113. *Remarks on the Topological Group of Measure Preserving Transformation.*

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I. **Introduction.** Let I be the unit interval and m be the Lebesgue measure. Let G be the group of all measure preserving transformations of I onto itself. For any $S \varepsilon G$, measurable set A, and positive number ε, define neighbourhood $N(S)$ of S as follows:

$$N(S) = N(S, A, \epsilon) = \{T : m(S(A) \ominus T(A)) < \varepsilon, \quad m(S^{-1}(A) \ominus T^{-1}(A)) < \varepsilon\}.$$

With this topology G is a complete topological group. The purpose of this note is to prove the following two properties of G.

Theorem 1. *G is simple,. i.e. G contains no closed normal subgroup except G and the identity E of G.*

Theorem 2. *G is arcwise connected.*

II. **Preliminaries.** The following definitions and results of P. R. Halmos[1] are used in the sequel.

1. A measure preserving transformation T is called nowhere periodic if $m\{x : x \varepsilon I, \ T^n x = x \text{ for some } n\} = 0$.[2]

2. If both T and S have exactly the same period n, then T and S are conjugate.[3]

3. The conjugate class of any nowhere periodic measure preserving transformation is everywhere dense in G.[4]

III. **Proof of Theorem 1.** Let us denote by N any closed normals ubgroup of G.

Lemma 1. If N contains a transformation of period n[5], $n \geqslant 2$, then N contains a nowhere periodic transformation.

Proof. We shall prove this lemma in three steps: (i) $n = 2$, (ii) $n = 3$ and (iii) $n \geqslant 4$.

(i) $n = 2$. Let S and T be the transformations $Sx = -x$ and $Tx = -x + \gamma$ where γ is an irrational number, then both S and T are of period 2 and $Rx = STx = x + \gamma$[6]. By 2 of II both S and T

1) P. R. Halmos: In general a measure preserving transformation is mixing, Ann. of Math., 45, 1944, pp. 786–792.

2) P. R. Halmos, loc. cit. p. 787.

3) P. R. Halmos, loc. cit. p. 789.

4) P. R. Halmos, loc. cit. p. 789.

5) In this note we shall call T to be of period n when T has exactly the period n.

6) Cf. P. R. Halmos and J. von Neumann: Operator methods in classical mechanics II, Ann. of Math., 43, 1942, pp. 332–350.

belongs to N so $R = ST$ belongs to N. Since R is nowhere periodic, N contains a nowhere periodic transformation.

Further it is easy to prove that if N contains a transformation S which is of period 2 on some measurable set A of positive measure and which is the identity transformation on $I-A$, then N contains a transformation R of period 2.

(ii) $n=3$. Let T be the transformation $Tx = x + \frac{1}{3}$. Let S be the transformation such that

$$Sx = \begin{cases} x + \frac{2}{3} & \text{for} \quad 0 < x \leqslant \frac{1}{3}, \\ x + \frac{5}{6} & \text{for} \quad \frac{1}{3} < x \leqslant \frac{1}{2}, \quad \frac{2}{3} < x \leqslant \frac{5}{6}, \\ x + \frac{1}{2} & \text{for} \quad \frac{1}{2} < x \leqslant \frac{2}{3}, \quad \frac{5}{6} < x \leqslant 1. \end{cases}$$

Then both T and S are of period 3, thus by 2 of II both T and S belong to N. Since ST belongs to N and is of period 2 on $A = \left\{x : 0 \leqslant x \leqslant \frac{2}{3}\right\}$ and is the identity transformation on $I-A$, N contains a transformation of period 2 (by (i)).

(iii) $n \geqslant 4$. Let T be the transformation $Tx = x + \frac{1}{n}$. Let S be the transformation such that

$$Sx = \begin{cases} x - \frac{1}{n} & \text{for} \quad \frac{4}{n} < x \leqslant 1, \\ x - \frac{2}{n} & \text{for} \quad \frac{2}{n} < x \leqslant \frac{4}{n}, \\ x + \frac{1}{n} & \text{for} \quad \frac{1}{n} < x \leqslant \frac{2}{n}, \\ x + \frac{n-1}{n} & \text{for} \quad 0 < x \leqslant \frac{1}{n}. \end{cases}$$

Then both S and T are of period n, so by 2 of II both S and T belong to N. Since ST is of period 3 on $B = \left\{x : 0 \leqslant x \leqslant \frac{3}{n}\right\}$ and is the identity transformation on $I-B$, we can prove by (ii) and (i) that N contains a transformation of period 2.

Thus the cases (ii) and (iii) have been reduced to the case (i). The proof of Lemma 1 is completed.

Lemma 2. If N contains a transformation $T \neq E$, then N contains a nowhere periodic transformation.

Proof. Let T be a transformation such that $T \neq E$. Then there exists the decomposition $I = \bigcup_{n=1}^{n=\infty} I_n$[7] of I such that $\{I_n\}$ are mutually disjoint invariant sets and T is of period n on I_n and nowhere periodic on I_∞. By the assumption the set $\Lambda = \{n : m(I_n) \neq 0, \ n \neq 1\}$ is not empty. Applying Lemma 1 to the group G_n of all measure preserving transformations on I_n for every $n \varepsilon \Lambda$, $n \neq \infty$, we can prove that N contains a nowhere periodic transformation on $I - I_\infty$.

Since any transformation which is nowhere periodic both on $I - I_\infty$ and I_∞ is nowhere periodic on I, N contains a nowhere periodic transformation. Thus the proof of Lemma 2 is completed.

If N contains an element different from the identity then by Lemma 2 N contains a nowhere periodic transformation. Therefore by 3 of II N coincides with G.

IV. Proof of Theorem 2.

Lemma 3. Let $I = A_1 \cup A_2$, $I = B_1 \cup B_2$ be any decompositions of I such that $m(A_1) = m(A_2) = m(B_1) = m(B_2) = \frac{1}{2}$ and $m(A_1 \frown A_2) = m(B_1 \frown B_2) = 0$. Then there exists a one-parameter subgroup U_t, $0 \leqslant t \leqslant 1$, such that $U_0 = E$, $U_1(A_1) = B_1$ and $U_1(A_2) = B_2$.

Proof. Put $A_1 \frown B_1 = C_1$, $A_2 \frown B_2 = C_2$. It is easy to prove that there exists a transformation S which is of period 2 on $I - (C_1 \cup C_2)$, $S(A_1 - C_1) = B_1 - C_1$, $S(A_2 - C_2) = B_2 - C_2$, and which is the identity transformation on $C_1 \cup C_2$. Using the fact that a transformation of period 2 is conjugate to the transformation R: $Rx = x + \frac{1}{2}$ (by 2 of II), we can find one-parameter subgroup U_t, $0 \leqslant t \leqslant 1$, such that $U_0 = E$, $U_1 = S$.

Thus the proof of the lemma is completed.

Let T be any measure preserving transformation and let I_0, I_1 be a dyadic set of rank 1. Applying Lemma 2 to I_0, I_1 and $T(I_0)$, $T(I_1)$, we get a one-parameter subgroup $V_t^{(1)}$, $0 \leqslant t \leqslant 1$, such that $V_0^{(1)} = E$, $V_1^{(1)}(I_0) = T(I_0)$ and $V_1^{(1)}(I_1) = T(I_1)$. Put

$$U_t^{(1)} = \begin{cases} V_{2t}^{(1)} & \text{for } 0 \leqslant t \leqslant \frac{1}{2}, \\ V_1^{(1)} & \text{for } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Let I_{00}, I_{01}, I_{10}, I_{11} be a dyadic set of rank 2. Applying Lemma 3 to $U_1^{(1)}(I_{00})$, $U_1^{(1)}(I_{01})$, $T(I_{00})$, $T(I_{01})$ and $U_1^{(1)}(I_{10})$, $U_1^{(1)}(I_{11})$, $T(I_{10})$, $T(I_{11})$, we get an arc $V_t^{(2)}$, $0 \leqslant t \leqslant 1$, such that $V_0^{(2)} = U_1^{(1)}$ and $V_1^{(2)}(I_{\varepsilon_1 \varepsilon_2}) = T(I_{\varepsilon_1 \varepsilon_2})$, $\varepsilon_i = 0,1$. Put

$$U_t^{(2)} = \begin{cases} U_{2t}^{(1)} & \text{for } 0 \leqslant t \leqslant \frac{1}{2}, \\ V_{4(t-\frac{1}{2})}^{(2)} & \text{for } \frac{1}{2} \leqslant t \leqslant \frac{3}{4}, \\ V_1^{(2)} & \text{for } \frac{3}{4} \leqslant t \leqslant 1. \end{cases}$$

7) The notation $\bigcup_{n=1}^{n=\infty} I_n$ means the sum of $I_1, I_2, \dots, I_n, \dots$ and I_∞.

We get successively the family of continuous arcs $\{U_t^{(n)}\}$ which satisfy the following properties;

(i) $U_t^{(n)}$ is continuous,

(ii) $U_0^{(n)} = E$, $U_1^{(n)}(I_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n}) = T(I_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n})$,

(iii) $U_t^{(n+1)}(I_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n}) = U_t^{(n)}(I_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n})$

where $I_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n}$, $\varepsilon_i = 0, 1$, $i = 1, 2, \ldots. n$, is a dyadic set of rank n. It is obvious that $U_t^{(n)}$ converges uniformly with $n \to \infty$. Put $U_t = \lim_{n\to\infty} U_t^{(n)}$, then U_t, $0 \leqslant t \leqslant 1$, is also a continuous arc and $U_0 = E$, and $U_1 = T$.

Remark.[8] "Arc" of this proposition cannot be replaced by a "one-parameter subgroup". In fact in any small neighbourhood $N(E)$ of the identity E there exists an element T of G through which no one-parameter subgroup passes. For example, an ergodic transformation $T \in N(E)$ with a rational proper value $\neq 1$ has this property.

8) This remark is due to Mr. H. Anzai.