29. On a Topological Method in Semi-Ordered Linear Spaces.

By Ichiro AMEMIYA.

(Comm. by K. KUNUGI, M.J.A., March 12, 1951.)

In Banach spaces, we always obtain a continuous linear functional as the limit of a weakly converging sequence of continuous linear functionals. And this property is based on a fact that a complete metric space is of second category. In continuous semi-ordered linear spaces, bounded (continuous or universally continuous) linear functionals have the same property[1]. To investigate the relation of these two cases, first in §1 we will define a kind of topology in abstract spaces by which we obtain a topological space having the property akin to that of second category one under some condition. In §2 applying it to semi-ordered linear spaces we will show that we can discuss the problem mentioned above by the topological method.

We shall make use of notations in the books of H. Nakano[2].

§1. Cell-topology.

Let R be an abstract space. For a family $\mathfrak{L}$ of subsets of R we denote by $\bar{\mathfrak{L}}$ the least totally aditive family including $\mathfrak{L}$ and the null set 0, and by $\mathfrak{T}$ the family of all the set X such that $C \in \bar{\mathfrak{L}}$ implies $XC \in \bar{\mathfrak{L}}$. Then we can see easily that $\mathfrak{T}$ satisfies the topological conditions[3] and hence we obtain a topology in R by which the family of all the open sets coincides with $\mathfrak{T}$. For brevity we will call it the topology by a *cell-system* $\mathfrak{L}$ and a set belonging in $\mathfrak{L}$ a *cell*. If $\mathfrak{L}$ satisfies the following condition:

$$(1) \quad \mathfrak{L} \ni C_\nu \ (\nu = 1, 2 \ldots.) \ C_1 \supset C_2 \supset \cdots, \text{ implies } \prod_{\nu=1}^{\infty} C_\nu \neq 0,$$

then a cell system $\mathfrak{L}$ is said to be *complete*.

Let R be a topological space by a complete cell-system $\mathfrak{L}$ in the sequel. Then R has the following important property:

Theorem 1.I. *For the sequence of closed sets* $B_\nu \ (\nu = 1, 2 \ldots.)$, *if every* B_ν *includes no cells, then the union* $\sum_{\nu=1}^{\infty} B_\nu$ *also includes no cells.*

Proof. If $\sum_{\nu=1}^{\infty} B_\nu \supset C \in \mathfrak{L}$ then there exist $C_\nu \in \mathfrak{L}$ $(\nu = 1, 2 \ldots.)$ such that $B_1' C \supset C$, $B_2' C_1 \supset C_2 \ldots . B_\nu' C_{\nu-1} \supset C_\nu \ldots.$ because B_ν' is open and $\bar{\mathfrak{L}} \ni B_\nu' C_{\nu-1} \neq 0$. Therefore by (1) we obtain that $0 \neq C \prod_{\nu=1}^{\infty} C_\nu \subset C \prod_{\nu=1}^{\infty} B_\nu' = C(\sum_{\nu=1}^{\infty} B_\nu)'$ and come to the contradiction.

For continuous functions on R we obtain by this theorem the following two theorems:

Theorem 1.2. *For a system of continuous functions* $f_\lambda(\lambda \in \Lambda)$, *if we have* $\sup_{\lambda \in \Lambda} |f_\lambda(x)| < +\infty$ *for every* $x \in R$ *then* $f_\lambda(\lambda \in \Lambda)$ *are uniformly bounded in some cell.*

Proof. Putting $B_\nu = \{x : \sup_{\lambda \in \Lambda} |f_\lambda(x)| \leqq \nu\}$ for every $\nu = 1, 2. \dots$ we have a sequence of closed sets B_ν and $\sum_{\nu=1}^{\infty} B_\nu = R$, then by the previous theorem B_ν includes a cell for some ν.

Theorem 1.3. *For a sequence of continuous functions* $f_\nu(\nu = 1, 2. \dots)$ *if there exists the limit* $\lim_{\nu \to \infty} f_\nu(x)$ *for every* $x \in R$, *then for every real number* $\varepsilon > 0$ *there exists the cell* C *and number* ν *such that for every* $x \in C$ *and numbers* $\mu, \rho \geqq \nu$ *we have* $|f_\mu(x) - f_\rho(x)| \leqq \varepsilon$.

Proof. Putting $B_\nu = \{x : \sup_{\mu, \rho \geqq \nu} |f_\mu(x) - f_\rho(x)| \leqq \varepsilon\}$ for every $\nu = 1, 2, \dots.$ we can prove the theorem similarly.

§ 2. Application to semi-ordered linear spaces.

Let R be a semi-ordered linear space. A set of positive elements A will be called an *ideal* if the conditions: 1) $A \ni 0$ 2) $a \in A$, $b \geqq a$ implies $b \in A$ 3) $a, b \in A$ implies $a \frown b \in A$ are satisfied. Taking as the cell-system, all the set of elements $[a, b] = \{x : a \leqq x \leqq b\}$ for $b - a \in A$, we obtain a topology in R. We will denote by R_A the topological space thus obtained. We can prove easily that this cell-system is complete for every ideal A if R is continuous.

In R_A for the continuity of linear functionals we obtain the following theorem:

Theorem 2.1. *In order that a linear functional* L *of* R *is continuous in* R_A, *it is necessary and sufficient that for every real number* $\varepsilon > 0$ *there exists an element* $a \in A$ *such that we have* $|L(x)| < \varepsilon$ *for every* $0 \leqq x \leqq a$.

Proof. If L is continuous then $\{x : |L(x)| < \varepsilon\}$ is open and contains 0, and hence includes some cell $C = [0, a]$.

Conversely if L satisfies the condition of the theorem, then for any real number α the set $X = \{x : L(x) > \alpha\}$ is open in R_A, because for any element y and any cell C such that $y \in CX$ namely $L(y) > \alpha + 2\varepsilon$ for some real number $\varepsilon > 0$ and $C = [y - b, y + c]$ for $b + c \in A$, if $|L(x)| < \varepsilon$ for $0 \leqq x \leqq a$, then since $a \frown b + a \frown c \geqq a \frown (b + c) \in A$ putting $C_1 = [y - a \frown b, y + a \frown c]$ we obtain a cell C^1 such that $y \in C_1 \subset C$, and for any element $z \in C_1$ since $|y - z| \leqq a$ we have $L(z) = L(y) + L(z - y) > \alpha + 2\varepsilon - 2\varepsilon = \alpha$ namely $z \in X$. For the set $\{x : L(x) < \alpha\}$ we can prove similarly that it is open.

We will say that an ideal A is a *simple ideal*, if A contains an element a such that A is the least ideal that includes αa for all real number α, and A is a *σ-ideal* if there exists a sequence $a_\nu \in R\ (\nu = 1, 2\ldots.)$ such that $a_\nu \downarrow_{\nu=1}^{\infty} 0$ and A is the least ideal that includes this sequence. Then a simple ideal is a σ-ideal, and by the previous theorem we can see easily that in order that a linear functional L is bounded (or continuous, universally continuous) it is necessary and sufficient that L is continuous in R_A for every simple ideal A (or every σ-ideal A, every ideal A such that the meet $\bigwedge A$ is 0)[4] and hence our question can be reduced to that of continuous linear functionals on R_A, and for it applying the theorem 1.2 and 1.3 with some variation on account of the linearity we can obtain immediately:

Theorem 2.2. *If R is continuous and for a system of continuous linear functionals $L_\lambda (\lambda \in \Lambda)$ on R_A if we have $\sup_{\lambda \in \Lambda} |L_\lambda(x)| < +\infty$ for every $x \in R$, then there exists an element $a \in A$ such that*

$$\sup_{0 \leq x \leq a} \sup_{\lambda \in \Lambda} |L_\lambda(x)| < +\infty$$

Theorem 2.3. *If R is continuous and for a sequence of continuous linear functionals $L_\nu\ (\nu = 1, 2\ldots.)$ on R_A if we have the limit $L(x) = \lim_{\nu \to \infty} L_\nu(x)$ for every $x \in R$, then for every real number $\varepsilon > 0$ there exists an element $a \in A$ such that we have $\sup_{0 \leq x \leq a} |L_\nu(x)| \leq \varepsilon$ for every $\nu = 1, 2\ldots.$ and hence $L(x)$ is also continuous on R_A.*

References.

1) H. Nakano: Modulared semi-ordered linear spaces, theorem 18.4 and 19.6.

2) [1] H. Nakano: Modulared semi-ordered linear spaces, Tokyo mathematical book series, Vol. I (1950).

[2] H. Nakano: Modern spectral theory, Tokyo mathematical book series, Vol. II (1950).

3) [2] P. 2.

4) [1] § 18, 19, and 22.