21. On the Affinely Connected Space Admitting a Group of Affine Motions.

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§ 1. Introduction.

In an n dimensional space A_n, with coefficients of affine connection $\Gamma^{\lambda}_{\mu\nu}$, we consider an infinitesimal deformation

$$\bar{x}^{\lambda}=x^{\lambda}+\xi^{\lambda}(x)dt\,, \tag{1}$$

where ξ^{λ} are the components of a contravariant vector field defining the deformation[1]. Then an affinor $XT^{\lambda}_{\cdot\mu\nu}$ derived from any given affinor $T^{\lambda}_{\cdot\mu\nu}$ in A_n by

$$XT^{\lambda}_{\cdot\mu\nu}\equiv\xi^{\alpha}T^{\lambda}_{\mu\nu;\alpha}-\xi^{\lambda}_{\cdot\alpha}T^{\alpha}_{\cdot\mu\nu}+\xi^{\alpha}_{\cdot\mu}T^{\lambda}_{\cdot\alpha\nu}+\xi^{\alpha}_{\cdot\nu}T^{\lambda}_{\cdot\mu\alpha} \tag{2}$$

is called the Lie derivative of $T^{\lambda}_{\cdot\mu\nu}$. In (2) we use the abbreviation

$$\xi^{\lambda}_{\cdot\mu}\equiv\xi^{\lambda}{}_{;\mu}+(\Gamma^{\lambda}_{\mu\nu}-\Gamma^{\lambda}_{\nu\mu})\xi^{\nu}\,, \tag{3}$$

where the semi-colon denotes the covariant derivative.

Although $\Gamma^{\lambda}_{\mu\nu}$ are not the components of an affinor, we can derive the Lie derivative by

$$X\Gamma^{\lambda}_{\mu\nu}\equiv\xi^{\lambda}{}_{,\mu,\nu}+\xi^{\alpha}\Gamma^{\lambda}_{\mu\nu,\alpha}-\Gamma^{\alpha}_{\mu\nu}\xi^{\lambda}{}_{,\alpha}+\Gamma^{\lambda}_{\alpha\nu}\xi^{\alpha}{}_{,\mu}+\Gamma^{\lambda}_{\mu\alpha}\xi^{\alpha}{}_{,\nu} \tag{4}$$

which is an affinor. In this expression a comma denotes a partial derivative.

An infinitesimal deformation for which the Lie derivative of the affine connection vanishes is called an affine motion. In this case the intrinsic relations of the affine connection are conserved by the deformation.

In the following let us consider the case of symmetric connection with $n\geqq 4$, hence we get

$$\xi^{\lambda}_{\cdot\mu}=\xi^{\lambda}{}_{;\mu}\,.$$

Then the condition that $X\Gamma^{\lambda}_{\mu\nu}$ vanish can be expressed in the form

$$\xi^{\lambda}{}_{;\mu;\nu}=-R^{\lambda}_{\cdot\mu\nu\omega}\xi^{\omega}\,, \tag{5}$$

where $R^{\lambda}_{\cdot\mu\nu\omega}$ is the curvature affinor:

$$R^{\lambda}_{\cdot\mu\nu\omega}\equiv\Gamma^{\lambda}_{\mu\nu,\omega}-\Gamma^{\lambda}_{\mu\omega,\nu}+\Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\alpha\omega}-\Gamma^{\alpha}_{\mu\omega}\Gamma^{\lambda}_{\alpha\nu}\,. \tag{6}$$

The problem whether an A_n admits an r-parameter group of

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1) See K. Yano: "Groups of Transformations in Generalised Spaces", Akademeia Press (1949).

affine motions will be solved by studying the integrability condition of the system of partial differential equations (5).

If (5) is completely integrable, that is, we can find a solution with any given initial values of ξ^λ and $\xi^\lambda{}_{,\mu}$ at x_0^λ, then the A_n admits an n^2+n-parameter group of affine motions and vice versa.

To find the integrability conditions, we differentiate (5) and obtain, in the form of Lie delivatives,

$$\begin{aligned} XR^{\lambda}_{\cdot\mu\nu\omega} &\equiv R^{\lambda}_{\cdot\mu\nu\omega;\,\sigma}\xi^\sigma - R^{\alpha}_{\cdot\mu\nu\omega}\xi^\lambda{}_{;\,\alpha} + R^{\lambda}_{\cdot\alpha\nu\omega}\xi^\alpha{}_{;\,\mu} \\ &\quad + R^{\lambda}_{\cdot\mu\alpha\omega}\xi^\alpha{}_{;\,\nu} + R^{\lambda}_{\cdot\mu\nu\alpha}\xi^\alpha{}_{;\,\omega} \\ &= 0\,, \end{aligned} \tag{7a}$$

$$\begin{aligned} XR^{\lambda}_{\cdot\mu\nu\omega;\,\sigma} &\equiv R^{\lambda}_{\cdot\mu\nu\omega;\,\sigma;\,\rho}\xi^\rho - R^{\alpha}_{\cdot\mu\nu\omega;\,\sigma}\xi^\lambda{}_{;\,\alpha} + R^{\lambda}_{\cdot\alpha\nu\omega;\,\sigma}\xi^\alpha{}_{;\,\mu} \\ &\quad + R^{\lambda}_{\cdot\mu\alpha\omega;\,\sigma}\xi^\alpha{}_{;\,\nu} + R^{\lambda}_{\cdot\mu\nu\alpha;\,\sigma}\xi^\alpha{}_{;\,\omega} + R^{\lambda}_{\cdot\mu\nu\omega;\,\alpha}\xi^\alpha{}_{;\,\sigma} \\ &= 0\,, \end{aligned} \tag{7b}$$

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If the first N sets of these equations for the unknowns ξ^α and $\xi^\alpha{}_{;\,\beta}$ have solutions at x_0^λ, and the $N+1$ st set is always fulfilled by any of them, then (5) is solvable with such initial conditions.

The necessary and sufficient condition that the initial conditions can be taken arbitrarily, that is, with the degree of freedom n^2+n, is already known. Curvature affinor must vanish and the space is flat. The group of affine motions has n^2+n parameters.

Some years ago it was reported that there is no group of affine motions with parameters less than n^2+n and more than n^2 [2]. The present author was very much interested in this theorem, and as he could not read the original paper he studied the space with n^2-parameter group. The result is given in the following.

§ 2. The equations necessarily satisfied in the space with n^2-parameter group of affine motions.

First, we can see that the equations that restrict the degree of freedom to n^2 are of the form

$$\xi^\kappa{}_{;\,\tau}V^\tau_{\kappa i} + \xi^\kappa V_{\kappa i} = 0, \qquad (i=1,2,\ldots,n) \tag{8}$$

for these should be derived from (7a), (7b) and so on which are linear in ξ^κ and $\xi^\kappa{}_{;\,\tau}$, the total number of independent equations being n. On the other hand as the initial conditions $(\xi^\kappa)_0$ and $(\xi^\kappa{}_{;\,\tau})_0$ should be able to be chosen arbitrarily at x_0^λ aside from the equations (8), the integrability conditions (7) must be fulfilled by (8).

As (7a) is equivalent to

$$R^{\lambda}_{\cdot\mu\nu\omega;\,\sigma}\xi^\sigma - \xi^\kappa{}_{;\,\tau}[R^{\tau}_{\cdot\mu\nu\omega}\delta^\lambda_\kappa - R^{\lambda}_{\cdot\kappa\nu\omega}\delta^\tau_\mu - R^{\lambda}_{\cdot\mu\kappa\omega}\delta^\tau_\nu - R^{\lambda}_{\cdot\mu\nu\kappa}\delta^\tau_\omega] = 0\,,$$

we get

$$R^{\lambda}_{\cdot\mu\nu\omega;\,\sigma} = -A^{i\lambda}_{\cdot\cdot\mu\nu\omega}V_{\sigma i}\,, \tag{9}$$

2) I. P. Egorov: Doklady Akad. Nauk (N. S.) SSSR., **57**, 867 (1947).

$$R^{\tau}_{\cdot\mu\nu\omega}\delta^{\lambda}_{\kappa}-R^{\lambda}_{\cdot\kappa\nu\omega}\delta^{\tau}_{\mu}-R^{\lambda}_{\cdot\mu\kappa\omega}\delta^{\tau}_{\nu}-R^{\lambda}_{\cdot\mu\nu\kappa}\delta^{\tau}_{\omega}=A^{i\lambda}_{\cdot\cdot\mu\nu\omega}V^{\tau}_{\kappa i}\,, \tag{10}$$

where $A^{i\lambda}_{\cdot\cdot\mu\nu\omega}$ with $i=1, 2, \ldots, n$ are n unknown affinors. Analogously we get from (7b) and (8)

$$R^{\lambda}_{\cdot\mu\nu\omega;\,\sigma;\,\rho}=-A^{i\lambda}_{\cdot\cdot\mu\nu\omega\sigma}V_{\rho i}\,, \tag{11}$$

$$R^{\tau}_{\cdot\mu\nu\omega;\,\sigma}\delta^{\lambda}_{\kappa}-R^{\lambda}_{\cdot\kappa\nu\omega;\,\sigma}\delta^{\tau}_{\mu}-R^{\lambda}_{\cdot\mu\kappa\omega;\,\sigma}\delta^{\tau}_{\nu}-R^{\lambda}_{\cdot\mu\nu\kappa;\,\sigma}\delta^{\tau}_{\omega}-R^{\lambda}_{\cdot\mu\nu\omega;\,\kappa}\delta^{\tau}_{\sigma}$$
$$=A^{i\lambda}_{\cdot\cdot\mu\nu\omega\sigma}V^{\tau}_{\kappa i}\,. \tag{12}$$

Although in (8) and in these equations we use the index i as the number of vectors or affinors, we may interprete it as a contravariant or covariant index for we can take any n independent linear combinations of equations (8).

In order to find the properties of $R^{\lambda}_{\cdot\mu\nu\omega}$ satisfying the equations (9), (10), (11), (12) and the like, we first solve (9) and (10), especially (10), and get the algebraic forms of $R^{\lambda}_{\cdot\mu\nu\omega}$, $V^{\kappa}_{\tau i}$, and $A^{i\lambda}_{\cdot\cdot\mu\nu\omega}$.

§ 3. Solutions of some equations.

When there is a vector u^{κ} that makes the determinant $|V^{\lambda}_{\kappa i}u^{\kappa}|$ non zero, we multiply (10) by u^{κ} and contract. Then operating the inverse matrix of $V^{\tau}_{\kappa i}u^{\kappa}$, we can eliminate $A^{i\lambda}_{\cdot\cdot\mu\nu\omega}$ from (10). After some calculations not so simple, we get as the final conclusion of (10)

$$R^{\lambda}_{\cdot\mu\nu\omega}=RB_{\mu}(\delta^{\lambda}_{\nu}B_{\omega}-\delta^{\lambda}_{\omega}B_{\nu})\,, \tag{13}$$

or else $R^{\lambda}_{\cdot\mu\nu\omega}=0$.

If $|V^{\lambda}_{\kappa i}u^{\kappa}|=0$ holds for any vector u^{κ}, we can conclude again after some calculations that the curvature affinor has the form

$$R^{\lambda}_{\cdot\mu\nu\omega}=(1/3)(B_{\mu\nu}\delta^{\lambda}_{\omega}-B_{\mu\omega}\delta^{\lambda}_{\nu}+2B_{\omega\nu}\delta^{\lambda}_{\mu})\,, \tag{14}$$

with $B_{\mu\nu}=-B_{\nu\mu}$, or else it vanishes. But as $R^{\lambda}_{\cdot\mu\nu\omega}$ must satisfy Bianchi's equation

$$R^{\lambda}_{\cdot\mu\nu\omega;\,\sigma}+R^{\lambda}_{\cdot\mu\omega\sigma;\,\nu}+R^{\lambda}_{\cdot\mu\sigma\nu;\,\omega}=0\,,$$

we get from (14) $B_{\mu\nu;\,\omega}-B_{\mu\omega;\,\nu}=0$, and $B_{\mu\nu}$ being skew symmetric in μ and ν, we get $B_{\mu\nu;\,\omega}=0$. Hence we have

$$B_{\mu\nu;\,\omega;\,\sigma}-B_{\mu\nu;\,\sigma;\,\omega}=-R^{\alpha}_{\cdot\mu\omega\sigma}B_{\alpha\nu}-R^{\alpha}_{\cdot\nu\omega\sigma}B_{\mu\alpha}=0\,,$$

from which we can conclude $B_{\mu\nu}=0$, $R^{\lambda}_{\cdot\mu\nu\omega}=0$ again.

§ 4. Properties of the curvature affinor.

We discard the case $R^{\lambda}_{\cdot\mu\nu\omega}=0$, for, then the space admits n^2+n-parameter group. We study the case where (13) holds.

This equation can be written in the form

$$R^{\lambda}_{\cdot\mu\nu\omega}=A_{\mu}(\delta^{\lambda}_{\nu}A_{\omega}-\delta^{\lambda}_{\omega}A_{\nu}) \tag{15}$$

and applying this to an equation derived from (10) we get

$$V^{\lambda}_{\kappa\mu}=A_{\kappa}B^{\lambda}_{\mu}\,. \tag{16}$$

As $|B^{\lambda}_{\mu}|$ does not vanish we can take some linear combinations of (8) as the starting equations and simplify (8) and (16) as following :

$$A_{\kappa}\xi^{\kappa}{}_{;\alpha}+V_{\kappa\alpha}\xi^{\kappa}=0\,, \tag{17}$$

$$V^{\lambda}_{\kappa\mu}=A_{\kappa}\delta^{\lambda}_{\mu}\,. \tag{18}$$

Then we obtain

$$A^{\tau\lambda}_{\cdot\cdot\mu\nu\omega}=(\delta^{\lambda}_{\omega}A_{\nu}-\delta^{\lambda}_{\nu}A_{\omega})\delta^{\tau}_{\mu}+A_{\mu}(\delta^{\tau}_{\nu}\delta^{\lambda}_{\omega}-\delta^{\tau}_{\omega}\delta^{\lambda}_{\nu})\,, \tag{19}$$

and from (9)

$$V_{\sigma\mu}=A_{\mu;\sigma}\,, \tag{20}$$

hence (17) becomes

$$XA_{\mu}=0\,. \tag{21}$$

This is a remarkable property.

Then applying these results to (12) we find after some calculations

$$A_{\mu;\nu}=aA_{\mu}A_{\nu} \tag{22}$$

and

$$\begin{aligned}-A^{\tau\lambda}_{\cdot\cdot\mu\nu\omega\sigma}=2a(\delta^{\lambda}_{\nu}A_{\omega}-\delta^{\lambda}_{\omega}A_{\nu})(\delta^{\tau}_{\mu}A_{\sigma}+\delta^{\tau}_{\sigma}A_{\mu})\\ +2aA_{\mu}A_{\sigma}(\delta^{\lambda}_{\nu}\delta^{\tau}_{\omega}-\delta^{\lambda}_{\omega}\delta^{\tau}_{\nu})\end{aligned} \tag{23}$$

with unknown a, as $A_{\mu}=0$ must be discarded. Hence we get

$$A_{\mu;\nu;\omega}=a_{;\omega}A_{\mu}A_{\nu}+2a^{2}A_{\mu}A_{\nu}A_{\omega}\,,$$

and applying this relation to the equation

$$A_{\mu;\nu;\omega}-A_{\mu;\omega;\nu}=-R^{\alpha}_{\cdot\mu\nu\omega}A_{\alpha}=0$$

we find $a_{;\omega}=b\mathrm{A}_{\omega}$.

Then we get from (15)

$$R^{\lambda}_{\cdot\mu\nu\omega;\sigma;\rho}=(2\mathrm{b}+6a^{2})R^{\lambda}_{\cdot\mu\nu\omega}A_{\sigma}A_{\rho}\,,$$

but as on the other hand we get from (11) and (23)

$$R^{\lambda}_{\cdot\mu\nu\omega;\sigma;\rho}=6a^{2}(\delta^{\lambda}_{\nu}A_{\omega}-\delta^{\lambda}_{\omega}A_{\nu})A_{\mu}A_{\sigma}A_{\rho}$$

we can conclude $b=0$, hence $a=$const.

It can be easily shown that $XR^{\lambda}_{\cdot\mu\nu\omega;\sigma;\rho}$ and so on also vanish if the conditions obtained above, that is,

$$\left.\begin{aligned}&R^{\lambda}_{\cdot\mu\nu\omega}=A_{\mu}(\delta^{\lambda}_{\nu}A_{\omega}-\delta^{\lambda}_{\omega}A_{\nu})\,,\\ &A_{\mu;\nu}=aA_{\mu}A_{\nu}\,,\\ &a_{;\omega}=0\end{aligned}\right\} \tag{24}$$

are fulfilled. (24) expresses just the space with n^2-parameter group of affine motions.

The result obtained above that the space is flat when $|V^{\lambda}_{\kappa i}u^{\kappa}|$ always vanishes enables us also to conclude the non-existence of groups with parameters more than n^2 and less than n^2+n.