26. *Probability-theoretic Investigations on Inheritance. VII$_2$. Non-Paternity Problems.*

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2. General formulae on probabilities of proving non-paternity. We now enter into our main discourse. Let us consider, as usual, an inherited character consisting of m allelomorphic genes A_i ($i=1, \dots, m$) with an equilibrium distribution given by (1.1). Though the case of mixed mother-child combination is rather general, we first treat, as a model, that of pure one; the former will be discussed in a subsequent section.

In general, we denote by

$$V(ij;\ hk) \tag{2.1}$$

the probability of proving non-paternity of a putative father, chosen at random with respect to type, against a given pair of a mother A_{ij} and her child A_{hk}. Among such quantities, only those are significant in which h or k coincides with at least one of i and j; otherwise, they may be regarded, according to impossibility of mother-child combinations, as to be equal to unity, but such a convention will become really a matter of indifference in the following lines. Let us again, as in (1.1) of IV, denote by $\pi(ij;\ hk)$ the probability of appearing of such a mother-child combination. The probability of the composed event that such a combination arises and then the proof of non-paternity can be established, is thus given by the product

$$P(ij;\ hk)=\pi(ij;\ hk)V(ij;\ hk). \tag{2.2}$$

It vanishes unless h or k coincides with at least one of i and j, regardless of the determination of value of (2.1), since then $\pi(ij;\ hk)$ so does.

If we sum up the probabilities $P(ij;\ hk)$ over all possible types A_{hk} of children, then we get the *sub-probability* of proving non-paternity against the type A_{ij} of mother, which will be denoted by

$$P(ij)=\sum_{h,k}P(ij;\ hk). \tag{2.3}$$

The probability of proving non-paternity against a fixed mother of type A_{ij} is then given by

$$P(ij)\,/\,\bar{A}_{ij}. \tag{2.4}$$

If we further sum up the probabilities $P(ij)$ over all types A_{ij} of mothers, we get the *whole probability* of proving non-paternity which will be denoted by P; i.e.,

$$P=\sum_{i,j} P(ij)=\sum_{i,j,h,k} P(ij;\ hk), \tag{2.5}$$

both summations extending over all possible respective sets of suffices.

In order to determine the value of (2.1) in an explicit form, we begin with the case of mother-child combination $(A_{ii};\ A_{ii})$. Then, anyone of a type not containing the gene A_i, i.e., of any type among $A_{hk}\ (h,k\neq i)$ can deny to be a true father. Hence, we obtain

$$\begin{aligned} V(ii;\ ii)&=\sum_{\substack{h,k\neq i\\ h\leq k}} \bar{A}_{hk}=\sum_{\substack{h=1\\ h\neq i}}^{m} \bar{A}_{hh}+\sum_{\substack{h,k\neq i\\ h<k}} \bar{A}_{hk} \\ &=\sum_{h\neq i} p_h^2+\sum_{\substack{h,k\neq i\\ h<k}} 2p_hp_k=\sum_{h,k\neq i} p_hp_k=(1-p_i)^2. \end{aligned} \tag{2.6}$$

The same result may also be derived by considering the complementary probability of the event that a type contains at least one gene A_i; in fact, we thus get again

$$V(ii;\ ii)=1-(p_i^2+\sum_{h\neq i} 2p_ip_h)=(1-p_i)^2.$$

Next, given a mother-child combination $(A_{ii};\ A_{ij})\ (j\neq i)$, the types $A_{hk}\ (h,k\neq j)$ are deniable, and hence we obtain

$$V(ii;\ ij)=\sum_{h,k\neq j} p_hp_k=(1-p_j)^2 \qquad (j\neq i); \tag{2.7}$$

the consideration of a complementary probability will, of course, lead also to the same result. In similar manners, we obtain the following results:

$$V(ij;\ ii)=\sum_{h,k\neq i} p_hp_k=(1-p_i)^2 \qquad (j\neq i), \tag{2.8}$$

$$V(ij;\ ij)=\sum_{h,k\neq i,j} p_hp_k=(1-p_i-p_j)^2 \qquad (j\neq i), \tag{2.9}$$

$$V(ij;\ ih)=\sum_{k,l\neq h} p_kp_l=(1-p_h)^2 \qquad (j,h\neq i;\ h\neq j). \tag{2.10}$$

The comparison of (2.6) with (2.7) and with (2.8) shows that the last two results remain valid also in case $j=i$. In particular, for a child A_{ii}, the probability in question is always equal to $(1-p_i)^2$ regardless of the types of mother. Further, as seen from (2.8) and (2.10), the result (2.10) remains valid also for $h=i$. In spite of such reducibilities, we write these probabilities separately, constructing the following table.

Mother \ Child	A_{ii}	A_{ij} $(j\neq i)$
A_{ii}	$(1-p_i)^2$	$(1-p_j)^2$

Mother \ Child	A_{ii}	A_{ij}	A_{ih} $(h\neq i, j)$
A_{ij} $(i\neq j)$	$(1-p_i)^2$	$(1-p_i-p_j)^2$	$(1-p_h)^2$

The quantities $\pi(ij;hk)$ having been already evaluated in § 1 of IV, the sub-probability of proving non-paternity, given in (2.2), against every mother-child combination can immediately be obtained. We get, for instance,

$$\begin{aligned} P(ii;ii)&=\pi(ii;ii)\,V(ii;ii)=p_i^3(1-p_i)^2, \\ P(ii;ij)&=\pi(ii;ij)\,V(ii;ij)=p_i^2p_j(1-p_j)^2 \quad (i\neq j). \end{aligned} \tag{2.11}$$

We now calculate the sub-probabilities defined in (2.3). First, for homozygotic mother A_{ii}, we have

$$P(ii)=P(ii;ii)+\sum_{j\neq i}P(ii;ij). \tag{2.12}$$

In view of the second relation (2.11), we get

$$\begin{aligned} \sum_{j\neq i}P(ii;ij)&=p_i^2\sum_{j\neq i}p_j(1-p_j)^2=p_i^2\Big(\sum_{j=1}^{m}p_j(1-p_j)^2-p_i(1-p_i)^2\Big) \\ &=p_i^2(1-2S_2+S_3-p_i(1-p_i)^2), \end{aligned} \tag{2.13}$$

where the notation for power-sum defined in (1.2) has been used. Thus, remembering also the first relation (2.11), we get, for (2.12),

$$P(ii)=p_i^2(1-2S_2+S_3). \tag{2.14}$$

Next, for heterozygotic mother $A_{ij}\,(i\neq j)$, we have

$$\begin{aligned} P(ij)=&P(ij;ii)+P(ij;jj)+P(ij;ij) \\ &+\sum_{h\neq i,\,j}(P(ij;ih)+P(ij;jh)). \end{aligned} \tag{2.15}$$

From the results on π's and V's, we get

$$\begin{aligned} &P(ij;ii)+P(ij;jj)+P(ij;ij) \\ &=p_ip_j(p_i(1-p_i)^2+p_j(1-p_j)^2+(p_i+p_j)(1-p_i-p_j)^2), \end{aligned} \tag{2.16}$$

$$\begin{aligned} &\sum_{k,\,k\neq i}(P(ij;ih)+P(ij;jh))=2p_ip_j\sum_{h\neq i,\,j}p_h(1-p_h)^2 \\ &=2p_ip_j\Big(\sum_{h=1}^{m}p_h(1-p_h)^2-p_i(1-p_i)^2-p_j(1-p_j)^2\Big) \\ &=2p_ip_j(1-2S_2+S_3-p_i(1-p_i)^2-p_j(1-p_j)^2), \end{aligned} \tag{2.17}$$

whence it follows

$$P(ij)=p_ip_j(2(1-2S_2+S_3)-4p_ip_j+3p_ip_j(p_i+p_j)) \quad (i\neq j). \tag{2.18}$$

The sub-probabilities having been thus obtained in (2.14) and (2.18), the whole probability will be calculated by means of (2.5). For that purpose, we now introduce a conventional notation defined by

$$\begin{cases} P(\mathring{ii})=[P(ij)]^{p_j=p_i}, \\ P(\mathring{ij})=P(ij) \end{cases} \qquad (j\neq i). \tag{2.19}$$

It should be noticed that, in general, $P(\mathring{ii})\neq P(ii)$, that is to say, $P(ij)\,(i\neq j)$ does not simply reduce to $P(ii)$, by putting $p_j=p_i$, as seen from (2.14) and (2.18). Now, making use of the convention defined in (1.5), we get, in view of general formula given by (1.7),

$$P=\sum_{i=1}^{m} P(ii)+\sum_{i,j}{}' P(ij)=\sum_{i=1}^{m} P(ii)+\frac{1}{2}\Big(\sum_{i,j=1}^{m} P(\overset{\circ}{ij})-\sum_{i=1}^{m} P(\overset{\circ}{ii})\Big)$$

Consequently, we get further, by means of (2.14) and (2.15),

$$\begin{aligned}P&=S_2(1-2S_2+S_3)+\sum_{i,j=1}^{m} p^i p_j\Big((1-2S_2+S_3)-2p_ip_j+\frac{3}{2}p_ip_j(p_i+p_j)\Big)\\&\qquad-\sum_{i=1}^{m} p_i^2((1-2S_2+S_3)-2p_i^2+3p_i^3)\\&=S_2(1-2S_2+S_3)+1-2S_2+S_3-2S_2^2+3S_2S_3\\&\qquad-(S_2(1-2S_2+S_3)-2S_4+3S_5),\end{aligned}$$

whence it follows the desired expression for the *whole probability*, stating that

$$(2.20)\qquad P=1-2S_2+S_3-2S_2^2+2S_4+3S_2S_3-3S_5.$$

Cf. also a later paper VIII.

—*To be continued*—