122. A Note on f-completeness

By Shouro KASAHARA
Kobe University

(Comm. by K. KUNUGI, M.J.A., July 12, 1954)

In a recent paper [2], A. W. Ingleton introduced a concept, *spherically completeness*, which is important for the extension of continuous linear mappings of a non-Archimedean normed space into another one. For a locally flat topological linear space whose topology is defined by a family of non-Archimedean semi-norms, the author has given a concept, *f-completeness* [3], on the extension property.

It is our purpose in this note to prove some conspicuous properties on f-completeness.

Throughout this note, we will denote by K a non-Archimedean valued field of which the valuation v is non-trivial, and assume that the locally flat linear spaces have the same K as the underlying field of scalars, and moreover by *f-complete space* we shall mean a locally flat linear space which is f-complete with respect to each of the non-Archimedean semi-norms defining the topology.

Let (E_ι) be a family of locally flat linear spaces, and let us consider the product space $F=\Pi_\iota E_\iota$, and denote by f_ι the projection of F to E_ι. Then it is clear that the topology of the linear space F is defined by the family of non-Archimedean semi-norms $p_\alpha\circ f_\iota$, where for any ι, p_α runs over the family of non-Archimedean semi-norms defining the topology of E_ι. That is, *the product space of a family of locally flat linear spaces is locally flat.*

The following proposition can be readily verified.

Proposition 1. (a) *The product of a family of f-complete spaces is also f-complete.* (b) *If W is a closed subspace*[1] *of an f-complete space E, then the quotient space E/W is f-complete.*

The part (a) of the proposition is clear.

Let p^* be the non-Archimedean semi-norm of E/W corresponding to a non-Archimedean semi-norm p of the space E. Then the inverse image of any p^*-flat variety in E/W by the canonical mapping π of E onto E/W is a p-flat variety in E, and hence the part (b) is clear.

Proposition 2. *Let W be an f-complete subspace of a Hausdorff linear space E; then W admits a topological supplement,*[2] *and is therefore closed.*

---

1) In this note "subspace" always means "linear subspace".
2) Cf. (1) p. 16.

It follows from the f-completeness of W that the identity mapping of W onto itself is extended to a mapping: $E \to W$; and hence W admits a topological supplement, and is closed since E is Hausdorff space.

Proposition 3. *If W is a closed subspace of a locally flat linear space E, and if W and E/W are both f-complete, then E is f-complete.*

Let $\{C_\varepsilon\}$ be a collection of p-flat varieties in E totally ordered by inclusion, and let us employ the notation in the proof of Proposition 1. Then it is clear that the image of any p-flat variety by the canonical mapping π is a p^*-flat variety in E/W, and the collection $\{\pi(C_\varepsilon)\}$ is a totally ordered set by inclusion. It follows that there is a common point $\pi(x_0)$ to all $\pi(C_\varepsilon)$; that is to say, x_0+W meets C_ε for all ε. Thus we have $(C_\varepsilon - x_0) \frown W \neq \phi$.

Whereas $\{C_\varepsilon - x_0\}$ is a totally ordered set by inclusion, and *a fortiori* $\{(C_\varepsilon - x_0)\frown W\}$ is also; and since $(C_\varepsilon - x_0)\frown W$ is a $p \mid W$-flat variety in W, there exists a y_0 in W such that y_0 is contained in every $(C_\varepsilon - x_0)\frown W$. Thus x_0+y_0 is in every C_ε, proving the proposition.

Proposition 4. *Let E be a non-Archimedean normed space over K. If E is spherically complete, then it is complete.*

Let $\mathfrak{F}$ be a Cauchy filter on E. For every neighborhood N of 0 in E and every set $A \in \mathfrak{F}$, let us consider the set $A+N$. Then, it is not hard to see that these sets form a Cauchy filter. In fact, for any neighborhood N of 0, there is an $A \in \mathfrak{F}$ such that for any x and y in A, $x-y \in N$; and hence $A+N=x+N$. Now since the family $\{x+N\}$ is a nest of spheres (cf. [2]) on E, it has a common point, which is a limit of the filter $\mathfrak{F}$.

Proposition 5. *If E is a non-Archimedean normed space over K, and if it has the extension property, then every closed subspace of E has the extension property.*

Let W be a closed subspace of E; we shall show that it is spherically complete.

Any nest of spheres $\mathfrak{S}$ on W may be considered as the set consisting of the intersections of W and each element of a nest of spheres $\mathfrak{S}_0$ on E; but then, by the assumption, $\mathfrak{S}_0$ has a common point x_0 in E.

For the sake of convenience, we denote by U_ε the set of point for which $\|x\| \leqq \varepsilon$, where $\| \ \|$ is the non-Archimedean norm. Then, every element of $\mathfrak{S}_0$ is described in the form: x_0+U_ε. Moreover, let us denote by δ the greatest lower bound of ε corresponding to each element of $\mathfrak{S}_0$.

If $\delta=0$, the element x_0 is clearly contained in W, since the intersection of any neighborhood of x_0 and the closed subspace W is nonvoid.

On the other hand, if $\delta>0$, then for any $\lambda>\delta$, there is an $x\in W$ such that $x\in x_0+U_\lambda$, i.e., $(x_0+U_\lambda)\frown W\neq\phi$. Since W is closed, we have $(x_0+U_\delta)\frown W\neq\phi$. Thus the proof is complete.

Proposition 6. *If W is an f-complete subspace of a locally flat linear space E, then every continuous mapping of W into any topological space can be extended to a continuous mapping whose domain is the whole of E.*

In fact, since W is f-complete, the identity mapping of W onto itself is extended to a mapping of E into W: W is a retract of E.

Let E and F be two locally flat linear spaces with the family of non-Archimedean semi-norms (p_α) and (q_β) respectively. It has shown in [(3), lemma 2] that a linear mapping u of E into F is continuous if and only if, for any $q\in(q_\beta)$, there exist a $p\in(p_\alpha)$ and a positive number a such that

$$(*)\qquad q(u(x))\leqq a\cdot p(x)$$

for all x in E.

Now regarding to such a linear mapping, we have

Proposition 7. *Let E and F be as above, and W an f-complete subspace of E. Then every continuous linear mapping u of W into F can be extended to a linear mapping of E into F satisfying the inequality $(*)$ for same p and a.*

In fact, if we denote by I^* the extension to E of the identity mapping of W, we have $p(I^*(x))\leqq p(x)$ for any $p\in(p_\alpha)$ and $x\in E$. Let $u^*=u\circ I^*$; then evidently u^* is linear and $u^*\mid W=u$. Now, since u is continuous, for any $q\in(q_\beta)$, there exist a $p\in(p_\alpha)$ and a positive number a and $(*)$ holds for all x in W. It remains to prove the inequality $(*)$ for u^*. For any x in E, we have

$$q(u^*(x))=q(u\circ I^*(x))\leqq a\cdot p(I^*(x)),$$

and hence
$$q(u^*(x))\leqq a\cdot p(x).$$

Let us now consider the field K instead of F; then the lemma 2 in [3] may be stated as follows: a linear function f defined on E is continuous if and only if, for any $p\in(p_\alpha)$ there is a positive number a such that

$$(**)\qquad v(f(x))\leqq a\cdot p(x)$$

for all $x\in E$.

As an immediate corollary of Proposition 7, we obtain

Proposition 8. *Let W be an f-complete subspace of a locally flat linear space E. Then every continuous linear function f defined on W can be extended to a linear function on the whole of E, satisfying the inequality $(**)$ for same a.*

Moreover, in view of Proposition 5, the following is apparent.

Proposition 9. *Let E be a non-Archimedean normed space having the extension property, and W a closed subspace of E. Then, every continuous linear function defined on W possesses an extension of the same norm whose domain is the whole of E.*

References

[1] Bourbaki, N.: Éléments de Mathématiques, Livre V, Chaps. I et II. Paris (1953).

[2] Ingleton, A. W.: The Hahn-Banach theorem for non-Archimedean-valued fields, Proc. Cambridge Philos. Soc., **48**, 41–45 (1952).

[3] Kasahara, S.: A note on locally flat topological linear spaces, to appear in Math. Japonicae, **3**, No. 2.