65. On Infinite-dimensional Representations of Semi-simple Lie Algebras and Some Functionals on the Universal Enveloping Algebras. I

By Shôichirô SAKAI

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During the past few years, Harish-Chandra[4,5,6] has obtained the very important results on the representations of semi-simple Lie groups on Banach spaces, and R. Godement[3] has obtained elementary and elegant proofs for some of them with many new results. Let G be a connected semi-simple Lie group, $\mathfrak{G}_0$ the Lie algebra of G, and G_0 the adjoint group of $\mathfrak{G}_0$. Then it is well-known that G_0 has the form $K_0\cdot S_0$, where K_0 is a maximal compact subgroup and S_0 a solvable subgroup and $K_0 \cap S_0=(e)$. Let K be the inverse image of K_0 in G, S some solvable subgroup of G isomorphic to S_0 with $G=K\cdot S$. In his theory of spherical functions, Godement essentially assumed the compactness of K, and has shown that there is a one-to-one correspondence between irreducible unitary representations of G and finite dimensional irreducible unitary representations of his algebra $L^c(d)$ (unpublished); this result is more useful for the determination of all irreducible unitary representations of G than the corresponding result due to Chandra. However, as K is in general the direct product of a compact subgroup and a vector subgroup, it is desirable to find a way which makes the Godement's restriction stated above unnecessary. The object of this paper is to extend the considerations of the author to the semi-simple Lie algebra and to make it adequate for this requirement.

Let $\mathfrak{G}_0$ be a real Lie ring, $\mathfrak{k}_0$ be a subring of $\mathfrak{G}_0$ and G_0 be the adjoint group[1] of $\mathfrak{G}_0$, K_0 be the analytic subgroup[1] of G_0 corresponding to K_0.[17] We shall assume that K_0 is compact. Let $\mathfrak{G}$ and $\mathfrak{k}$ be the complexification of $\mathfrak{G}_0$ and $\mathfrak{k}_0$ respectively, and $U(\mathfrak{G})$ and $U(\mathfrak{k})$ be the universal enveloping algebra of $\mathfrak{G}$ and $\mathfrak{k}$ respectively, then $U(\mathfrak{k})$ can be considered as a subalgebra of $U(\mathfrak{G})$.

Since the elements of G_0 are automorphisms of $\mathfrak{G}_0$, they are uniquely extended to automorphisms of $U(\mathfrak{G})$, which we shall denote by $\alpha \rightarrow \varepsilon_x \alpha \varepsilon_{x^{-1}} = \mathrm{ad}\,(x)\alpha (x \in G_0)$, and call the correspondence $x \rightarrow \mathrm{ad}\,(x)$ the adjoint representation of G_0.[10]

Let Ω be all equivalence classess of finite-dimensional irreducible representations of the ring $\mathfrak{k}_0$, then Ω can be considered to be all equivalence classes of finite-dimensional irreducible representations of $U(\mathfrak{k})$. Let Ω' be the sub-family composed of all elements which induce the representations of the group K_0. We identify the element of Ω' with the equivalence class of irreducible representations of the group K_0.

Since K_0 is compact, $\mathfrak{G}=\sum_{\tilde{d}\in\Omega'}\mathfrak{G}(\tilde{d})$[3] and so, from the theory of the Kronecker product of representations, we can easily show that $U(\mathfrak{G})=\sum_{\tilde{d}\in\Omega'} U(\mathfrak{G})(\tilde{d})$ where $\sum$ denotes the direct sum.

Let $\tilde{d}_0$ be the identity representation and put $U(\mathfrak{G})(\tilde{d}_0)=U^o(\mathfrak{G})$, then $U^o(\mathfrak{G})$ is the subalgebra of $U(\mathfrak{G})$ of all elements which commute with $U(\mathfrak{k})$.

If $\alpha=\alpha^o+\sum\alpha_i\ (\alpha^o\in U^o(\mathfrak{G}),\ \alpha_i\in U(\mathfrak{G})(\tilde{d}_i))$, then the mapping $\alpha\to\alpha^o$ is an idempotent operator from $U(\mathfrak{G})$ on $U^o(\mathfrak{G})$, which satisfies the following relation.

(i) $(\alpha^o\beta)^o=\alpha^o\beta^o,\ (\beta\alpha^o)^o=\beta^o\alpha^o$ for $\alpha,\beta\in U(\mathfrak{G})$

(ii) $(\gamma\alpha)^o=(\alpha\gamma)^o$ for $\alpha\in U(\mathfrak{G})$ and $\gamma\in U(\mathfrak{k})$.

Moreover put $\tilde{U}=\sum_{\tilde{d}\neq d_0} U(\mathfrak{G})(\tilde{d})$, then $\tilde{U}$ consists of linear combinations of $[\gamma,\alpha]=\gamma\alpha-\alpha\gamma(\gamma\in U(\mathfrak{k}),\ \alpha\in U(\mathfrak{G}))$.[2]

Now let $x_1,\dots,x_n$ be a base of $\mathfrak{G}_0$, and define as

$$x_i^*=-x_i\ (i=1,\dots,n)\qquad (\sqrt{-1}\,x_i)^*=-\sqrt{-1}\,x_i^*.$$

Then this $*$-operation is uniquely extended to a conjugate linear anti-automorphism on $U(\mathfrak{G})$, which we shall call the adjoint operation on $U(\mathfrak{G})$. If $\alpha^*=\alpha$, we call α self-adjoint.

If $\alpha\in U(\mathfrak{G})(d)$ and the representation of $\mathfrak{k}_0$ induced on $\mathrm{ad}\,(U(\mathfrak{k}))\alpha$ is irreducible, then we have $(\varepsilon_k\alpha^*\varepsilon_{k-1})=(\varepsilon_k\alpha\varepsilon_{k-1})^*=(\sum_j m^d_{ji}(k)\alpha_j)^*=\sum_j \overline{m^d_{ji}(k)}\alpha_j^*$. Hence α^* belongs to $U(\mathfrak{G})(d^*)$, where d^* is the contragradient representation of d, therefore we have $(\alpha^*)^o=(\alpha^o)^*$.

Put $P=\{\beta|\beta=\sum_l\lambda_l\alpha_l^*\alpha_l\ \lambda_l\geqq 0,\ \alpha_l\in U(\mathfrak{G})\}$, and call the elements of P to be positive. Let $\alpha=\alpha^o+\sum\alpha_i$, then $\alpha^*\alpha=\alpha^{o*}\alpha^o+\sum_i\alpha_i^*\alpha^o+\sum_i\alpha^{o*}\alpha_i+\sum_{i,j}\alpha_i^*\alpha_j$. If $d_i\neq d_j$, $d_i^*\times d_j$ can not contain the identity representation, therefore

$$(\alpha^*\alpha)^o=\alpha^{o*}\alpha^o+\sum_i(\alpha_i^*\alpha_i)^o.$$

Since (the general form of α_i is) $\alpha_i=\sum_{p,q}\lambda^i_{pq}\beta_{pq}$, $(p,q=1,2,\dots\dim(d_i))$ where λ^i_{pq} are complex numbers and $\varepsilon_k\beta_{pq}\varepsilon_{k-1}=\sum_r m^{d_i}_{rq}(k)\beta_{pr}$ $(r=1,2,\dots\dim(d_i))$, we can easily show that

$$(\alpha_i^*\alpha_i)^o=\sum_{q,r}\{\sum_p\bar{\lambda}^i_{pq}\beta^*_{pr}/\sqrt{\dim(d_i)}\}\{\sum_p\lambda^i_{pq}\beta_{pr}/\sqrt{\dim(d_i)}\}.$$

Hence $(\alpha_i^*\alpha_i)^o$ and so $(\alpha^*\alpha)^o$ belongs to P. Therefore we have the following proposition.

Proposition 1. *$(\alpha^*)^o=(\alpha^o)^*$ and P is invariant under the O-operation.*

Definition 1. A linear functional φ on $U(\mathfrak{G})$ is called to be $\mathfrak{k}_0$-*invariant* if it satisfies the following:

$$\varphi(\gamma\alpha)=\varphi(\alpha\gamma) \quad \text{for} \quad \alpha \in U(\mathfrak{G}) \quad \text{and} \quad \gamma \in U(\mathfrak{k}).$$

Definition 2. A linear functional φ on $U(\mathfrak{G})$ is called to be *positive* if it satisfies the following:

$$\varphi(\alpha)\geqq 0 \quad \text{for} \quad \alpha \in P.$$

Definition 3. A linear subspace V of $U(\mathfrak{G})$ is called to be $\mathfrak{k}_0$-*invariant* if

$$\alpha \in V \text{ means } [x, \alpha] \in V \quad \text{for} \quad x \in \mathfrak{k}_0.$$

Proposition 1 means that in order that a $\mathfrak{k}_0$-invariant linear functional φ is positive, it is necessary and sufficient that $\varphi(\alpha)\geqq 0$ for $\alpha \in P \cap U^o(\mathfrak{G})$.

Now let $\mathfrak{M}_0$ be a left ideal of $U^o(\mathfrak{G})$ and put $\mathfrak{M}=\{\alpha|(\beta\alpha)^o \in \mathfrak{M}_0, \alpha \in U(\mathfrak{G})$ and all $\beta \in U(\mathfrak{G})\}$, then $\mathfrak{A}$ is a $\mathfrak{k}_0$-invariant left ideal of $U(\mathfrak{G})$. We obtain the following proposition.

Proposition 2. *If $\mathfrak{N}$ is a $\mathfrak{k}_0$-invariant left ideal such that $\mathfrak{N}\cap U^o(\mathfrak{G})=\mathfrak{M}_0$, then $\mathfrak{N}\subset\mathfrak{M}$.*

Proof. As $\mathfrak{M}$ is $\mathfrak{k}_0$-invariant, $\mathfrak{N}=\sum_{\tilde{d}\in\Omega'}\mathfrak{N}(\tilde{d})$. If $\alpha \in \mathfrak{N}$ and $\alpha \bar{\in} \mathfrak{M}$, there exists an element $\beta(\in U(\mathfrak{G}))$ such that $(\beta\alpha)^o \bar{\in} \mathfrak{M}_0$. However $(\beta\alpha)^o \in \mathfrak{N}$. This contradicts the assumption.

In particular, if $\mathfrak{k}_0=\mathfrak{G}_0$, then $U^o(\mathfrak{k})$ is the center of $U(\mathfrak{k})$, and any two-sided ideal of $U(\mathfrak{k})$ is $\mathfrak{k}_0$-invariant. Moreover in this case if $\mathfrak{M}_0$ is an ideal of $U^o(\mathfrak{k})$, $\mathfrak{M}$ is also an ideal of $U(\mathfrak{k})$, so that if $\mathfrak{M}_0$ is a maximal ideal of $U^o(\mathfrak{k})$, $\mathfrak{M}$ is maximal. Therefore we have the following proposition, which is to be valid for any semi-simple Lie algebra.

Proposition 3. *If $\mathfrak{M}$ is a maximal ideal of $U(\mathfrak{k})$, then $\mathfrak{M}\cap U^o(\mathfrak{k})$ is a maximal ideal of $U^o(\mathfrak{k})$ and the mapping $\mathfrak{M}\rightarrow\mathfrak{M}\cap U^o(\mathfrak{k})$ is the one-to-one correspondence between the maximal ideals of $U(\mathfrak{k})$ and the maximal ideals of $U^o(\mathfrak{k})$.*

Now let $\mathfrak{G}_0$ and G_0 be the real Lie ring at the beginning and its adjoint group and suppose that $\mathfrak{G}_0$ is semi-simple. Now let $\{\pi, V\}$ be an irreducible representation of $\mathfrak{G}_0$ (and so $U(\mathfrak{G})$) on a not necessarily finite-dimensional vector space over the complex field, and assume that

$$V=\sum_{d\in\Omega} V(d) \quad \text{and} \quad \dim V(d)<\infty \quad \text{for all} \quad d \in \Omega.$$

We shall call such an irreducible representation quasi-simple as in Harish-Chandra.[5] Since the above sum $\sum$ is a direct sum, we can consider the idempotent operator $E(d)$ from V on $V(d)$ and the operator $E(d)\pi(\alpha)E(d)$ on $V(d)$ $(\alpha \in U(\mathfrak{G}))$. Since $\{\pi, V\}$ is irreducible, $\{E(d)\pi(\alpha)E(d)|\alpha \in U(\mathfrak{G})\}$ forms an irreducible family[9] of operators on $V(d)$.

Lemma 1. *For arbitrary $\alpha, \beta \in U(\mathfrak{G})$, there exists a $\gamma(\in U(\mathfrak{G}))$ such that $E(d)\pi(\gamma)E(d)\pi(\beta)E(d)=E(d)\pi(\gamma)E(d)$.*

Proof. Since $\pi(\beta)V(d) \subset \sum_{i=1}^{r} V(d_i)$ where d_i depends on β and d, there exists, by the generalized Burnside's theorem, a $\delta \in U(\mathfrak{k})$ satisfying $E(d)\pi(\alpha)E(d)=\pi(\delta)\pi(\beta)E(d)$, so that we have $E(d)\pi(\alpha)E(d)\pi(\beta)E(d)=E(d)\pi(\alpha\delta\beta)E(d)$.

The above lemma means that $\{E(d)\pi(\alpha)E(d)|\alpha \in U(\mathfrak{G})\}$ is the full operators on $V(d)$. Moreover $\pi(\gamma)E(d)\pi(\alpha)E(d)-E(d)\pi(\alpha)E(d)\pi(\gamma)=E(d)\pi([\gamma, \alpha])E(d)(\gamma \in U(\mathfrak{k}), \alpha \in U(\mathfrak{G}))$. Hence if $E(d)\pi(\alpha)E(d)$ commutes with $\pi(\gamma)$, then $E(d)\pi([\gamma, \alpha])E(d)=0$ and so $E(d)\pi(\sum_{j=1}^{p}\lambda_j[\gamma^j_{n_j}, \ldots [\gamma^j_1,\alpha]])=0$ ($\gamma^j_i \in U(\mathfrak{k})$ and λ_j complex numbers). Let $\alpha=\alpha^o+\sum_{i=1}^{m}\alpha_i$, then, by the generalized Burnside's theorem, $\alpha_i(i=1, \ldots, m)$ have the form $\sum \lambda_{ji}[\gamma^j_{n_{ji}}, [\gamma^j_{n_{ji}-1}, \ldots [\gamma^j_{1_i}, \alpha]]$. So if $E(d)\pi(\alpha)E(d)$ commutes with $\pi(\gamma)$, then $E(d)\pi(\alpha)E(d)=E(d)\pi(\alpha^o)E(d)$. Put $\mathfrak{A}=U(\mathfrak{k})U^o(\mathfrak{G})$. The correspondence $u(\in \mathfrak{A}) \rightarrow E(d)\pi(u)E(d)$ is a representation of the algebra $\mathfrak{A}$ on $V(d)$, which we shall denote by $\{\bar{\pi}_d, V(d)\}$.

From the above consideration we can conclude the following theorem.

Theorem 1. *The representation $\{\bar{\pi}_d, V(d)\}$ of $\mathfrak{A}$ induced by a quasi-simple irreducible representation of $U(\mathfrak{G})$ is irreducible.*

Remark. The above result has been shown by R. Godement[3] in the case of semi-simple Lie groups with some additional restrictions. The above theorem implies that this restriction is unnecessary.

Next we shall define:

$$\mathfrak{M}_0^{d_1}=\{\alpha|\pi(\alpha)V(d_1)=0, \quad \alpha \in U^o(\mathfrak{G})\},$$
$$\mathfrak{M}^{d_1}=\{\alpha|(\beta\alpha)^o \in \mathfrak{M}_0^{d_1}, \quad \alpha \in U(\mathfrak{G}) \text{ and all } \beta \in U(\mathfrak{G})\},$$

and

$$\mathfrak{M}'^{d_1}=\{\alpha|\pi(\alpha)V(d_1)=0, \quad \alpha \in U(\mathfrak{G})\},$$

for some $d_1(\in \Omega)$ such that $V(d_1)\neq(0)$.

$\mathfrak{M}_0^{d_1}$ is a two-sided maximal ideal of $U^o(\mathfrak{G})$, and $\mathfrak{M}^{d_1}$ and $\mathfrak{M}'^{d_1}$ are $\mathfrak{k}_0$-invariant left ideals of $U(\mathfrak{G})$.

Theorem 2. $\mathfrak{M}^{d_1}=\mathfrak{M}'^{d_1}$.

Proof. $\mathfrak{M}'^{d_1}\cap U^o(\mathfrak{G})=\mathfrak{M}_0^{d_1}$ and so $\mathfrak{M}'^{d_1}\subset\mathfrak{M}^{d_1}$, from Proposition 2. If $\alpha \in \mathfrak{M}'^{d_1}$, $\pi(\alpha)E(d_1)\neq 0$ and by the irreducibility of $\{\pi, V\}$ there exists an element $\gamma(\in U(\mathfrak{G}))$ such that $E(d_1)\pi(\gamma)\pi(\alpha)E(d_1)\neq 0$. Moreover from the irreducibility of $\{\bar{\pi}_{d_1}, V(d_1)\}$ there exists a $\delta \in \mathfrak{A}$ such that $S_p(\pi(\delta)E(d_1)\pi(\gamma\alpha)E(d_1))=S_p(E(d_1)\pi(\delta\gamma\alpha)E(d_1))\neq 0$. Put $\varphi^{\pi}_{d_1}(\alpha)=S_p(E(d_1)\pi(\alpha)E(d_1))$ for $\alpha \in U(\mathfrak{G})$, then $\varphi^{\pi}_{d_1}(\alpha)=\varphi^{\pi}_{d_1}(\alpha^o)$. Therefore we have $\varphi^{\pi}_{d_1}(\delta\gamma\alpha)=\varphi^{\pi}_{d_1}((\delta\gamma\alpha)^o)\neq 0$, so that $\pi((\delta\gamma\alpha)^o)E(d_1)\neq 0$, which means $(\delta\gamma\alpha)^o \bar{\in} \mathfrak{M}_0^{d_1}$ and so $\alpha \in \mathfrak{M}^{d_1}$. This completes the proof.

Let e_i $(i=1, 2, \ldots, n)$ be a base of $V(d_1)$ and put $\mathfrak{M}_{e_i}=\{\alpha|\pi(\alpha)e_i=0,\ \alpha \in U(\mathfrak{G})\}$, then $\mathfrak{M}_{e_i}$ $(i=1, 2, \ldots, n)$ are maximal left ideals of $U(\mathfrak{G})$. If we denote π_i $(i=1, 2, \ldots, n)$ the canonical representations of $U(\mathfrak{G})$ on $U(\mathfrak{G})/\mathfrak{M}_{e_i}$, they are equivalent to π.

We shall consider the representation $\pi'=\sum_{i=1}^{n}\oplus\pi_i$ on $V'=\sum_{i=1}^{n}\oplus U(\mathfrak{G})/\mathfrak{M}_{e_i}$, then $\pi(\alpha)V(d_1)=0$ $(\alpha \in U(\mathfrak{G}))$, if and only if $\pi'(\alpha)e=0$ for the vector $e=(e_1, \ldots, e_n) \in V'$.

Moreover by the irreducibility of $\{\bar{\pi}_{d_1}, V(d_1)\}$, we can easily show the following proposition.

Proposition 4. *The canonical representation of $U(\mathfrak{G})$ on $U(\mathfrak{G})/\mathfrak{M}^{d_1}$ is equivalent to π'.*

Remark. We notice that this proposition implies the following Theorem of Harish-Chandra:[5] In order that two quasi-simple irreducible representations $\{\pi_1, V_1\}$ and $\{\pi_2, V_2\}$ of $U(\mathfrak{G})$ are equivalent, it is necessary and sufficient that $\varphi_{d_1}^{\pi_1}(\alpha)=\varphi_{d_1}^{\pi_2}(\alpha)$ for all $\alpha \in U^o(\mathfrak{G})$ and for some $d \in \Omega$ such that $V(d)\neq(0)$.

As a consequence of the above proposition, we see that the representation $\{\bar{\pi}'_d, V'(d)\}$ of $\mathfrak{A}$ on $V'(d)$ is equivalent to $\sum_{i=1}^{n}\oplus\bar{\pi}_{id}$. If we denote the representation of $U^o(\mathfrak{G})$ on $V(d)$ by $\{\tilde{\pi}_d, V(d)\}$, then $U^o(\mathfrak{k})\subset U^o(\mathfrak{G})$, so that if $d\neq d_1$, $\{\tilde{\pi}_d, V(d)\}$ is not equivalent to $\{\tilde{\pi}_{d_1}, V(d_1)\}$, by the Proposition 3. This means that if M is an invariant subspace of V' under $\pi'(U^o(\mathfrak{G}))$, $M=\sum_{d\in\Omega} M\cap V'(d)$.

On the other hand, since $\mathfrak{M}^{d_1}$ is $\mathfrak{k}_0$-invariant, $\mathfrak{M}^{d_1}=\sum_{\tilde{d}\in\Omega'}\overline{\mathfrak{M}^{d_1}(\tilde{d})}$[11] for the adjoint representation, so that $U(\mathfrak{G})/\mathfrak{M}^{d_1}=\sum_{\tilde{d}\in\Omega'}\overline{U(\mathfrak{G})(\tilde{d})/\mathfrak{M}^{d_1}(\tilde{d})}$ $=\sum_{\tilde{d}\in\Omega'}\overline{(U(\mathfrak{G})/\mathfrak{M}^{d_1})(\tilde{d})}$ for the representation of $U(\mathfrak{k})$ induced, by the adjoint representation, on the factor space $U(\mathfrak{G})/\mathfrak{M}^{d_1}$, which we shall call the adjoint representation on $U(\mathfrak{G})/\mathfrak{M}^{d_1}$.

If $(\alpha)_{\mathfrak{M}^{d_1}}$[12] $(\in \overline{(U(\mathfrak{G})/\mathfrak{M}^{d_1})(\tilde{d})})$ and $u(\in U^o(\mathfrak{G}))$, $(u\alpha)_{\mathfrak{M}^{d_1}}$ and $(\alpha u)_{\mathfrak{M}^{d_1}}$ belongs to $\overline{(U(\mathfrak{G})/\mathfrak{M}^{d_1})(\tilde{d})}$. Hence $\overline{(U(\mathfrak{G})/\mathfrak{M}^{d_1})(\tilde{d})}$ is invariant under $\pi'(U^o(\mathfrak{G}))$ and we have

$$\overline{(U(\mathfrak{G})/\mathfrak{M}^{d_1})(\tilde{d})}=\sum_{d\in\Omega}\overline{(U(\mathfrak{G})/\mathfrak{M}^{d_1})(\tilde{d})}\cap U(\mathfrak{G})/\mathfrak{M}^{d_1}(d)\ldots(\mathrm{A}).$$

It turns out that $\dim\overline{(U(\mathfrak{G})/\mathfrak{M}^{d_1})(\tilde{d})}<\infty$ and that in order that $\alpha(\in U(\mathfrak{G})(\tilde{d}))$ belongs to $\mathfrak{M}^{d_1}$, it is sufficient that $(\beta^*\alpha)^o \in \mathfrak{M}_0^{d_1}$ for all $\beta \in U(\mathfrak{G})(\tilde{d})$. Henceforward we shall assume that $\mathfrak{M}_0^{d_1}$ is a self-adjoint ideal of $U^o(\mathfrak{G})$; i.e. if $\alpha \in \mathfrak{M}_0^{d_1}$, $\alpha^* \in \mathfrak{M}_0^{d_1}$. Let $(\alpha_i)_{\mathfrak{M}^{d_1}}$ $(i=1, 2, \ldots, r)$ be a base of $\overline{(U(\mathfrak{G})/\mathfrak{M}^{d_1})(\tilde{d}_\mu)}$. Then in order that $\alpha(\in U(\mathfrak{G})(\tilde{d}_\mu))$ belongs to $\mathfrak{M}^{d_1}$, it is sufficient that $(\alpha_i^*\alpha)^o$ $(i=1, 2, \ldots, r)$ belong to $\mathfrak{M}_0^{d_1}$.

From the preceding considerations on $\{\tilde{\pi}'_d, V'(d)\}$ and on (A), we can obtain the following

Theorem 3. *If* $\mathfrak{M}_0^{d}$ $(\neq U^o(\mathfrak{G}))$ *for some* $d_1 \in \Omega$ *is self-adjoint, all* $\mathfrak{M}_0^{d}$*'s are self-adjoint.*

Corollary. *If* $\varphi_{d_1}^{\pi}$ $(\neq 0)$ *for some* $d_1 \in \Omega$ *is self-adjoint,*[13] *all* φ_d^{π}*'s are self-adjont.*

Next we shall state some lemmas for the following Theorem 4. As $\mathfrak{M}_0^{d_1}$ is self-adjoint by our assumption, $\mathfrak{M}_0^{d_1}$ is a self-adjoint maximal ideal of $U^o(\mathfrak{k})$. Therefore we can easily show that d_1 is unitary,[15] so that, by Theorem 3, all d's which occur in π are unitary.

Let the elements of Ω, which occur in π, be $d_1, d_2, d_3, \dots$ and let u_n^i $(i \leq n,\ i, n=1, 2, \dots)$ be the elements of $U^o(\mathfrak{k})$[14] such that $\pi'(u_n^i)=E'(d_i)$ on $\sum_{i=1}^{n} V'(d_i)$ and let v_n^j $(j \leq n;\ i, n=1, 2, \dots)$ be the elements of $U^o(\mathfrak{k})$ such that $\pi'(v_n^j)=\sum_{i=1}^{j} \pi'(u_n^i)$. Since all d_i are unitary, we can assume that all u_n^i and v_n^j are self-adjoint. Moreover let the elements of Ω', which occur in the adjoint representation on $U(\mathfrak{G})/\mathfrak{M}^{d_1}$, be $\tilde{d}_0, \tilde{d}_1, \dots$. It follows, by (A), that for an arbitrarily fixed number m, there exists a number $t(m)$ and a v_n^m such that

$$\sum_{i=1}^{m} V'(d_i) \subset \sum_{q=1}^{t(m)} \overline{(U(\mathfrak{G})/\mathfrak{M}^{d_1})(\tilde{d}_q)} \qquad \text{and}$$

$$\pi'(v_n^m)\Big(\sum_{q=1}^{t(m)} (U(\mathfrak{G})/\mathfrak{M}^{d_1})(\tilde{d}_q)\Big)=\sum_{i=1}^{m} V'(d_i),$$

in other words:

$$\sum_{i=1}^{m} V'(d_i)=\{(\beta)_{\mathfrak{M}^{d_1}} | \beta=v_n^m\gamma,\ \gamma \in \sum_{q=1}^{t(m)} U(\mathfrak{G})(\tilde{d}_q)\}.$$

If α is an arbitrarily fixed element of $U(\mathfrak{G})$ and n' $(\geq n)$ is a sufficiently large number, we have the following relations:

$$\Big(\sum_{i=1}^{m} E'(d_i)\Big)\, \pi'(\alpha)\Big(\sum_{i=1}^{m} E'(d_i)\Big)=\pi'(v_{n'}^m)\pi'(\alpha) \quad \text{on} \quad \sum_{i=1}^{m} V'(d_i),$$

and

$$\Big(\sum_{i=1}^{m} E'(d_i)\Big)\pi'(\alpha^*)\Big(\sum_{i=1}^{m} E'(d_i)\Big)=\pi'(v_{n'}^m)\pi'(\alpha^*) \quad \text{on} \quad \sum_{i=1}^{m} V'(d_i).$$

On the other hand, we have

$$\pi'(v_{n'}^m)\pi'(\alpha)\pi'(v_n^m)=\pi'(v_n^m)\pi'(\alpha)\pi'(v_{n'}^m)$$
$$=\pi'(v_{n'}^m\, \alpha\, v_{n'}^m)=\pi'(v_{n'}^m\alpha) \quad \text{on} \quad \sum_{i=1}^{m} V'(d_i).$$

From the above facts with some additional considerations, we obtain the following Lemma.

Lemma 2. *If* $\big(\sum_{i=1}^{m} E(d_i)\big)\pi(\alpha)\big(\sum_{i=1}^{m} E(d_i)\big)=0(\alpha \in U(\mathfrak{G}))$, $\big(\sum_{i=1}^{m} E(d_i)\big)\pi(\alpha^*)\big(\sum_{i=1}^{m} E(d_i)\big)=0$.

By the analogous method with the Lemma 1, it can be shown that $\{(\sum_{i=1}^{m} E(d_i))\pi(\alpha)(\sum_{i=1}^{m} E(d_i)) | \alpha \in U(\mathfrak{G})\}$ are the full operators on

$\sum_{i=1}^{m} V(d_i)$. Furthermore, from Lemma 2, it is easily shown that the mapping $(\sum_{i=1}^{m} E(d_i))\pi(\alpha)(\sum_{i=1}^{m} E(d_i)) \to (\sum_{i=1}^{m} E(d_i))\pi(\alpha^*)(\sum_{i=1}^{m} E(d_i))$ is a conjugate linear anti-automorphism. Therefore by the well-known theorem on the automorphisms of simple algebras,[16] we obtain that $(\sum_{i=1}^{m} E(d_i))\pi(\alpha^*)(\sum_{i=1}^{m} E(d_i)) = H_m\{(\sum_{i=1}^{m} E(d_i))\pi(\alpha)(\sum_{i=1}^{m} E(d_i))\}^{\sigma} H_m^{-1}$, where H_m denotes a linear operator on $\sum_{i=1}^{m} V(d_i)$ and A^{σ} donotes the adjoint operator of A in the sense of finite dimensional vector space.

Proposition 5. *H_m is a self-adjoint operator for all m.*

Finally we assume that $\{\tilde{\pi}_{d_1}, V(d_1)\}$ is unitary.[15] Then the representation $\{\overline{\pi}_{d_1}, V(d_1)\}$ of $\mathfrak{A}$ is also unitary,[15] so that we have, by Lemma 2, $(E(d_1)\pi(\alpha)E(d_1))^{\sigma} = E(d_1)\pi(\alpha^*)E(d_1)$ for all $\alpha \in U(\mathfrak{G})$ and so $H_1 = 1$.

From some more considerations together with Proposition 5, it turns out that if dim $(d_1)=1$, all H_m are positive self-adjoint operators. By this fact, we can easily show that $\varphi_d^{\pi}(\alpha^*\alpha) = Sp(E(d)\pi(\alpha^*\alpha)E(d)) \geqq 0$, for all $\alpha \in U(\mathfrak{G})$ and all $d \in \Omega$.

Now we conclude the following

Theorem 4. *Suppose dim $(d_1)=1$. Then in order that the functional $\varphi_{d_1}^{\pi}$ is positive, it is necessary and sufficient that $\varphi_{d_1}^{\pi}(u^*u) \geqq 0$ for all $u \in U^o(\mathfrak{G})$. Moreover if $\varphi_{d_1}^{\pi}(\neq 0)$ is positive, all φ_d^{π}'s are positive.*

Remark 1. It seems to be almost certain that the restriction dim $(d_1)=1$ in the above theorem is unnecessary.

In another paper, we shall discuss the problem with the complete proof of Theorem 4.

Remark 2. In the general semi-simple Lie group G, we can show, by a slight modification of Harish-Chandra's Theorem[6)7)] that in order that a quasi-simple irreducible representation $\{\pi, \mathfrak{H}\}$ of G is infinitesimally equivalent to a unitary irreducible representation, it is necessary and sufficient that some spherical fuction φ_d^{π} $(\neq 0)$ is positive in our sense. Therefore the above theorem gives a sufficient condition in order that $\{\pi, \mathfrak{H}\}$ is infinitesimally unitary.

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7) S. Sakai: On the representations of semi-simple Lie groups. Proc. Japan Acad., **30**, 14 (1954).

8) $\mathfrak{G}(\tilde{d})$ is a subspace of $\mathfrak{G}$ composed of all elements which transform under $\mathrm{ad}(K_0)$ according to $\tilde{d}$.

9) There not exist non-trivial subspaces which are invariant under $\{E(d)\pi(\alpha)E(d)|\alpha \in U(\mathfrak{G})\}$.

10) The representation of $\mathfrak{G}_0$ corresponding to the adjoint representation is the form $ad(x)\alpha=[x, \alpha]=x\alpha-\alpha x$ $(x \in \mathfrak{G}_0,\ \alpha \in U(\mathfrak{G}))$.

11) To distinguish the adjoint representation from π we denote $\overline{\mathfrak{M}}^{d_1}(\tilde{d})$.

12) $(\alpha)_{\mathfrak{M}}d_1$ denotes the canonical image of $\alpha(\in U(\mathfrak{G}))$ in $U(\mathfrak{G})/\mathfrak{M}^{d_1}$.

13) A linear functional φ is said to be self-adjoint, if $\overline{\varphi(\alpha^*)}=\varphi(\alpha)$ for all $\alpha \in U(\mathfrak{G})$.

14) The existence of such u_n^i is assured by the generalized Burnside's theorem, and proposition 3.

15) In general, a finite-dimensional representation $\{\tilde{\pi}, \tilde{V}\}$ of an algebra A with adjoint operation is said to be unitary, if it satisfies that $\tilde{\pi}(\alpha^*)=(\pi(\alpha))^\sigma$ where $(\pi(\alpha))^\sigma$ is the adjoint operator of $\pi(\alpha)$. $d(\in \Omega)$ is said to be unitary, if it contains a unitary representation of $U(\mathfrak{k})$.

16) E. Artin, C. Nesbitt, and R. Thrall: Rings with minimum condition.

17) Let $\mathfrak{c}$ be the center of $\mathfrak{G}_0$, then K_0 is the analytic subgroup of G_0 corresponding to a ring $(\mathfrak{k}_0+\mathfrak{c})/\mathfrak{c}$.