36. *Notes on Fourier Analysis (XXXIX).* *Convergence and Summability of Orthogonal Series.*

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§ 1. Let $\{\varphi_n(x)\}$ be any normalized orthogonal system (N.O.S) in (a, b), that is

$$(1) \qquad \int_a^b \varphi_i(x)\ \varphi_j(x) = \begin{cases} 0, & (i \neq j) \\ 1, & (i = j). \end{cases}$$

Rademacher and Menchoff proved that if

$$(2) \qquad \sum_{n=1}^{\infty} a_n^2 \log^2 n < \infty$$

then the orthogonal series

$$(3) \qquad \sum_{n=1}^{\infty} a_n\ \varphi_n(x)$$

converges almost everywere in (a, b). Generalizing this, Kantorovitch[1] proved the following maximal theorem:

Theorem 1. Let $\{\varphi_n(x)\}$ be N.O.S. in (a, b), then

$$(4) \qquad \int_a^b \sup_n \left| \sum_{k=1}^{n} \frac{a_k}{\log(k+1)} \varphi_k(x) \right|^2 dx \leqq A \sum_{n=1}^{\infty} a_n^2,$$

where A is an absolute constant.

From the theorem of Rademacher-Menchoff, it is evident that

$$(5) \qquad \sum_{n=1}^{\infty} a_n^2\ n^{2\alpha} < \infty \qquad (\alpha > 0)$$

implies the almost everywhere convergence of

$$(6) \qquad \sum_{n=1}^{\infty} a_n\ \varphi_n(x).$$

More generally if (5) holds, then (6) is $(C, -\alpha+\delta)$, $\delta > 0$, summable almost everwhere. (See Kaczmrz[2], Zygmund[3]). Recently Cheng[4]

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1) Kantorovitch, L.: Some theorems on the almost everywhere convergence, Comptes Rendus Acad. Sci. URSS., **14** (1937), 537-540.

2) Kaczmarz, S.: Uber die Konvergenz der Reihen von Orthogonalfunktionen, Math. Zeitschr., **23** (1925).

3) Zygmund, A.: Une remarque sur un théorème de M. Kaczmarz, Math. Zeitschr., **25** (1926), 297-298.

4) Cheng. M. T.: Cesàro summability of orthogonal series, Duke Math. Journ., **14** (1947), 401-404.

proved that if $\{\varphi_n(x)\}$ is uniformly bounded and

$$\int_a^b \varphi_n(x)dx = 0 \qquad (n = 1, 2, \dots),$$

then (5) implies the $(C, -\alpha)$ summability of (6), where $1/2 < \alpha < 1$.

The restriction $1/2 < \alpha < 1$ seems curious. Concerning this point we shall prove the following theorem.

Theorem 2. Let $\{\varphi_n(x)\}$ be N.O.S. If

$$(5) \qquad \sum_{n=1}^{\infty} a_n^2 \, n^{2\alpha} < \infty$$

then

$$(6) \qquad \sum_{n=1}^{\infty} a_n \, \varphi_n(x)$$

is $(C, -\alpha)$ summable almost everywhere, where $0 < \alpha < 1$.

This is generalized as follows.

Theorem 3. Let $\{\varphi_n(x)\}$ be N.O.S. If we denote by $N_n^{(\alpha)}(x)$ the n-th $(C, -\alpha)$ -mean of the series

$$\sum_{n=1}^{\infty} a_n \, n^{-\alpha} \, \varphi_n(x),$$

then

$$\int_a^b \sup_n \, | N_n^{(\alpha)}(x) |^2 \, dx \leq A_\alpha \sum_{n=1}^{\infty} a_n^2,$$

where A_α is an absolute constant depending only on α.

The method of proof of Theorem 3 gives incidentally a new proof of Theorem 1. We prove Theorem 1 in § 2 and Theorem 3 in § 3.

§ 2. In this paragraph, we denote by $t_n(x)$, and $\tau_n^{(\beta)}(x)$ $(\beta > -1)$ the n-th partial sum and the (C, β) -mean of the series

$$(7) \qquad \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{\log(n+1)}} \varphi_n(x),$$

respectively, and put $\tau_n^{(1)}(x) = \tau_n(x)$. The partial sum and (C, β) mean of

$$(6) \qquad \sum_{n=1}^{\infty} a_n \, \varphi_n(x)$$

are denoted by $s_n(x)$ and $\sigma_n^{(\beta)}(x)$, respectively, and put

$$\sigma_n^{(1)}(x) = \sigma_n(x).$$

Then we have the following lemmas.

Lemma 1. Let $\alpha \geqq 0$, and $\sum_{n=1}^{\infty} a_n^2 < \infty$, then

$$\int_a^b \sup_n \left[\sum_{k=1}^{n} |\tau_k^{(\alpha)}(x)|^2/n\right] dx \leqq A_\alpha \sum_{n=1}^{\infty} a_n^2. \tag{8}$$

Proof. It is sufficient to prove the case $\alpha = 0$. By Riesz-Fischer's theorem, $t_n(x) = \tau_n^{(0)}(x)$ converges in mean to a function $F(x) \in L^2(a, b)$. Then

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_a^b |t_n(x) - F(x)|^2 dx = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=n+1}^{\infty} \frac{a_k^2}{\log(k+1)} \tag{9}$$

$$= \sum_{k=1}^{\infty} \frac{a_k^2}{\log(k+1)} \sum_{n=1}^{k} \frac{1}{n} \leqq A \sum_{n=1}^{\infty} a_n^2.$$

On the other hand we have

$$\frac{1}{n} \sum_{k=1}^{n} |t_k(x)|^2 \leqq \frac{2}{n} \sum_{k=1}^{n} |t_k(x) - F(x)|^2 + 2\,|F(x)| \tag{10}$$

$$\leqq \sum_{k=1}^{\infty} \frac{|t_k(x) - F(x)|^2}{k} + 2\,|F(x)|^2$$

and then

$$\int_a^b \sup_n \frac{1}{n} \sum_{k=1}^{n} |t_k(x)|^2 dx \leqq A \int_a^b \sum_{n=1}^{\infty} \frac{|t_n(x)^2 - F(x)|^2}{n} dx + A \int_a^b |F(x)|^2 dx$$

$$\leqq A \sum_{n=1}^{\infty} a_n^2 + A \sum_{n=1}^{\infty} a_n^2 / \log(n+1) \leqq 2A \sum_{n=1}^{\infty} a_n^2$$

by (9), which is the required.

Lemma 2. Let $0 < \alpha < 1$. We have

$$\int_a^b \sum_{k=1}^{\infty} \frac{\left|\sigma_k^{\left(-\frac{\alpha+1}{2}\right)}(x) - \sigma_k^{\left(-\frac{\alpha+1}{2}+1\right)}(x)\right|^2}{k^{\alpha+1}} dx \leqq A_\alpha \sum_{k=1}^{\infty} a_k^2, \tag{11}$$

and

$$\int_a^b \sum_{k=1}^{\infty} \frac{\left|\sigma_k^{\left(-\frac{1}{2}\right)}(x) - \sigma_k^{\left(\frac{1}{2}\right)}(x)\right|^2}{k} dx \leqq A \sum_{k=1}^{\infty} a_k^2 \log(k+1), \tag{12}$$

provided that the right-hand side series of (11) and (12) converge.

Proof. Since

$$\sigma_n^{(\beta-1)}(x) - \sigma_n^{(\beta)}(x) = \frac{1}{\beta A_n^{(\beta)}} \sum_{k=1}^{n} k A_{n-k}^{(\beta-1)} a_k \varphi_k(x),$$

we have

$$\int_a^b \left|\sigma_n^{\left(-\frac{1+\alpha}{2}\right)}(x) - \sigma_n^{\left(-\frac{1+\alpha}{2}+1\right)}(x)\right|^2 dx \tag{13}$$

$$= \frac{1}{\beta^2 \left[A_n^{\left(-\frac{1+\alpha}{2}+1\right)}\right]^2} \sum_{k=1}^{n} k^2 \left[A_{n-k}^{\left(-\frac{1+\alpha}{2}\right)}\right]^2 a_k^2.$$

Consequently,

$$\sum_{n=1}^{\infty}\int_a^b \frac{|\sigma_n^{\left(-\frac{1+\alpha}{2}\right)}(x)-\sigma_n^{\left(-\frac{1+\alpha}{2}+1\right)}(x)|^2}{n^{\alpha+1}}\,dx \leqq A_\alpha \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=1}^{n}\frac{a_k^2}{(n-k+1)^{\alpha+1}}$$

$$\leqq A_\alpha \sum_{k=1}^{\infty} k^2 a_k^2 \sum_{n=k+1}^{\infty}\frac{1}{n^2(n-k)^{\alpha+1}}$$

$$\leqq A_\alpha \left\{ \sum_{k=1}^{\infty} k^2 a_k^2 \sum_{n=k+1}^{2k}\frac{1}{n^2(n-k)^{\alpha+1}} + \sum_{k=1}^{\infty} k^2 a_k^2 \sum_{n=2k+1}^{\infty}\frac{1}{n^2(n-k)^{\alpha+1}}\right\}$$

$$\leqq A_\alpha \left\{ \sum_{k=1}^{\infty} a_k^2 \sum_{n=k+1}^{2k}\frac{1}{(n-k)^{\alpha+1}} + \sum_{k=1}^{\infty} a_k^2 \sum_{n=2k+1}^{\infty}\frac{1}{n^3}\right\}.$$

If $\alpha > 0$, then

$$\sum_{n=1}^{\infty}\int_a^b \frac{|\sigma_n^{\left(-\frac{1+\alpha}{2}\right)}(x)-\sigma_n^{\left(-\frac{1+\alpha}{2}+1\right)}(x)|^2}{n^{\alpha+1}}\,dx \leqq A_\alpha \sum_{n=1}^{\infty} a_n^2, \tag{14}$$

and if $\alpha = 0$, then

$$\sum_{n=1}^{\infty}\int_a^b \frac{|\sigma_n^{\left(-\frac{1}{2}\right)}(x)-\sigma_n^{\left(\frac{1}{2}\right)}(x)|^2}{n}\,dx \leqq A \sum_{n=1}^{\infty} a_n^2 \log(n+1). \tag{15}$$

Lemma 3. If $\sum_{n=1}^{\infty} a_n^2 < \infty$, then

$$\int_a^b \sup_n \frac{|t_n(x)|^2}{\log(n+1)}\,dx \leqq A \sum_{n=1}^{\infty} a_n. \tag{16}$$

Proof. Since

$$t_n(x) = \sum_{k=1}^{n} \tau_k^{\left(-\frac{1}{2}\right)}(x)\, A_k^{\left(-\frac{1}{2}\right)} A_{n-k}^{\left(-\frac{1}{2}\right)}, \tag{17}$$

we have

$$|t_n(x)|^2 \leqq \left\{ \sum_{k=1}^{n} |\tau_k^{\left(-\frac{1}{2}\right)}(x)|^2 \right\} \left\{ \sum_{k=1}^{n} | A_k^{\left(-\frac{1}{2}\right)} A_{n-k}^{\left(-\frac{1}{2}\right)} |^2 \right\} \tag{18}$$

$$\leqq A \sum_{k=1}^{n}\left[\tau_k^{\left(-\frac{1}{2}\right)}(x)\right]^2 \sum_{k=1}^{n}\frac{1}{k(n-k+1)} \leqq \frac{A\log(n+1)}{n}\sum_{k=1}^{n}\left[\tau_k^{\left(-\frac{1}{2}\right)}(x)\right]^2.$$

Then, since

$$|t_n(x)|^2/\log(n+1) \leqq \frac{A}{n}\sum_{k=1}^{n}\left[\tau_n^{\left(-\frac{1}{2}\right)}(x)\right]^2 \tag{19}$$

$$\leqq A \sum_{k=1}^{n} |\tau_k^{\left(-\frac{1}{2}\right)}(x) - \tau_k^{\left(\frac{1}{2}\right)}(x)|^2 k^{-1} + A \sum_{k=1}^{n} |\tau_k^{\left(\frac{1}{2}\right)}(x)|^2/n$$

we have

$$(20)\qquad \int_a^b \sup_n \frac{|t_n(x)|^2}{\log(n+1)}dx \leqq A\int_a^b \sum_{n=1}^{\infty} \frac{|\tau^{(-\frac{1}{2})}(x)-\tau_n^{(\frac{1}{2})}(x)|^2}{n}dx$$

$$+A\int_a^b \sup_n \frac{\sum_{k=1}^{n}|\tau_k^{(\frac{1}{2})}(x)|^2}{n}dx.$$

By Lemma 1 and 2, we get this lemma.

Lemma 4. If $\sum_{n=1}^{\infty} a_n^2 < \infty$, then we have

$$(21)\qquad \int_a^b \sup_n |\tau_n(x)|^2 dx \leqq A\sum_{n=1}^{\infty} a_n^2 .$$

Proof.

$$\int_a^b \sup_n |\tau_n(x)|^2 dx \leqq \int_a^b \sup_n \left(\frac{\sum_{k=1}^{n}|t_k(x)|}{n}\right)^2 dx$$

$$\leqq A\int_a^b \sup_n \left(\frac{\sum_{k=1}|t_k(x)|^2}{n}\right)dx \leqq A\sum_{n=1}^{\infty} a_n^2 ,$$

by Lemma 1.

Proof of Theorem 1. Put

$$\lambda_k = 1/\sqrt{\log(k+1)},$$

then we have

$$(23)\qquad \sum_{k=1}^{n} \frac{a_k}{\log(k+1)}\varphi_k(x) = \sum_{k=1}^{n} \lambda_k \frac{a_k}{\sqrt{\log(k+1)}}\varphi_k(x)$$

$$= \sum_{k=1}^{n-1} t_k(x)\Delta\lambda_k + \lambda_n t_n(x)$$

$$= \sum_{k=1}^{n-2} (k+1)\tau_k(x)\Delta^2\lambda_k + n\tau_{n-1}(x)\Delta\lambda_{n-1} + \lambda_n t_n(x),$$

where $\Delta\lambda_n = \lambda_n - \lambda_{n-1} = O(1/n\log^{3/2}(n+1))$ and $\Delta^2\lambda_n = O(1/n^2\log^{3/2}(n+1))$.

Thus we have

$$\sup_n \left|\sum_{k=1}^{n} \frac{a_k}{\log(k+1)}\varphi_k(x)\right|^2 \leqq A \sup_n \left|\sum_{k=1}^{n} (k+1)\tau_k(x)\Delta^2\lambda_k\right|^2$$

$$+A\sup_n |\tau_n(x)|^2/\log^{3/2}(n+1)$$

$$+A\sup_n |t_n(x)|^2/\log(n+1)$$

$$\leqq A\sup_n |\tau_n(x)|^2 + A\sup_n |t_n(x)|^2/\log(n+1),$$

and then we get the theorem by Lemma 4 and 3.

§ 3. *Lemma 5.* Let $\sum_{n=1}^{\infty} a_n^2 < \infty$, then

$$\int_a^b \sup_n \left| \frac{\sigma_n^{(-\alpha)}(x)}{n^{-\alpha}} \right|^2 dx \leqq A_\alpha \sum_{n=1}^{\infty} a_n^2,$$

where $0 < \alpha < 1$.

Proof. If we denote by $s_n^{(-\alpha)}(x)$ the n-th $(C, -\alpha)$ sum of the series $\sum_{n=1}^{\infty} a_n \varphi_n(x)$, then we have

$$|s_n^{(-\alpha)}(x)|^2 = \left| \sum_{k=1}^{n} \sigma_k^{(-\beta)}(x) A_k^{(-\beta)} A_{n-k}^{(-\alpha+\beta-1)} \right|^2$$

$$\leqq \sum_{k=1}^{n} |\sigma_k^{(-\beta)}(x)|^2 \sum_{k=1}^{n} [A_k^{(-\beta)}]^2 [A_{n-k}^{(-\alpha+\beta-1)}]^2$$

$$\leqq \sum_{k=1}^{n} |\sigma_k^{(-\beta)}(x)|^2 \sum_{k=1}^{n} \frac{1}{k^{2\beta}} \frac{1}{(n-k-1)^{2(\alpha-\beta+1)}}.$$

Now if we put $2\beta = 2(\alpha-\beta+1)$, then $\beta = (\alpha+1)/2$.
The second factor of the last expression is

$$\sum_{k=1}^{n} \frac{1}{k^{\alpha+1}} \frac{1}{(n-k+1)^{\alpha+1}} \leqq \sum_{k=1}^{[n/2]} + \sum_{[n/2]+1}^{n} \leqq A_\alpha / n^{\alpha+1}.$$

Consequently we have

$$|s_n^{(-\alpha)}(x)|^2 \leqq A_\alpha \sum_{k=1}^{n} |\sigma_k^{\left(-\frac{1+\alpha}{2}\right)}(x)|^2 (1/n^{\alpha+1}) \tag{25}$$

$$\leqq \frac{A_\alpha}{n^{\alpha+1}} \sum_{k=1}^{n} |\sigma_k^{\left(-\frac{1+\alpha}{2}\right)}(x) - \sigma_k^{\left(-\frac{1+\alpha}{2}+1\right)}(x)|^2 + \frac{A_\alpha}{n^{1+\alpha}} \sum_{k=1}^{n} |\sigma_k^{\left(-\frac{1+\alpha}{2}+1\right)}(x)|^2$$

$$\leqq A_\alpha \sum_{k=1}^{n} \frac{|\sigma_k^{\left(-\frac{1+\alpha}{2}\right)}(x) - \sigma_k^{\left(-\frac{1+\alpha}{2}+1\right)}(x)|^2}{k^{1+\alpha}} + \frac{A_\alpha}{n^{\alpha+1}} \sum_{k=1}^{n} |\sigma_k^{\left(-\frac{1+\alpha}{2}+1\right)}(x)|^2.$$

Since $0 < (1+\alpha)/2 < 1$ and $0 < -(1+\alpha)/2+1 < 1$, we get Lemma 5 from Lemma 1 and 2.

Lemma 6. Let $\sum_{n=1}^{\infty} k_n$ be given and put $s_n^{(-\delta)} = \sum_{\nu=1}^{n} A_{n-\nu}^{(-\delta)} k_\nu$

and $$s_{m,n}^{(-\delta)} = \sum_{\nu=1}^{m} A_{n-\nu}^{(-\delta)} k_\nu \qquad (m < n),$$

then $$|s_{m,n}^{(-\delta)}| \leqq \max_{1 \leqq \nu \leqq m} |s_\nu^{(-\delta)}|, \qquad \text{where } 0 \leqq \delta < 1.$$

This is well known. (See Hardy-Riesz[5].)

5) Hardy, G. H. and Riesz, M.: General theory of Dirichlet series, Cambridge Tract, 1915.

Lemma 7. If we put

$$T_n(x) = \sum_{k=1}^{n} a_k k^{-\alpha} \varphi_k(x),$$

then

$$(26) \qquad \int_a^b \sup_n |T_n(x)|^2 \, dx \leqq A_\alpha \sum_{n=1}^{\infty} a_n^2,$$

provided that the right-hand side series converges.

This is an easy consequence of theorem 1.

Proof of Theorem 3. Let $N_n^{(\alpha)}(x)$ be the n-th $(C, -\alpha)$-mean of the series

$$\sum_{n=1}^{\infty} n^{-\alpha} a_n \varphi_n(x),$$

then

$$(27) \qquad N_n^{(\alpha)}(x) = \frac{1}{A_n^{(-\alpha)}} \sum_{\nu=1}^{n} A_{n-\nu}^{(-\alpha)} \nu^{-\alpha} a_\nu \varphi_\nu(x)$$

$$= \frac{1}{A_n^{(-\alpha)}} \left[\sum_{\nu=1}^{[n/2]} + \sum_{[n/2]+1}^{n} \right] A_{n-\nu}^{(-\alpha)} \nu^{-\alpha} a_\nu \varphi_\nu(x) = P_n(x) + Q_n(x),$$

say. Now,

$$(28) \qquad P_n(x) = \frac{1}{A_n^{(-\alpha)}} \sum_{\nu=1}^{[n/2]} A_{n-\nu}^{(-\alpha)} \nu^{-\alpha} a_\nu \varphi_\nu(x)$$

$$= \frac{1}{A_n^{(-\alpha)}} \sum_{\nu=1}^{[n/2]-1} A_{n-\nu}^{(-\alpha-1)} \sum_{\mu=1}^{\nu} \mu^{-\alpha} a_\mu \varphi_\mu(x) + \frac{A_{n-[n/2]}^{(-\alpha)}}{A_n^{(-\alpha)}} \sum_{\nu=1}^{[n/2]} \nu^{-\alpha} a_\nu \varphi_\nu(x).$$

If we put $\sup_n |T_n(x)| = T(x)$, then

$$(29) \qquad |P_n(x)| \leqq T(x) A_\alpha n^\alpha \sum_{\nu=1}^{[n/2]} (n-\nu)^{-\alpha-1} + A_\alpha T(x) \leqq A_\alpha T(x).$$

On the other hand

$$Q_n(x) = \frac{1}{A_n^{(-\alpha)}} \sum_{\nu=[n/2]+1}^{n} A_{n-\nu}^{(-\alpha)} \nu^{-\alpha} a_\nu \varphi_\nu(x)$$

$$= \frac{1}{A_n^{(-\alpha)}} \left\{ \sum_{\nu=[n/2]+1}^{n-1} \Delta \nu^{-\alpha} \sum_{\mu=1}^{\nu} A_{n-\mu}^{(-\alpha)} a_\mu \varphi_\mu(x) + n^{-\alpha} \sum_{\nu=[n/2]}^{n} A_{n-\nu}^{(-\alpha)} a_\nu \varphi_\nu(x) \right.$$

$$\left. - [n/2]^{-\alpha} \sum_{\nu=1}^{[n/2]} A_{n-\nu}^{(-\alpha)} a_\nu \varphi_\nu(x) \right\}$$

Let us put $M(x) = \sup_n \left| \sum_{\nu=1}^{n} A_{n-\nu}^{(-\alpha)} \nu^{-\nu} a_\nu \varphi_\nu(x) \right|$, then from Lemma 6, we have

$$(30) \qquad |Q_n(x)| \leqq A_\alpha M(x).$$

Thus we get the theorem from (27), (29), (30), Lemma 5 and 7.

Theorem 2 is an easy consequence of this theorem.