26. Convergence of Fourier Series

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1. G. H. Hardy and J. E. Littlewood [1] proved the following theorem concerning the convergence of Fourier series at a point.

Theorem HL. *If*

$$\int_0^t \left|\varphi_x(u)\right| du = o\left(t \Big/ \log \frac{1}{t}\right) \qquad (t\to 0) \tag{1}$$

and

$$\int_0^t |\, d(u^\Delta \varphi_x(u))\,| = O(t) \qquad (\Delta>1), \tag{2}$$

then the Fourier series of $f(t)$ converges at $t=x$.

Recently G. Sunouchi [2] proved the following

Theorem S. *If (1) holds and*

$$\lim_{k\to\infty} \limsup_{h\to 0} \int_{(hk)^{1/\Delta}}^{\eta} \left| \frac{\varphi_x(t)-\varphi_x(t+h)}{t} \right| dt = 0 \qquad (\Delta>1,\ \eta>0), \tag{3}$$

then the Fourier series of $f(t)$ converges at $t=x$.

The object of this paper is to prove a convergence theorem similar to Theorem S, replaced the first condition by the weaker in order and the second condition by the stronger. More precisely we prove the following

Theorem 1. *Let $0<\alpha<1$. If*

$$\varphi_x(t)-\varphi_x(t') = o\left(1 \Big/ \left(\log \frac{1}{|t-t'|}\right)^\alpha\right) \qquad (t, t'\to 0) \tag{4}$$

and

$$\lim_{n\to\infty} \int_{\pi e^{(\log n)^\alpha}/n}^{\eta} \left| \frac{\varphi_x(t)-\varphi_x(t+\pi/n)}{t} \right| dt = 0 \qquad (\eta>0), \tag{5}$$

then the Fourier series of $f(t)$ converges at $t=x$.

As S. Izumi and G. Sunouchi [3] have proved, in the case $\alpha \geqq 1$ the Fourier series of $f(t)$ converges uniformly at $t=x$ without the second condition.

Theorem 2. *Let $\alpha>0$. If*

$$\varphi_x(t)-\varphi_x(t') = o\left(1 \Big/ \left(\log\log \frac{1}{|t-t'|}\right)^\alpha\right) \qquad (t, t'\to 0) \tag{6}$$

and

$$\lim_{n\to\infty} \int_{\pi e^{(\log\log n)^\alpha}/n}^{\eta} \left| \frac{\varphi_x(t)-\varphi_x(t+\pi/n)}{t} \right| dt = 0 \qquad (\eta>0) \tag{7}$$

then the Fourier series of $f(t)$ converges at $t=x$.

In Theorems 1 and 2, if the conditions (5) and (7) are replaced by

$$a_n=O(e^{(\log n)^\alpha}/n) \quad (0<\alpha<1), \quad a_n=O(e^{(\log\log n)^\alpha}/n) \quad (\alpha>1)$$

respectively, then the Fourier series converges uniformly at x, where a_n is the n-th Fourier cosine coefficients of $\varphi_x(t)$ (cf. [4]).

2. Proof of Theorem 1. We assume $x=0$ and $f(0)=0$, and further put $\varphi_0(t)=\varphi(t)$.

$$\begin{aligned} s_n(0)&=\frac{1}{\pi}\int_0^\pi \varphi(t)\frac{\sin nt}{t}\,dt+o(1)\\ &=\frac{1}{\pi}\left[\int_0^{\pi/n}+\int_{\pi/n}^{\pi e^{\beta(\log n)^\alpha}/n}+\int_{\pi e^{\beta(\log n)^\alpha}/n}^{\pi}\right]+o(1)\\ &=\frac{1}{\pi}\Big[I+J+K\Big]+o(1), \end{aligned}$$

say, where β is the least number $\geqq 1$ such that $e^{\beta(\log n)^\alpha}$ is an odd integer. We can see $I=o(1)$ and

$$\begin{aligned} J&=\int_{\pi/n}^{\pi e^{\beta(\log n)^\alpha}/n}\varphi(t)\frac{\sin nt}{t}\,dt=\sum_{k=1}^{\rho-1}\int_{k\pi/n}^{(k+1)\pi/n}\varphi(t)\frac{\sin nt}{t}\,dt\\ &=\int_{\pi/n}^{2\pi/n}\sum_{k=0}^{\rho-2}(-1)^k\varphi\Big(t+\frac{k\pi}{n}\Big)\frac{\sin nt}{t+k\pi/n}\,dt \end{aligned}$$

where $\rho=e^{\beta(\log n)^\alpha}$. By the first mean value theorem for $\pi/n\leqq\theta\leqq 2\pi/n$,

$$\begin{aligned} J&=-2\sum_{k=0}^{\rho-2}\frac{(-1)^k}{n\theta+k\pi}\varphi(\theta+k\pi/n)\\ &=-\frac{2}{\pi}\sum_{k=0}^{(\rho-3)/2}\frac{1}{2k+1}\left[\varphi\Big(\frac{2k\pi}{n}+\theta\Big)-\varphi\Big(\frac{(2k+1)\pi}{n}+\theta\Big)\right]+o(1)\\ &=o\Big(\frac{1}{(\log n)^\alpha}\sum_{k=0}^{(\rho-3)/2}\frac{1}{2k+1}\Big)=o\Big(\frac{1}{(\log n)^\alpha}\log\rho\Big)=o(1). \end{aligned}$$

For the proof of $K=o(1)$, we divide the integral into two parts such that

$$K=\left[\int_{\pi\rho/n}^{\eta}+\int_{\eta}^{\pi}\right]\varphi(t)\frac{\sin nt}{t}\,dt=K_1+K_2,$$

where η is a positive number $<\pi$, then we have easily $K_2=o(1)$, since $\varphi(t)$ is Lebesgue integrable. And

$$\begin{aligned} K_1&=\int_{\pi\rho/n}^{\eta}\varphi(t)\frac{\sin nt}{t}\,dt\\ &=\left[-\int_{\pi\rho/n}^{\eta}+\int_{\eta-\pi/n}^{\eta}-\int_{\pi\rho/n-\pi/n}^{\pi\rho/n}\right]\varphi\Big(t+\frac{\pi}{n}\Big)\frac{\sin nt}{t+\pi/n}dt\\ &=-\int_{\pi\rho/n}^{\eta}\varphi\Big(t+\frac{\pi}{n}\Big)\frac{\sin nt}{t+\pi/n}\,dt+o(1)\\ &=\frac{1}{2}\int_{\pi\rho/n}^{\eta}\Big\{\frac{\varphi(t)-\varphi(t+\pi/n)}{t}\Big\}\sin nt\,dt\\ &\qquad+\frac{1}{2}\int_{\pi\rho/n}^{\eta}\varphi(t+\pi/n)\Big\{\frac{1}{t}-\frac{1}{t+\pi/n}\Big\}\sin nt\,dt+o(1), \end{aligned}$$

where

$$\frac{\pi}{n}\int_{\pi\rho/n}^{\eta}\frac{\varphi(t+\pi/n)}{t(t+\pi/n)}\sin nt\,dt=o(1).$$

Thus we have $K=o(1)$. Therefore the theorem is proved.

3. **Proof of Theorem 2.** Similarly as in the proof of the previous theorem, we assume $x=0$, $f(0)=0$ and further we divide the integral into three parts such that

$$\begin{aligned} s_n(0)&=\frac{1}{\pi}\int_0^{\pi}\varphi(t)\frac{\sin nt}{t}\,dt+o(1)\\ &=\frac{1}{\pi}\left[\int_0^{\pi/n}+\int_{\pi/n}^{\pi e^{\beta(\log\log n)^{\alpha}}/n}+\int_{\pi e^{\beta(\log\log n)^{\alpha}}/n}^{\pi}\right]+o(1)\\ &=\frac{1}{\pi}\Big[I+J+K\Big]+o(1)\end{aligned}$$

where β is the least number $\geqq 1$ such that $e^{\beta(\log\log n)^{\alpha}}$ is an odd integer. Similarly as in the proof of Theorem 1, we can prove that $I=o(1)$, $J=o(1)$, and $K=o(1)$.

4. *More generally we can prove the following*

Theorem 3. *If* $\lambda(n)\to\infty$,

$$\varphi_x(t)-\varphi_x(t')=o\left(1\Big/\lambda\left(\frac{1}{|t-t'|}\right)\right) \qquad (t,t'\to 0)$$

and

$$\lim_{n\to\infty}\int_{\pi e^{\lambda(n)}/n}^{\eta}\left|\frac{\varphi(t)-\varphi(t+\pi/n)}{t}\right|dt=0 \qquad (\eta>0),$$

then the Fourier series of $f(t)$ *converges at* $t=x$.

The theorem has the significance in the case $\lambda(n)=O(\log n)$ only.

References

[1] G. H. Hardy and J. E. Littlewood: Some new convergence criteria for Fourier series, Annali di Pisa, **3**, 43–62 (1934).

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[4] M. Satô: Uniform convergence of Fourier series. III, Proc. Japan Acad., **30**, 809–813 (1954).