60. On Closed Mappings

By Sitiro HANAI
Osaka University of the Liberal Arts and Education

(Comm. by K. KUNUGI, M.J.A., April 12, 1954)

If S and E are T_1-spaces, a single-valued mapping $f(S)=E$ is said to be closed provided that the image of every closed set in S is closed in E. It is interesting to know how the topology of E is affected by the topology of S under f. Concerning this question, G. T. Whyburn and A. V. Martin have recently investigated and obtained some results.[1]

In this note, we will consider the case when the topology of E affected by the topology (under some restrictions) of S under f becomes metrizable.

1. We will firstly prove the following

Theorem 1. Let S be a perfectly separable Hausdorff space and let E a compact space.[2] If $f(S)=E$ is a closed mapping such that $f^{-1}(p)$ is compact for every point p of E, then E is a separable metric space.

To establish this theorem, we prove the following lemmas.

Lemma 1. Let S be a perfectly separable Hausdorff space. If $f(S)=E$ is a closed continuous mapping such that $f^{-1}(p)$ is compact for every point p of E, then E is perfectly separable.

Proof. Let $\{U_n\}(n=1, 2, 3, \dots)$ be a countable basis of open sets of S. For each finite subset $(n_1, n_2, \dots, n_m)$ of $(1, 2, 3, \dots)$, let $(\sum_{i=1}^{m} U_{n_i})_0$ be the union of all $f^{-1}(p)$ such that $\sum_{i=1}^{m} U_{n_i} \supset f^{-1}(p)$. Then $(\sum_{i=1}^{m} U_{n_i})_0$ is an open inverse set, and the family $\{(\sum_{i=1}^{m} U_{n_i})_0\}$ of all such sets is evidently countable.

Now let O be an open set of E and $p \in O$, then $f^{-1}(O) \supset f^{-1}(p)$ and $f^{-1}(O)$ is open in S because f is continuous. Then $f^{-1}(O)=\sum_{j=1}^{\infty} U_{n_j}$ where $\{U_{n_j}\} \subset \{U_n\}(n=1, 2, 3, \dots)$. Since $f^{-1}(p)$ is compact, there exists a finite subset $\{U_{n_k}\}(k=1, 2, \dots, l)$ of $\{U_{n_j}\}(j=1, 2, 3, \dots)$ such that $\sum_{k=1}^{l} U_{n_k} \supset f^{-1}(p)$, hence $(\sum_{k=1}^{l} U_{n_k})_0 \supset f^{-1}(p)$. As f is closed and

1) G. T. Whyburn: Open and closed mappings, Duke Math. Jour., **17**, 69–74 (1950). A. V. Martin: Decompositions and quasi-compact mappings, (abstract), Bull. Amer. Math. Soc., **59**, 397 (1953).

2) We use "compact" in the sense of "bicompact".

continuous, $f\{(\sum_{k=1}^{l} U_{n_k})_0\}$ is open in E. Hence the family of open sets $[f\{(\sum_{i=1}^{m} U_{n_i})_0\}]$ is a countable basis of open sets of E. Thus we get the lemma.

Lemma 2. Let S be a Hausdorff space and let E a compact space. If $f(S)=E$ is a closed mapping such that $f^{-1}(p)$ is compact for each point p of E, then f is continuous.

Proof. Suppose on the contrary that f is not continuous. Then there exists a point $a \in S$ at which f is not continuous. Let $\mathfrak{F}$ be the filter of neighborhoods of a and $\mathfrak{F}'$ be the filter whose base is $f(\mathfrak{F})$. Then $\mathfrak{F}'$ does not converge to $f(a)\equiv p$. Hence there exists an open neighborhood $V(p)$ such that $V(p) \bar{\in} \mathfrak{F}'$. Let Φ' be an ultrafilter which contains $\mathfrak{F}'$, then $CV(p) \bar{\in} \Phi'$ where $CV(p)$ denotes the complement of $V(p)$. Since E is compact by assumption, Φ' converges to a point q. Then $p \neq q$ because $CV(p) \in \Phi'$, and hence $a \bar{\in} f^{-1}(q)$. As $f^{-1}(q)$ is compact and S is a Hausdorff space, there exist open neighborhoods $U(a)$ and $U\{f^{-1}(q)\}$ of a and $f^{-1}(q)$ respectively such that $U(a)U\{f^{-1}(q)\}=\phi$. Then $\overline{U(a)}\ f^{-1}(q)=\phi$ and hence $q \bar{\in} f\{\overline{U(a)}\}$.

On the other hand, since $f\{U(a)\} \in \Phi'$ and $\Phi' \to q$, we have $V(q)$ $f\{U(a)\} \neq \phi$ for every open neighborhood $V(q)$ of q. Hence $q \in \overline{f\{U(a)\}}$. Because f is closed, we have $\overline{f\{U(a)\}} \subset f\{\overline{U(a)}\}$, which contradicts the fact that $q \bar{\in} f\{\overline{U(a)}\}$.

Proof of Theorem 1. By lemmas 1 and 2, it is evident that E is perfectly separable. To prove that E is metrizable, we have only to prove that E is a Hausdorff space because E is a compact space. Let p and q be any distinct two points of E. Then $f^{-1}(p)\, f^{-1}(q)=\phi$, and $f^{-1}(p)$ and $f^{-1}(q)$ are compact sets by assumption. Since S is a Hausdorff space, there exist open neighborhoods $U\{f^{-1}(p)\}$ and $U\{f^{-1}(q)\}$ of $f^{-1}(p)$ and $f^{-1}(q)$ respectively such that $U\{f^{-1}(p)\}$ $U\{f^{-1}(q)\}=\phi$.

As f is closed continuous, there exist open inverse sets $U_0\{f^{-1}(p)\}$ and $U_0\{f^{-1}(q)\}$ such that $U\{f^{-1}(p)\} \supset U_0\{f^{-1}(p)\} \supset f^{-1}(p)$ and $U\{f^{-1}(q)\} \supset U_0\{f^{-1}(q)\} \supset f^{-1}(q)$. Hence E is a Hausdorff space.

Remark. Theorem 1 is also proved by use of Lemma 2, Theorem 3 in the following section, and T. Iwamura's theorem.[3)]

2. In this section, we will consider the case when S is a metric space.

We begin with proving the following:

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3) T. Iwamura: Remarks on closed mappings and compactness, Natural Science Report of the Ochanomizu Univ., **1**, 6–8 (1951).

Theorem 2. If S is a metric space and $f(S)=E$ is a closed continuous mapping, then E is a metric space.

Proof. (1) E satisfies the first countability axiom. In fact, let $O^{(n)}(x)$ ($n=1, 2, 3, \ldots$) be open spheres with the center x and the radius $\frac{1}{n}$ for each point x of S. For any point p of E, we have $\sum_{x\in f^{-1}(p)} O^{(n)}(x) \supset f^{-1}(p)$. Since f is closed continuous, in the same way as Lemma 1, we can see that $(\sum_{x\in f^{-1}(p)} O^{(n)}(x))_0$ is an open inverse set. Let $O_0^{(n)}(p) \equiv (\sum_{x\in f^{-1}(p)} O^{(n)}(x))_0$ and let $G^{(n)}(p) \equiv f\{O_0^{(n)}(p)\}$, then $G^{(n)}(p)$ is an open neighborhood of p.

Now we will prove that $\{G^{(n)}(p)\}$ (1, 2, 3, ...) is a basis of the neighborhood system of p. Let $O(p)$ be any open neighborhood of p, then $f^{-1}\{O(p)\}$ is open in S because f is continuous. Let d be the distance between $Cf^{-1}\{O(p)\}$ and $f^{-1}(p)$, then $\sum_{x\in f^{-1}(p)} O^{(n)}(x) Cf^{-1}\{O(p)\} = \phi$ for an n such that $\frac{1}{n} < d$. Hence $O_0^{(n)}(p) \subset f^{-1}\{O(p)\}$. Then we have $G^{(n)}(p) \subset O(p)$. Thus E satisfies the first countability axiom. Further it is easy to see that $G^{(1)}(p) \supset G^{(2)}(p) \supset \cdots$ and $\prod_n G^{(n)}(p) = p$.

(2) For any point p of E and any index n, there exists some index $m=m(p, n)$ such that $G^{(m)}(p)G^{(m)}(q) \neq \phi$ implies $G^{(m)}(q) \subset G^{(n)}(p)$. For, let d' be the distance between $Cf^{-1}\{G^{(n)}(p)\}$ and $f^{-1}(p)$, and let m an integer such that $\frac{3}{m} < d'$. If $G^{(m)}(p)G^{(m)}(q) \neq \phi$, then $O_0^{(m)}(p)O_0^{(m)}(q) \neq \phi$. For any point z of $O_0^{(m)}(q)$, there exists $y \in f^{-1}(q)$ such that $z \in O^{(m)}(y)$. Since $O_0^{(m)}(p)$ and $O_0^{(m)}(q)$ are open inverse sets, there exists a point r such that $f^{-1}(r) \subset O_0^{(m)}(p)O_0^{(m)}(q)$. Then there exists a point w of $O^{(m)}(y)f^{-1}(r)$. Hence $w \in O^{(m)}(x)$ for some point x of $f^{-1}(p)$. For this point x, we can easily see that $z \in O^{(n)}(x)$. As $O_0^{(m)}(q) \subset O_0^{(n)}(p)$ follows from that $O_0^{(m)}(q)$ is an inverse open set, we get $G^{(m)}(q) \subset G^{(n)}(p)$.

By (1), (2) and A. H. Frink's theorem,[4] E is metrizable. This completes the proof of the theorem.

The following theorem due to A. V. Martin[5] is easily verified by use of Lemma 1 and Theorem 2.

Theorem 3. If S is a separable metric space and $f(S)=E$ is a closed continuous mapping such that $f^{-1}(p)$ is compact for each point p of E, then E is a separable metric space.

By Lemma 2 and Theorem 3, we have also the following:

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4) A. H. Frink: Distance functions and the metrization problem, Bull. Amer. Math. Soc., **44**, 133-142 (1937).

5) A. V. Martin: Loc. cit.

Corollary. Let S be a separable metric space and let E a compact space. If $f(S)=E$ is a closed mapping such that $f^{-1}(p)$ is compact for each point p of E, then E is a separable metric space.

Theorem 4. If S is a locally compact metric space and $f(S)=E$ is a closed continuous mapping such that $f^{-1}(p)$ is compact for each point p of E, then E is a locally compact metric space.

Proof. By Theorem 2, E is metrizable. Accordingly we need only to prove that E is locally compact. For this purpose, let p be any point of E and let U any open neighborhood of p, then $f^{-1}(U)$ is open in S. For each point x of $f^{-1}(p)$, there exists an open neighborhood $O(x)$ of x such that $f^{-1}(U)\supset\overline{O(x)}$ and $\overline{O(x)}$ is compact because S is a locally compact metric space. Since $f^{-1}(p)$ is compact, there exist open neighborhoods $O(x_i)$ of finite points x_i $(i=1, 2, \dots, n)$ of $f^{-1}(p)$ respectively such that $f^{-1}(U)\supset\sum_{i=1}^{n}O(x_i)\supset f^{-1}(p)$ where each $O(x_i)$ satisfies the same condition as the above $O(x)$. Then $f^{-1}(U)\supset \sum_{i=1}^{n}\overline{O(x_i)}\supset\sum_{i=1}^{n}O(x_i)\supset f^{-1}(p)$.

Now let $O\equiv(\sum_{i=1}^{n}O(x_i))_0$, then $U\supset f(\overline{O})\supset f(O)\ni p$ and $f(O)$ is open because f is closed and continuous. As $\sum_{i=1}^{n}\overline{O(x_i)}\supset\overline{O}$ and $\sum_{i=1}^{n}\overline{O(x_i)}$ is compact, $\overline{O}$ is compact. Hence $f(\overline{O})=\overline{f(O)}$ is compact. This completes the proof of the theorem.