84. Probability-theoretic Investigations on Inheritance. XII_2. Probability of Paternity.

By Yûsaku KOMATU.
Department of Mathematics, Tokyo Institute of Technology and
Department of Legal Medicine, Tokyo Medical and Dental University.

(Comm. by T. FURUHATA, M.J.A., July 12, 1952.)

3. Paternity on two-children family.

Similar problems as above will also be discussed with respect to two-children family. Let a fixed mother-children combination $(A_{ij}; A_{hk}, A_{fg})$ be given, and C_1 be a cause that a presented man is really a father of the children and C_2 be another cause that he is not their father. We suppose here again that the probabilities a priori of these mutually exclusive causes are both equal to 1/2. The probability of an event that, under the cause C_1, a mating $A_{ab} \times A_{ij}$ produces the children A_{hk} and A_{fg} has already outlined in § 3 of IV, which will be denoted by

$$\lambda(ab,\ ij;\ hk,\ fg). \tag{3.1}$$

On the other hand, under the cause C_2, a mother A_{ij}, together with a common father, produces children A_{hk} and A_{fg} with the probability

$$\pi(ij; hk, fg)/\bar{A}_{ij}. \tag{3.2}$$

Hence, in view of the Bayes' theorem, for a given mother-children combination $(A_{ij}; A_{hk}, A_{fg})$, the probability a posteriori of a man A_{ab} to be a true father, i. e., his *probability of paternity*, is expressed by

$$\Lambda(ij; hk, fg; ab) = \frac{\lambda(ab,\ ij; hk, fg)}{\lambda(ab,\ ij; hk, fg) + \pi(ij; hk, fg)/\bar{A}_{ij}}. \tag{3.3}$$

The value of the last expression is determined for every possible quadruple as follows; different letters indicating different genes.

$$\begin{aligned}
&\Lambda(ii;\ ii,\ ii;\ ii) = \frac{2}{2+p_i(1+p_i)}, && \Lambda(ii;\ ii,\ ii;\ ih) = \frac{1}{1+2p_i(1+p_i)};\\
&\Lambda(ii;\ ii,\ ih; ih) = \frac{1}{1+2p_ip_h}; && \Lambda(ii; ih,\ ih; hh) = \frac{2}{2+p_h(1+p_h)},\\
&\Lambda(ii; ih,\ ih; ih) = \frac{1}{1+2p_h(1+p_h)}, && \Lambda(ii; ih,\ ih; hk) = \frac{1}{1+2p_h(1+p_h)};\\
&\Lambda(ii; ih,\ ik; hk) = \frac{1}{1+2p_hp_k};
\end{aligned} \tag{3.4}$$

$$\Lambda(ij;\, ii,\, ii;\, ii)=\frac{2}{2+p_i(1+p_i)},\quad \Lambda(ij;\, ii,\, ii;\, ij)=\frac{1}{1+2p_i(1+p_i)},$$

$$\Lambda(ij;\, ii,\, ii;\, ih)=\frac{1}{1+2p_i(1+p_i)};\quad \Lambda(ij;\, ii,\, jj;\, ij)=\frac{1}{1+2p_ip_j};$$

$$\Lambda(ij;\, ii,\, ij;\, ii)=\frac{2}{2+p_i(1+p_i+p_j)},$$

$$\Lambda(ij;\, ii,\, ij;\, ij)=\frac{1}{1+p_i(1+p_i+p_j)},$$

$$\Lambda(ij;\, ii,\, ij;\, ih)=\frac{1}{1+2p_i(1+p_i+p_j)};$$

$$\Lambda(ij;\, ii,\, ih;\, ih)=\frac{1}{1+2p_ip_h};$$

$$\Lambda(ij;\, ii,\, jh;\, ih)=\frac{1}{1+2p_ip_h};$$

$$\Lambda(ij;\, ij,\, ij;\, ii)=\frac{2}{2+(p_i+p_j)\,(1+p_i+p_j)},$$

$$\Lambda(ij;\, ij,\, ij;\, ij)=\frac{2}{2+(p_i+p_j)\,(1+p_i+p_j)},$$

$$\Lambda(ij;\, ij,\, ij;\, ih)=\frac{1}{1+2(p_i+p_j)(1+p_i+p_j)}; \tag{3.5}$$

$$\Lambda(ij;\, ij,\, ih;\, ih)=\frac{1}{1+2p_h(p_i+p_j)}$$

$$\Lambda(ij;\, ij,\, ih;\, jh)=\frac{1}{1+2p_h(p_i+p_j)};$$

$$\Lambda(ij;\, ih,\, ih;\, ih)=\frac{1}{1+2p_h(1+p_h)},$$

$$\Lambda(ij;\, ih,\, ih;\, jh)=\frac{1}{1+2p_h(1+p_h)},$$

$$\Lambda(ij;\, ih,\, ih;\, hh)=\frac{1}{1+2p_h(1+p_h)},$$

$$\Lambda(ij;\, ih,\, ih;\, hk)=\frac{1}{1+2p_h(1+p_h)};$$

$$\Lambda(ij;\, ih,\, ik;\, hk)=\frac{2}{2+p_hp_k};$$

$$\Lambda(ij;\, ih,\, jh;\, ih)=\frac{1}{1+2p_h(1+p_h)},$$

$$\Lambda(ij;\, ih,\, jh;\, hk)=\frac{1}{1+2p_h(1+p_h)};$$

$$\Lambda(ij;\, ih,\, jk;\, hk)=\frac{1}{1+2p_hp_k}.$$

Thus, remembering an evident symmetry relation

$$(3.6) \qquad \Lambda(ij; hk, fg; ab) = \Lambda(ij; fg, hk; ab),$$

all the possible quadruples have been essentially exhausted except those with identically vanishing probability.

Similar results based upon phenotypes can be derived also in case where recessive genes are existent. The details in respective cases will here be omitted and left to the reader.

4. Paternity without regard to type of mother.

In preceding discussions on paternity, the type of mother has also been taken into account. On the other hand, corresponding to the problems on absolute non-paternity discussed in XI, we may consider the problems to determine the probabilities of paternity without regard to type of mother, when, against given child or children, a man cannot establish his absolute non-paternity.

Let a child be given and a man be presented. Two mutually exclusive causes are considered. Namely, the one is that the man is really a father of the child and the other is that his is not its father, which will be denoted by C_1 and C_2, respectively. Probabilities a priori of those causes will again be supposed to be equal to 1/2. Probability of an event, that, under the cause C_1, a man A_{ab} produces a child A_{hk} is expressed by

$$(4.1) \qquad \pi(ab;\ hk)/\bar{A}_{ab}.$$

In fact, while the quantity $\pi(ab; hk)$ has originally been introduced as the probability of mother-child combination, it remains to maintain its validity also as that of father-child combination. On the other hand, under the cause C_2, a child A_{hk} will be regarded to appear with a general probability of population, i. e.,

$$(4.2) \qquad \bar{A}_{hk}.$$

Hence, again in view of the Bayes' theorem, the probability a posteriori of a man A_{ab} to be a true father of a child A_{hk}, i. e., his *probability of paternity* is expressed by

$$(4.3) \qquad \Theta(hk;\ ab) = \frac{\pi(ab; hk)/\bar{A}_{ab}}{\pi(ab; hk)/\bar{A}_{ab} + \bar{A}_{hk}} = \frac{\pi(ab; hk)}{\pi(ab; hk) + \bar{A}_{ab}\bar{A}_{hk}}.$$

The following results will be obtained by actual calculations:

$$(4.4) \qquad \Theta(hh; hh) = \frac{p_h}{p_h + p_h^2} = \frac{1}{1 + p_h},$$

$$(4.5) \qquad \Theta(hh; hk) = \frac{p_h/2}{p_h/2 + p_h^2} = \frac{1}{1 + 2p_h} \qquad (k \neq h);$$

$$(4.6) \qquad \Theta(hk; hk) = \frac{(p_h + p_k)/2}{(p_h + p_k)/2 + 2p_h p_k} = \frac{p_h + p_k}{p_h + p_k + 4p_h p_k} \qquad (h \neq k),$$

(4.7) $\Theta(hk;\ kl)=\dfrac{p_k/2}{p_k/2+2p_hp_k}=\dfrac{1}{1+4p_h}$ $(h\neq k;\ l\neq h,\ k)$.

A symmetry character of π's, stated in (1.21) of IV, implies immediately that of Θ's; namely, a relation

(4.8) $\Theta(hk;ab)=\Theta(ab;hk)$

holds identically. Remembering this, all the possible pairs have thus been essentially exhausted. For instance, we see $\Theta(hk;\ kk)=1/(1+2p_k)$ $(h\neq k)$ from (4.5) and (4.8).

In respective cases of concrete blood types where recessive genes may exist, analogous results will be obtained. In fact, we have only to replace probability of father-child combination and general probability of population in above argument by the corresponding ones. Thus, the following tables will be constructed.

Child \ Man	O	A	B	AB
O	$\frac{1}{1+r}$	$\frac{1}{1+p+2r}$	$\frac{1}{1+q+2r}$	0
A	$\frac{1}{1+p+2r}$	$\frac{p^2+3pr+r^2}{p^2+3pr+r^2+p(p+2r)^2}$	$\frac{r}{r+(q+2r)(p+2r)}$	$\frac{p+r}{p+r+2p(p+2r)}$
B	$\frac{1}{1+q+2r}$	$\frac{r}{r+(p+2r)(q+2r)}$	$\frac{q^2+3qr+r^2}{q^2+3qr+r^2+q(q+2r)^2}$	$\frac{q+r}{q+r+2q(q+2r)}$
AB	0	$\frac{p+r}{p+r+2p(p+2r)}$	$\frac{q+r}{q+r+2q(q+2r)}$	$\frac{p+q}{p+q+4pq}$

Child \ Man	Q	q
Q	$\frac{1+uv}{1+u+3uv+uv^2}$	$\frac{1}{2+v}$
q	$\frac{1}{2+v}$	$\frac{1}{1+v}$

Child \ Man	Q	q_-	q_+
Q	$\frac{1+uv}{1+u+3uv+uv^2}$	$\frac{1}{2+v}$	$\frac{1}{2+v}$
q_-	$\frac{1}{2+v}$	$\frac{v^2+v_1v_2}{v^2+v_1v_2+v_1(v+v_2)^2}$	$\frac{1}{1+v+v_2}$
q_+	$\frac{1}{2+v}$	$\frac{1}{1+v+v_2}$	$\frac{1}{1+v_2}$

Let now two children A_{hk}, A_{fg} of the same family be given and a man A_{ab} be presented. Then, the probability a posteriori of the man to be a true father of the brethren-combination will be required. Similar argument as above will be valid here also. We have only to replace (4.1) and (4.2) by

(4.9) $\pi(ab;hk,fg)/\bar{A}_{ab}$,

(4.10) $\sigma(hk,fg)$,

respectively; $\sigma(hk,fg)$ denoting, of course, the brethren combination introduced in (1.1) of V. Hence, the Bayes' theorem implies the required probability a posteriori being expressed by

$$(4.11) \qquad \Theta(hk, fg; ab) = \frac{\pi(ab; hk, fg)/\bar{A}_{ab}}{\pi(ab; hk, fg)/\bar{A}_{ab} + \sigma(hk, fg)}.$$

Actual calculations will show the following results; different letters being supposed to denote different genes.

$$(4.12) \qquad \begin{aligned} &\Theta(ii,\ ii;\ ii) = \frac{2}{2+p_i(1+p_i)}, \quad \Theta(ii,\ ii;\ ih) = \frac{1}{1+2p_i(1+p_i)};\\ &\Theta(ii,\ jj;\ ij) = \frac{1}{1+2p_ip_j}, \quad \Theta(ii,\ ij;\ ii) = \frac{1}{1+p_i(1+p_i)},\\ &\Theta(ii,\ ij;\ ij) = \frac{1+p_i+p_j}{1+p_i+p_j+4p_ip_j(1+p_i)}, \quad \Theta(ii, ij; ih) = \frac{1}{1+4p_i(1+p_i)};\\ &\Theta(ii, jh; ij) = \frac{1}{1+4p_ip_j}, \quad \Theta(ii, jh; ih) = \frac{1}{1+4p_ip_h}; \end{aligned}$$

$$\Theta(ij,\ ij; ii) = \frac{1+p_j}{1+p_j+p_i(1+p_i+p_j+2p_ip_j)},$$

$$\Theta(ij,\ ij;\ ij) = \frac{(p_i+p_j)(1+p_i+p_j)}{(p_i+p_j)(1+p_i+p_j)+4p_ip_j(1+p_i+p_j+2p_ip_j)},$$

$$\Theta(ij, ij; ih) = \frac{1+p_j}{1+p_j+4p_i(1+p_i+p_j+2p_ip_j)};$$

$$\Theta(ij\ ih;\ ii) = \frac{1}{1+p_i(1+2p_i)},$$

$$(4.13) \quad \Theta(ij, ih; ij) = \frac{p_i+p_j}{p_i+p_j+4p_ip_j(1+2p_i)}$$

$$\Theta(ij, ih; ih) = \frac{p_i+p_h}{p_i+p_h+4p_ip_h(1+2p_i)},$$

$$\Theta(ij, ih; jh) = \frac{1+p_i}{1+p_i 4p+_j p_h(1+2p_i)}, \quad \Theta(ij, ih; ik) = \frac{1}{1+4p_i(1+2p_i)};$$

$$\Theta(ij, hk; ih) = \frac{1}{1+8p_ip_h}.$$

A symmetry relation

$$(4.14) \qquad \Theta(hk, fg; ab) = \Theta(fg,\ hk;\ ab)$$

follows immediately from (3.4) of IV and (1.3) of V. Thus, the results (4.12) and (4.13) exhaust all the possible triples essentially.

Similar results based upon phenotypes can also be derived, but the details will be left to the reader.

In conclusion, it would also be noticed that the problems and the results throughout the present chapter may be generalized to case of mixed combinations by the corresponding modifications.