, the locus of their intersection is an algebraic surface of the  $3n-2$ -th order, and it has  $3(n-1)$  asymptotic planes, of which, each  $n-1$  asymptotic planes are perpendicular to one of three directions.*

8. In the  $m$ -dimensional space, we can prove similar theorems.

**Theorem 10.** *In the  $m$ -dimensional space, supposing the points  $P_1, P_2, \dots, P_m$  to be the intersections of an algebraic hypersurface in the  $\infty^{m-1}$  system of algebraic hypersurfaces of the  $n$ -th order and  $m$  straight line  $g_1, g_2, \dots, g_m$  which meet at one point, the hyperplane  $P_1, P_2, \dots, P_m$  envelops an algebraic hypersurface of the  $mn$  ( $m-1$ )-th class which has the straight lines  $g_1, g_2, \dots, g_m$  as  $n-1$ -ple lines.*

**Theorem 11.** *In the  $m$ -dimensional space, draw  $m$  tangential hyperplanes, which are perpendicular to one of  $m$  fixed directions respectively to an algebraic hypersurfaces in the  $\infty^{m-1}$  system of algebraic hypersurfaces of the  $n$ -th class, then the locus of the points of intersection is an algebraic hypersurface of the  $[mn-(m-1)]$ -th order, and it has  $m(n-1)$  asymptotic hyperplanes, of which, each  $n-1$  asymptotic hyperplanes are perpendicular to one of  $m$  directions.*