76. On Selberg's Elementary Proof of the Prime-Number Theorem.

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Small Latin characters except x denote natural numbers; p represents a prime, and x denotes a real number $\geqq 1$.

A. Selberg obtained recently an elementary proof of the prime-number theorem using the following asymptotic formula:

$$\vartheta(x)\log x+\sum_{p\leq x}\vartheta\left(\frac{x}{p}\right)\log p = 2x\log x+O(x)\,, \tag{1}$$

where
$$\vartheta(x)=\sum_{p\leq x}\log p\,.$$

We shall give in this note a simple proof above for

$$\psi(x)\log x+\sum_{n\leq x}\psi\left(\frac{x}{n}\right)\Lambda(n) = 2x\log x+O(x)\,, \tag{2}$$

where we define

$$\psi(x)=\sum_{n\leq x}\Lambda(n)\,,\quad \Lambda(n)=\begin{cases}\log p & \text{for } n=p^l\,,\\ O & \text{otherwise.}\end{cases}$$

The formula (2) is as effective as (1) and may be used as a substitute for (1) in the proof of the prime-number theorem. (Of course we could prove directly, if we wished, the equivalence of the two formulae.)

We have clearly

$$\log n=\sum_{d\mid n}\Lambda(d)\,, \tag{3}$$

and hence, by Möbius' inversion-formula,

$$\Lambda(n)=\sum_{d\mid n}\mu(d)\log\frac{n}{d}\,. \tag{4}$$

We find further, using (3),

$$\begin{aligned}\sum_{n\leq x}\psi\left(\frac{x}{n}\right) &= \sum_{mn\leq x}\Lambda(m)=\sum_{n\leq x}\sum_{d\mid n}\Lambda(d)\\ &=\sum_{n\leq x}\log n=\int_1^x\log\xi\, d\xi+O(\log x)\\ &= x\log x-x+O(\log x)\,,\end{aligned} \tag{5}$$

$$\sum_{n\leq x}\Lambda(n)\left[\frac{x}{n}\right]=\sum_{mn\leq x}\Lambda(n)=x\log x+O(x)\,,$$

$$\sum_{n\leq x}\Lambda(x)\frac{x}{n}=\sum_{n\leq x}\Lambda(n)\left[\frac{x}{n}\right]+O(\sum_{n\leq x}\Lambda(n))$$

(6) $$=x\log x+O(x)+O(\psi(x))\,.$$

If $F(x)$ and $G(x)$ are any two functions defined for $x\geqq 1$, which are connected by the relation

(7) $$G(x)=\sum_{n\leq x}F\left(\frac{x}{n}\right)\log x\,,$$

then we have, noting (4),

$$\sum_{n\leq x}\mu(n)G\left(\frac{x}{n}\right)=\sum_{n\leq x}\mu(n)\sum_{m\leq \frac{x}{n}}F\left(\frac{x}{mn}\right)\log\frac{x}{n}$$

$$=\sum_{n\leq x}F\left(\frac{x}{n}\right)\sum_{d\mid n}\mu(d)\left(\log\frac{x}{n}+\log\frac{n}{d}\right)$$

$$=\sum_{n\leq x}F\left(\frac{x}{n}\right)\log\frac{x}{n}\sum_{d\mid n}\mu(d)+\sum_{n\leq x}F\left(\frac{x}{n}\right)\Lambda(n)$$

(8) $$=F(x)\log x+\sum_{n\leq x}F\left(\frac{x}{n}\right)\Lambda(n)\,,$$

since we have

$$\sum_{d\mid n}\mu(d)=\begin{cases}1 & \text{for}\quad n=1\,,\\ 0 & \text{for}\quad n>1\,.\end{cases}$$

We now put, in (7) and (8),

(9) $$F(x)=\psi(x)-x+C+1\,,$$

where C is Euler's constant. We have, using (5) and the well-known formula

$$\sum_{n\leq x}\frac{1}{n}=\log x+C+O\left(\frac{1}{x}\right),$$

the following results:

$$\sum_{n\leq x}\psi\left(\frac{x}{n}\right)\log x=x\log^2 x-x\log x+O(\log^2 x)\,,$$

$$\sum_{n\leq x}\frac{x}{n}\log x=x\log^2 x+Cx\log x+O(\log x)\,,$$

$$\sum_{n\leq x}(C+1)\log x=(C+1)x\log x+O(\log x)\,,$$

and hence

$$G(x)=\sum_{n\leq x}\left(\psi\left(\frac{x}{n}\right)-\frac{x}{n}+C+1\right)\log x$$

$$=O(\log^2 x)=O(\sqrt{x}).$$

We find therefore, by (8),

$$F(x)\log x+\sum_{n\leq x}F\left(\frac{x}{n}\right)\Lambda(n)$$

$$=O\left(\sum_{n\leq x}\sqrt{\frac{x}{n}}\right)=O\left(\sqrt{x}\int_0^x\frac{d\xi}{\sqrt{\xi}}\right)=O(x),$$

whence follows, by (9) and (6),

$$\psi(x)\log x+\sum_{n\leq x}\psi\left(\frac{x}{n}\right)\Lambda(n)$$

$$=x\log x+\sum_{n\leq x}\frac{x}{n}\Lambda(n)-(C+1)\log x-(C+1)\sum_{n\leq x}\Lambda(n)+O(x)$$

$$=2x\log x+O(x)+O(\psi(x)). \tag{10}$$

Since $\psi(x)$ and $\Lambda(n)$ are non-negative, we obtain, from (10),

$$\psi(x)(\log x+O(1))\leq O(x\log x),$$

and hence

$$\psi(x)=O(x). \tag{11}$$

Inserting (11) in (10), we find the desired formula (2).

It may be mentioned, as a subsidiary result, that the following known formula follows at once from (11) and (6):

$$\sum_{n\leq x}\frac{\Lambda(n)}{n}=\log x+O(1).$$

Reference.

Atle Selberg: An elementary proof of the prime-number theorem; Ann. of Math. (2), vol. 50 (1949), pp. 305-313.