111. On Spaces with a Complete Structure.

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The purpose of this note is to study the problem: Is it true that every completely regular space with a complete structure is homeomorphic to a closed subset of a Cartesian product of the space of real numbers with its usual topology?

Concerning the above problem, under a restriction with respect to cardinal numbers of the spaces, an affirmative answer will be given in this note.

§ 1. Definition 1.[1] Let us call the structure of a completely regular space X with the uniformity made up of all countable normal coverings of the space X the *e-structure* of X and denote by eX. Moreover we say the space with the complete e-structure to be *e-complete* and let us call a cardinal number $\mathfrak{m}$ to be *e-complete* if the discrete space with the potency $\mathfrak{m}$ is e-complete.

Definition 2.[2] Let X be a completely regular space and let $C(X, R)$ be the set of all real-valued continuous functions with domain X. Further-more let f be a function in $C(X, R)$. Then the set of points in X for which f vanishes is said to be a *Z-set* and is denoted by $Z(f)$. Finally let $Z(X)$ be the family of all Z-sets of X. Then a subfamily $\mathfrak{A}$, of the family $Z(X)$ is said to be a *CZ-maximal family* of X if $\mathfrak{A}$ enjoys the following four conditions:

a) $\mathfrak{A}$ is not an empty family,
b) $\mathfrak{A}$ does not contain a void set,
c) $\mathfrak{A}$ never contains countable subfamilies with total intersection void, and
d) $\mathfrak{A}$ is maximal with respect to the properties a), b) and c).

§ 2. **Lemma 1.** *Every CZ-maximal family of a completely regular space X is a Cauchy family of the e-structure eX. For any Cauchy family of the e-structure eX there exists a Cauchy family such that they are equivalent.*

Lemma 2.[3] *A completely regular space is homeomorphic to a closed subset of a Cartesian product of the reals if and only if for any CZ-maximal family the total intersection is not void.*

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1) J. W. Tukey: Convergence and uniformity in topology, Princeton University press, Princeton 1940; T. Shirota: On systems of structures of a completely regular space, Osaka Math. J. **2** (1950).

2) E. Hewitt: Rings of real valued continuous functions, Trans. Amer. Math. Soc. **64** (1948).

3) E. Hewitt: loc. cit, 2).

From Lemma 1 and 2 we have

Theorem 1. *A completely regular space X is e-complete if and only if it is h meomorphic to a closed subset of a Cartesian product of the reals.*

§ 3. Cardinal numbers and discrete spaces.

Lemma 3. *If two cardinal numbers* $\mathfrak{m}$ and $\mathfrak{n}$ *are e-complete, then* $\mathfrak{n}^{\mathfrak{m}}$ *is also e-complete. If every cardinal number less than a given cardinalnumber* $\mathfrak{m}$ *is e-complete and if* $\mathfrak{m}$ *is represented by the sum* $\sum_{\alpha\in A}\mathfrak{m}_\alpha$ *where the cardinal number* $|A|$ *of* A *is less than* $\mathfrak{m}$ *and* $\mathfrak{m}_\alpha < \mathfrak{m}$, *then* $\mathfrak{m}$ *is also e-complete.*

The above lemma can be proved by the method used by Ulam.[4]

§ 4. **Lemma 4.** *Let $\{\mathfrak{U}_n \mid n = 1, 2, 3, \dots\}$ be a normal sequence of open coverings of a completely regular space X and let $\mathfrak{U}_1 = \{U_\alpha \mid A\}$. Then there exists a family $\{E_{\beta_m} \mid \beta_m \in B_m \;\&\; m = 1, 2, \dots\}$ of subsets of X satisfying the following conditions:*

i) $\{E_{\beta_m} \mid \beta_m \in B_m \;\&\; m = 1, 2, 3, \dots\}$ *is a closed covering of X,*

ii) $E_{\beta_m^{(1)}} \neq E_{\beta_m^{(2)}}$ *for* $\beta_m^{(1)} \neq \beta_m^{(2)}$,

iii) E_{β_m} *is not void,*

iv) *every element of* $\mathfrak{U}_{m+3}$ *does not intersect two element of* $\{E_{\beta_m} \mid \beta_m \in B_m\}$ *at the same time,*

and

v) $s(E_{\beta_m}, \mathfrak{U}_{m+3}) \subset U_\beta$,

where B_m is a subset of the set A.

This lemma is due to H. A. Stone.[5]

Lemma 5. *Let X and Y be two completely regular spaces and let ϕ be a continuous mapping of X into Y. Then if $\mathfrak{A}$ is a CZ-maximal family of X, the subfamily of $Z(Y)$:*

$$\mathfrak{A}' = \{Z' \mid Z' \supset \phi(Z) \text{ for some } Z \in \mathfrak{A} \;\&\; Z' \in Z(Y)\}$$

is a CZ-maximal family of Y.

Lemma 6. *Let $\mathfrak{A}$ be a CZ-maximal family of a completely regular space X and let f be a function in $C(X, R)$ which is not constant and not negative such that $F_0 = Z(f)$. Then if $F_1 = \{x \mid f(x) \leq a\}$ where $a > 0$, the subfamily of $Z(F_1)$:*

$$\mathfrak{A}' = \{Z(g) \mid g \in C(F_1, R) \;\&\; Z(g) \supset Z \frown F_0 \neq 0 \text{ for some } Z \in \mathfrak{A}\}$$

is a CZ-maximal family of F_1.

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4) S. Ulam: Zur Masstheorie in der allgemeinen Mengenlehre, Fund. Math. **16** (1930).

5) A. H. Stone: Paracompactness and product space, Bull. Amer. Math. Soc. **54** (1948).

By virtue of Lemmas 4, 5 and 6 we have

Theorem 2. *Let X be a completely regular space whose cardinal number $|X|$ is e-comple'e. Then if there exists a complete structure over X, X is e-complete.*

Proof. Suppose that X admits a complete structure gX with the uniformity $\{\mathfrak{U}_\delta | D\}$. Let $\mathfrak{A}$ be a CZ-maximal family of X. Moreover let $\mathfrak{U} = \{U_\alpha | A\}$ be an arbitrary normal covering in $\{\mathfrak{U}_\delta | D\}$. Then there exists a normal sequence $\{\mathfrak{U}_n | n = 1, 2, 3, \dots\}$ such that

$$\mathfrak{U} >^* \mathfrak{U}_1 >^* \mathfrak{U}_2 >^* \cdots >^* \mathfrak{U}_n >^* \cdots .$$

According to Lemma 4 there exists a closed covering of X $\{E_{\beta_m} | B_m \subset A \ \& \ m = 1, 2, 3, \dots\}$ such that it satisfies of the conditions of this lemma. Let $F_m = \sum_{\beta_m \in B_m} E_{\beta_m}$. Then $\{F_m | m = 1, 2, 3, \dots\}$ is a closed covering. Since $\mathfrak{A}$ admits the condition c) of § 1, there exists a set $F_n \in \{F_m\}$ such that F_n is compatible with $\mathfrak{A}$. Let f be a continuous function such that $f(x) = 0$ for $x \in F_n$ and $f(x) = 2$ for $x \notin S(F_n, \mathfrak{U}_{n+5})$; moreover, let $Z_0 = \{x | f(x) \leqq 0\}$ and let $Z_1 = \{x | f(x) \leqq 1\}$. Then since $Z_0 \supset F_n$, $Z_0 \in \mathfrak{A}$ and by Lemma 6, the family

$$\mathfrak{A}' = \{Z(g) | g \in C(Z_1, R) \ \& \ Z(g) \supset Z \frown Z_0 \neq 0 \text{ for some } Z \in \mathfrak{A}\}$$

is a CZ-maximal family of Z_1.

Now, it is easy to show that $Z_1 = \sum_{\beta_n \in B_n} Z_{\beta_n}$, where $S(E_{\beta_n}, \mathfrak{U}_{n+5}) \supset Z_{\beta_n} \supset E_{\beta_n}$. Hence for two different indices α_n and β_n belonging to B_n, $S(Z_{\beta_n}, \mathfrak{U}_{n+5}) \frown S(Z_{\alpha_n}, \mathfrak{U}_{n+5}) \subset S(E_{\beta_n}, \mathfrak{U}_{n+4}) \frown S(E_{\alpha_n}, \mathfrak{U}_{n+4}) = 0$. Hence the mapping ϕ of Z_1 onto the discrete space B_n such that if $x \in Z_{\beta_n}$, $\phi(x) = \beta_n$, is continuous, and therefore by Lemma 5 the family of subsets of B_n

$$\mathfrak{A}'' = \{C | B_n \geqq C \geqq \phi(Z') \quad \text{for some} \quad Z' \in \mathfrak{A}'\}$$

is a CZ-maximal family of B_n. Since $|B_n| \leqq |A| \leqq |X|$ and since $|X|$ is e-complete, B_n is e-complete. Hence there exists a $\beta_n \in B_n$ such that $\{\beta_n\}$ is the total intersection of $\mathfrak{A}''$, i.e., $\beta_n \in \phi(Z')$ for any $Z' \in \mathfrak{A}'$. Then it is easy to see that for any $Z \in \mathfrak{A}$ $Z \frown Z_{\beta_n} \neq 0$, hence $Z_{\beta_n} \in \mathfrak{A}$. Moreover $Z_{\beta_n} \subset S(E_{\beta_n}, \mathfrak{U}_{n+5}) \subset S(E_{\beta_n}, \mathfrak{U}_{n+3}) \subset U_\beta \in \mathfrak{U}$ by the condition v) of Lemma 5. Thus we see that for any $\mathfrak{U}_\delta \in \{\mathfrak{U}_\delta | D\}$ there exists a $Z \in \mathfrak{A}$ such that $Z \subset U \in \mathfrak{U}_\delta$. This fact is equivalent to the statement that $\mathfrak{A}$ is a Cauchy family of gX.

By the assumption of our theorem gX is complete, hence there exists a limit point x of the Cauchy family $\mathfrak{A}$ of gX, i.e., x is the total intersection of $\mathfrak{A}$. Since $\mathfrak{A}$ is an arbitrary CZ-maximal family, X is e-complete by Lemma 1. Thus the proof is complete.

From Theorem 2 we have immediately the following

Theorem 3. *Let X be a fully normal T_1-space. Then if $|X|$ is an e-complete cardinal number, X is e-complete.*

For, any fully normal space admits a complete structure[6].

§ 5. From Theorem 1, Theorem 2 and Lemma 3 we obtain the following two theorems.

Theorem 4. *The following three statements are equivalent:*

a) *every completely regular space with complete structure is homeomorphic to a closed subset of a Cartesian product of the reals,*

b) *every cardinal number is e-complete,*

c) *every discrete space admits no measure completely additive on all subsets, vanishing for every point, assuming only value* 0 *and* 1 *and equal to* 1 *for the whole space*[7].

Theorem 5. *For spaces X whose cardinal numbers $|X|$ are weakly accessible from $\aleph_0$*[8] *in A. Tarski's sense,* i.e., $|X| \leqq 2^{\aleph_0}$, $\leqq \mathfrak{f}$ or $\leqq 2^{\mathfrak{f}}$ *etc., the following conditions are equivalent:*

a) *X is homeomorphic to a closed subset of a Cartesian product of the reals,*

b) *there exists a complete structure over X,*

c) *X is e-complete.*

The other properties of the e-complete space and the properties of $C(X, R)$ as well as the full proofs of the above theorems will be given in the Osaka Mathematical Journal.

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6) This was proved by the author in 1948.

7) S. Ulam: loc. cit., 4); E, Hewitt: Linear functionals on spaces of continuous functions, Fund. Math. **37** (1950).

8) A. Tarski: Über unerreichbare Kardinalzahlen, Fund. Math. **30** (1938); A. Tarski: Drei Überdekungssätze der allgemeinen Mengenlehre, Fund. Math. **30** (1938).