57. On the Integration of the Temporally Inhomogeneous Diffusion Equation in a Riemannian Space. II

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1. **Introduction.** In a preceding note with the same title,[1] the author devised an existence proof of the solution for the Cauchy's problem of the temporally inhomogeneous diffusion equation with C^∞ coefficients:

$$(1.1)\qquad \frac{\partial f(t,s,x)}{\partial t}-A_{tx}f(t,s,x)=0,\quad t>s,$$

$$\lim_{t\to s} f(t,s,x)=f(x)\in L_1(R)^{2)}\ \text{almost everywhere},$$

$$A_{tx}f(t,s,x)=g(x)^{-1/2}\frac{\partial^2}{\partial x^i\partial x^j}(g(x)^{1/2}a^{ij}(t,x)f(t,s,x))$$

$$-g(x)^{-1/2}\frac{\partial}{\partial x^i}(g(x)^{1/2}b^i(t,x)f(t,s,x))+c(t,x)f(t,s,x),$$

in a connected domain R of an m-dimensional, orientable C^∞ Riemannian space with the metric $ds^2=g_{ij}(x)dx^idx^j$.

The purpose of the present note is to show that the existence proof in [I] may be modified so as to yield the existence proof of the solution admitting the kernel representation

$$(1.2)\qquad f(t,s,x)=\int_R P(t,s,x,y)f(y)dy\quad \text{for every } f(x)\in L_1(R).$$

2. **The Proof of the Kernel Representation.** Let D denote a linear set of C^∞ functions with compact carriers such that D is $L_1(R)$-dense in $L_1(R)$. We regard $A_t=A_{tx}$ as an additive operator on $D\subseteqq L_1(R)$ to $L_1(R)$, and let $\overline{A}_t$ be the smallest closed extension of A_t. We assume that D is so chosen that the following Hypothesis is satisfied.

Hypothe is: Let, for sufficiently large integer n (independently of t), the resolvents

$$(2.1)\qquad I_t^{(n)}=(I-n^{-1}\overline{A}_t)^{-1}$$

1) Proc. Japan Acad., **30** (1954), No. 1, 19-23. This note will be referred to as [I]. At this juncture, the author wants to give the following corrigenda to [I]: (i) On page 23, lines 9–10, "the second term" and "the third term" should be read respectively as "the third term" and "the fourth term". (ii) On page 23, line 24, "is equivallent to" should be read as "equivalent, when $x\in V(x_0)$, to".

2) The Banach space of Borel measurable functions which are integrable, with respect to the measure $dx=g(x)^{1/2}dx^1\dots dx^m$, over R.

exist as bounded operators on $L_1(R)$ to $L_1(R)$ such that

(2.2) $I_t^{(n)}f(x)$ is non-negative and $\int_R I_t^{(n)}f(x)dx=\int_R f(x)dx$ when $f(x)\in L_1(R)$ is non-negative,

(2.3) $I_t^{(n)}f(x)$ is strongly continuous in t for every $f(x)\in L_1(R)$.

Then the following results were proved in [I] actually: Consider the approximate equation of (1.1) in $L_1(R)$

(2.4) $$\text{strong}\lim_{\delta\to 0}\delta^{-1}\{f^{(n)}(t+\delta,s,x)-f^{(n)}(t,s,x)\}=\bar{A}_t I_t^{(n)}f^{(n)}(t,s,x),\ t\geqq s,$$
$$\text{strong}\lim_{t\to s}f^{(n)}(t,s,x)=f(x)\in L_1(R).$$

Then we may choose a subsequence $\{n'\}$ of $\{n\}$ such that $\{f^{(n')}(t,s,x)\}$ converges, for every $t>s$, in the sense of the "distribution", to a solution of (1.1) satisfying the conditions below.

(2.5) $\int_R |f(t,s,x)|dx\leqq\int_R |f(x)|dx$ and $f(t,s,x)$ is non-negative when $f(x)$ is non-negative,

(2.6) for any $x_0\in R$ and for any sufficiently small vicinity $U(x_0)$, there exist a vicinity $V(x_0)\subseteqq U(x_0)$ and kernels $H(x,y,t,\tau)$, $K(x,y,t,\tau)$ such that (i) $H(x,y,t,\tau)$ and $K(x,y,t,\tau)$ are C^∞ for $t>\tau$ and $t\geqq\tau$ respectively, (ii) $H(x,y,t,\tau)$ and $K(x,y,t,\tau)$ both vanish if x or y is outside of $U(x_0)$, and (iii) when $x\in V(x_0)$ we have the representation
$$f(t,s,x)=\int_R H(y,x,t,s)f(y)dy+\int_s^t\left\{\int_R K(y,x,t,\tau)f(\tau,s,y)dy\right\}d\tau.$$

We may modify the choice of the subsequence $\{n'\}$ as follows. Firstly we remark that the mapping $f(x)\to f^{(n)}(t,s,x)$ is a bounded linear mapping on $L_1(R)$ to $L_1(R)$. This we see by the boundedness and the strong continuity in t of the operator $\bar{A}_t I_t^{(n)}=n(I_t^{(n)}-I)$. Actually it was proved in [I] that

(2.7) $\|f^{(n)}(t,s,x)\|\leqq\|f(x)\|$.

Thus, since $L_1(R)$ is separable, we may choose, by a diagonal method, a subsequence $\{n''\}$ of $\{n\}$ such that $\{f^{(n'')}(t,s,x)\}$ converges, for every $t>s$ and for every $f(x)\in L_1(R)$ simultaneously, in the sense of the "distribution" to a solution $f(t,s,x)$ of (1.1) satisfying (2.5) and (2.6). Hence, for any triple (t,s,x) with $t>s$ and $x\in V(x_0)$, the value $f(t,s,x)$ of this solution may be considered as an additive functional $F(f)=F_{t,s,x}(f)$ of $f(x)\in L_1(R)$. Moreover, we have, by (2.5) and (2.6),
$$|F_{t,s,x}(f)|\leqq\int_R |f(y)|dy\cdot\max_y|H(x,y,t,s)|$$
$$+(t-s)\int_R |f(y)|dy\cdot\max_{y,s\leqq\tau\leqq t}|K(x,y,t,\tau)|.$$

Therefore there exists a bounded measurable function in y depending upon (t, s, x), say $P(t, s, x, y)$, such that (1.2) holds good.

Remark 1. Comparing (1.2) with (2.6) and remembering (2.5), we see that the kernel $P(t, s, x, y)$ may be considered to be measurable in (t, s, x, y) for $t>s$. Thus, again by (1.2) and (2.6), $P(t, s, x, y)$ is equivalent, when $x \in V(x_0)$, to the function

$$H(x,y,t,s) + \int_s^t \left\{ \int_R K(z, x, t, \tau) P(\tau, s, z, y) dz \right\} d\tau.$$

Hence we see that $P(t, s, x, y)$ satisfies the equation

$$\frac{\partial P}{\partial t} - A_{tx} P = 0, \quad t>s.\text{[3]} \tag{2.8}$$

Remark 2. The original sequence $\{f^{(n)}(t, s, x)\}$ itself converges in the sense of the "distribution" to the solution $f(t, s, x)$ of (1.1) satisfying (1.2) and (2.5)–(2.6), if it is assured that the solution of (1.1) satisfying (2.5) is unique. A condition for the uniqueness was given in another paper.[4] However, the author is not so far able to prove that whether the above Hypothesis assures this uniqueness or not.

3) Cf. K. Yosida: On the fundamental solution of the parabolic equation in a Riemannian space, Osaka Math. J., **5** (1953), No. 1, 65–74. See also S. Itô: The fundamental solution of the parabolic equation in a differentiable manifold, ibid., 75–92.

4) See the papers referred to in 3).