11. *On the Integro-jump of a Function and Its Fourier Coefficients*

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1. Introduction. Suppose that $f(x)$ is periodic with period 2π and Lebesgue integrable in $(-\pi, \pi)$. Let the Fourier series of $f(x)$ be

$$f(x)\sim a_0/2+\sum_{n=1}^{\infty}(a_n \cos nx+b_n \sin nx)$$

and let

$$\bar{s}_n(x)=\sum_{\nu=1}^{n}(b_\nu \cos \nu x-a_\nu \sin \nu x)\equiv\sum_{\nu=1}^{n}B_\nu(x).$$

We denote by $\bar{\sigma}_n^\alpha(x)$ the n-th Cesàro mean of order α of the sequence $\{\bar{s}_n(x)\}$.

H. C. Chow showed the following

Theorem A.[1] *If there exists a number $L(x)$ such that*

$$(1.1)\qquad \int_0^t \psi(u)\,du=o(t),\quad \int_0^t |\psi(u)|\,du=O(t),\ \text{as } t\to 0,$$

where $\psi(t)=f(x+t)-f(x-t)-L(x)$, *then*

$$(1.2)\qquad \lim_{n\to\infty}\left[\bar{\sigma}_{2n}^\alpha(x)-\bar{\sigma}_n^\alpha(x)\right]=\frac{1}{\pi}\log 2\cdot L(x),\ \text{for } \alpha>0.$$

F. C. Hsiang proved also the following

Theorem B.[2] *If the integral*

$$(1.3)\qquad \int_0^t \frac{\psi(u)}{u^{1/\alpha}}\,du\ (1>\alpha>0),$$

exists, then

$$(1.4)\qquad \lim_{n\to\infty}\left[\bar{\sigma}_{2n}^1(x)-\bar{\sigma}_n^1(x)\right]=\frac{1}{\pi}\log 2\cdot L(x).$$

Concerning the sequence $\{nB_n(x)\}$, O. Szász[3] proved the following

Theorem C. *Under the assumption of Theorem A, we have*

$$(1.5)\qquad \lim_{n\to\infty} nB_n(x)=-\frac{1}{\pi}L(x)\ (C, 2).$$

Recently Kenzi Yano[4] showed that Theorem C is still valid even if $(C, 2)$ is replaced by $(C, 1+\alpha)$, for every $\alpha>0$.

It will not be of no interest to replace the conditions of Theorem

1) H. C. Chow: Journ. London Math. Soc., **16**, 23–27 (1941). In this theorem, the case $\alpha=1$ is O. Szász' theorem (Duke Math. Journ., **4**, 401–407 (1938)).
2) F. C. Hsiang: Bull. Calcutta Math. Soc., **44**, 55–58 (1952).
3) O. Szász: Trans. American Math. Soc., **50** (1942).
4) Kenzi Yano: Nara Joshidai Kiyô (in Jap.), **1** (1951).

A by that depending on the number α. In fact, we get the following

Theorem 1. *If* $0<\alpha<1$ *and*

$$\int_0^t \psi(u)\,du = o(t^{1/\alpha}), \quad \text{as } t\to 0, \tag{1.6}$$

then the relation (1.2) holds.

The condition (1.6) is more general than (1.3) in Theorem B.

Theorem 1 may be generalized in the following form:

Theorem 2. *If* $0<\beta<\gamma$, $\alpha=\beta/(\gamma-\beta+1)$, $0<\alpha<2$ *and*

$$\psi_\beta(t) \equiv \frac{1}{\Gamma(\beta)}\int_0^t (t-u)^{\beta-1}\psi(u)\,du = o(t^\gamma), \quad \text{as } t\to 0,$$

then the relation (1.2) holds.

Concerning the summability of the sequence $\{nB_n(x)\}$, we get similar theorems:

Theorem 3. *Under the assumption of Theorem 1, we have*

$$\lim_{n\to\infty} nB_n(x) = -\frac{1}{\pi}L(x) \quad (C,\ 1+\alpha). \tag{1.7}$$

Theorem 4. *Under the assumption of Theorem 2, we have the relation (1.7).*

2. For the proof of above theorems, we need the following lemmas.

Lemma 1.[5] *If* $\alpha>-1$ *and* $\bar{\tau}_n^\alpha(x)$ *denotes the n-th Cesàro mean of order* α *of the sequence* $\{nB_n(x)\}$, *then*

$$\bar{\tau}_n^\alpha(x) = n\{\bar{\sigma}_n^\alpha(x) - \bar{\sigma}_{n-1}^\alpha(x)\}.$$
$$\bar{\tau}_n^{\alpha+1}(x) = (\alpha+1)\{\bar{\sigma}_n^\alpha(x) - \bar{\sigma}_n^{\alpha+1}(x)\}.$$

Lemma 2.[6] *If* $g_n^\alpha(t)$ *denotes the n-th Cesàro mean of order* α *of the sequence* $\{g_n(t)\}$, *where* $g_n(t)=\cos nt\ (n\geqq 1)$ *and* $g_0(t)=1/2$, *then we have*

$$\left|\left(\frac{d}{dt}\right)^k g_n^\alpha(t)\right| \begin{cases} \leqq An^k & (k\geqq 0), \\ \leqq An^{-2}t^{-k-2} & (k\leqq \alpha-2), \\ \leqq An^{k-\alpha}t^{-\alpha} & (k>\alpha-2), \end{cases}$$

for $\alpha>0$, $0<t<\pi$ *and* $k=0, 1, 2, \dots$.

Lemma 3.[7] *If* $h_n^\alpha(t) = \sum_{\nu=n+1}^{2n} g_\nu^\alpha(t)/\nu$, *then*

$$\left|\left(\frac{d}{dt}\right)^k h_n^\alpha(t)\right| \begin{cases} \leqq An^k & (k\geqq 0), \\ \leqq An^{-2}t^{-k-2} & (k\leqq \alpha-1), \\ \leqq An^{k-\alpha-1}t^{-\alpha-1} & (k>\alpha-1), \end{cases}$$

for $\alpha>0$, $0<t<\pi$ *and* $k=1, 2, \dots$.

We shall prove Theorem 1. After H. C. Chow, we write

5) E. Kogbetliantz: Mémorial des Sciences Math., **5**, 23–30 (1931) (cf. Chow: loc. cit.).

6) Cf. Chow: Loc. cit. and A. Zygumund: Trigonometrical series, 258–259 (1935).

7) Cf. Chow: Loc. cit.

$$nB_n(x)=\frac{n}{\pi}\int_0^\pi \{f(x+t)-f(x-t)\}\sin nt\,dt$$
$$=-\frac{1}{\pi}\int_0^\pi \Big\{f(x+t)-f(x-t)\Big\}\frac{d}{dt}\cos nt\,dt,$$

then

$$\bar{\tau}_n^\alpha(x)=-\frac{1}{\pi}\int_0^\pi \Big\{f(x+t)-f(x-t)\Big\}\frac{d}{dt}g_n^\alpha(t)dt, \tag{2.1}$$

and hence, by Lemma 1,

$$\bar{\sigma}_{2n}^\alpha(x)-\bar{\sigma}_n^\alpha(x)=-\frac{1}{\pi}\int_0^\pi \Big\{f(x+t)-f(x-t)\Big\}\frac{d}{dt}h_n^\alpha(t)\,dt. \tag{2.2}$$

If we put $\Omega_n=-\frac{1}{\pi}\int_0^\pi \frac{d}{dt}h_n^\alpha(t)\,dt$, then

$$-\pi\Big[\bar{\sigma}_{2n}^\alpha(x)-\bar{\sigma}_n^\alpha(x)-\Omega_n L(x)\Big]=\int_0^\pi \psi(t)\frac{d}{dt}h_n^\alpha(t)\,dt=I,$$

say. Since the sequence $\{(-1)^n-1\}$ is summable (C,α) to -1 for every $\alpha>0$, it follows that $\Omega_n=\frac{1}{\pi}\log 2+o(1)$, as $n\to\infty$. Therefore it is sufficient to show that $I=o(1)$ as $n\to\infty$. We devide the integral I into two parts such that

$$I=\int_0^{k/n^{\alpha/(1+\alpha)}}+\int_{k/n^{\alpha/(1+\alpha)}}^\pi=I_1+I_2,$$

where, by Lemma 3,

$$|I_2|\leqq\frac{A}{n^\alpha}\int_{k/n^{\alpha/(1+\alpha)}}^\pi |\psi(t)|t^{-\alpha-1}\,dt\leqq\frac{A}{n^\alpha}\frac{n^\alpha}{k^{1+\alpha}}\int_0^\pi|\psi(t)|dt\leqq Ak^{-1-\alpha}.$$

On the other hand, we get, by integration by parts,

$$I_1=\Big[\psi_1(t)\frac{d}{dt}h_n^\alpha(t)\Big]_0^{k/n^{\alpha/(1+\alpha)}}-\int_0^{k/n^{\alpha/(1+\alpha)}}\psi_1(t)\Big(\frac{d}{dt}\Big)^2 h_n^\alpha(t)\,dt.$$

When k is fixed,

$$I_1=\Big[\psi_1(t)\frac{d}{dt}h_n^\alpha(t)\Big]_0^{1/n}+\Big[\psi_1(t)\frac{d}{dt}h_n^\alpha(t)\Big]_{1/n}^{k/n^{\alpha/(1+\alpha)}}$$
$$-\Big\{\int_0^{1/n}+\int_{1/n}^{k/n^{\alpha/(1+\alpha)}}\Big\}\psi_1(t)\Big(\frac{d}{dt}\Big)^2 h_n^\alpha(t)\,dt$$
$$=o\Big(n[t^{1/\alpha}]_0^{1/n}\Big)+o\Big(\frac{1}{n^\alpha}[t^{1/\alpha-\alpha-1}]_{1/n}^{k/n^{\alpha/(1+\alpha)}}\Big)$$
$$+o\Big(n^2\int_0^{1/n}t^{1/\alpha}\,dt\Big)+o\Big(n^{1-\alpha}\int_{1/n}^{k/n^{\alpha/(1+\alpha)}}t^{1/\alpha-\alpha-1}dt\Big)$$
$$=o(1)+o(1/n^{1/\alpha-1})+o(k^{1/\alpha-\alpha-1}/n^{1/(1+\alpha)})+o(1/n^{1/\alpha-1})$$
$$+o(1/n^{1/\alpha-1})+o(k^{1/\alpha-\alpha}/n^{\alpha-1+\frac{1-\alpha^2}{\alpha}\frac{\alpha}{1+\alpha}})$$
$$=o(1),$$

as $n\to\infty$. Hence we have

$$\overline{\lim_{n\to\infty}}\,\pi|\bar{\sigma}_{2n}^\alpha(x)-\bar{\sigma}_n^\alpha(x)-\Omega_n L(x)|\leqq Ak^{-1-\alpha}.$$

Letting $k \to \infty$, we get

$$\lim_{n\to\infty} [\bar{\sigma}_{2n}^{\alpha}(x) - \bar{\sigma}_{n}^{\alpha}(x)] = \frac{\log 2}{\pi} L(x),$$

which is the required.

As we may see by Lemma 3, the order of the kernel $\frac{d}{dt} h_n^{\alpha}(t)$ equals to the order of the Fejér kernel, and the proof of Theorem 2 is reduced to prove $I = o(1)$, as in Theorem 1. While the estimation of I is similar as in the proof of the Izumi and Sunouchi theorem concerning the (C, α) summability of Fourier series.[8] Hence we omit the detail, concerning the proof of Theorem 2.

Instead of Lemma 3 and (2.2) if we use Lemma 2 and (2.1), then we can prove Theorems 3 and 4 similarly as Theorems 1 and 2.

We end this paper by a remark: S. Izumi[9] showed that for the case $\alpha = 1$ in Theorem A, the second condition of (1.1) can be replaced by the following Lebesgue type condition:

$$\int_t^{\pi} \frac{|\psi(t+u) - \psi(u)|}{u} du = O(1), \text{ as } t \to 0.$$

However this fact holds not only for the case $\alpha = 1$, but also for the case $\alpha > 0$, which is obvious from K. Yano's argument.[10]

Added in Proof. We get the following[11]

Corollary. *If the integral*

$$\tilde{f}(x) = -\frac{1}{2\pi} \int_{\to 0}^{\pi} \theta(t) \cot \frac{t}{2} dt$$

exists as a "Cauchy integral" at the origin, where $\theta(t) = f(x+t) - f(x-t)$, *then one of the following conditions is sufficient for the* (C, α) $(\alpha > 0)$ *summability of the conjugate Fourier series of* $f(x)$;

1°. $$\int_0^t \theta(u)\, du = o(t) \quad (t \to 0) \quad \text{and} \quad \int_0^t |\theta(u)|\, du = O(t),$$

2°. $$\int_0^t \theta(u)\, du = o(t^{1/\alpha}), \quad (0 < \alpha < 1),$$

3°. $$\theta_{\beta}(t) \equiv \frac{1}{\Gamma(\beta)} \int_0^t \theta(u)\,(t-u)^{\beta-1}\, du = o(t^{\gamma})\ (t \to 0),$$

$(0 < \beta < \gamma,\ \alpha = \beta/(\gamma - \beta + 1),\ 0 < \alpha < 1)$.

This is the analogue of the Cesàro summability theorem of Fourier series.[12]

Proof is easy from Theorems C, 3, 4 and the following result due to Hardy-Littlewood.[13]

Lemma 4. *If* $\sum u_n$ *is summable* (A), *then a necessary and sufficient condition that it should be summable* (C, α), $\alpha > -1$, *is that the sequence* $\{nu_n\}$ *is summable* $(C, 1+\alpha)$ *to the value* 0.

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8) S. Izumi and G. Sunouchi: Tôhoku Math. Journ., (2) **1**, 313–326 (1950).
9) S. Izumi: Journ. Math. Soc., Japan, **1**, 226–231 (1949).
10) K. Yano: Loc. cit.
11) Cf. R. Mohanty and N. Nanda: Proc. American Math. Soc., **5**, 79–84 (1654).
12) Cf. S. Izumi and G. Sunouchi: Loc. cit.
13) G. H. Hardy and J. E. Littlewood: Journ. London Math. Soc., **6**, 283 (1931).