167. Vector-space Valued Functions on Semi-groups. III

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In this Note, we shall define the Maak function and prove the existence for almost periodic functions. We shall use the terminologies in my Note [5], [7]. The method is due to W. Maak [3].

V. Fundamental theorem on almost periodic function

Let $f(x)$ be an almost periodic function on a semi-group G with unit into a locally convex vector space E. For any nbd U of E, we have a minimal decomposition of G. The following propositions are clear.

Proposition 5.1. For any nbd U and an almost periodic function, G has a minimal decomposition.

Proposition 5.2. Let $\{A_i\}$ $i=1, 2,\dots, n$ be a minimal decomposition of G for any almost periodic function, then for a, b of G,

$$A_i \frown aGb \neq 0 \qquad (i=1, 2,\dots, n).$$

(For the details, see W. Maak [3].)

Theorem 12. For an almost periodic function on a semi-group, and any element x of G,

$$f(axb) \in U$$

implies

$$f(x) \in U.$$

Proof. Let V be a nbd of E, and $\{A_i\}$ a minimal decomposition of G for U. From Proposition 5.2, we can find A_i and h_i' of G such that

$$x \in A_i, \qquad ah_i'b \in A_i.$$

Hence

$$f(x)=\{f(x)-f(ah_i'b)\}+f(ah_i'b) \in V+U$$

this shows $f(x) \in U$.

From Theorem 12, we have the following

Corollary 12.1. Let $f(x)$ be almost periodic on a semi-group G. For any nbd U, let $\{A_i\}$ be a minimal decomposition of G. Then $a, b \in G$ and $x, y \in A_i$ implies

$$f(axb)-f(ayb) \in U.$$

By Proposition 5.2 and Corollary 12.1, we have

Theorem 13. Let $f(x)$ be almost periodic on a semi-group G. For any nbd U, and x, a, b of G, there is an element x' such that

$$f(cxd)-f(cax'bd) \in U$$

for every c, d of G.

VI. The existence and the uniqueness of Maak function

Let $f(x, y)$ be a function on $G\times G$ into E.

Definition 5. $f(x, y)$ is called *Maak function* on G, if

(18) $f(x, y)$ is almost periodic of x for every fixed y.

(19) $f(x, 1)\equiv f(x)$ for all x.

(20) $f(xa, ya)=f(x, y)$ for every x, y of G.

Let $f(x)$ be an almost periodic function G. By Theorem 13, for a given nbd U and y of G, there is an element y' such that

$$f(cd)-f(cyy'd)\in U$$

for all c, d of G.

Let $f_U(x, y)=f(x, y')$, then $f_U(x, y)$ is almost periodic of x for each y.

Lemma. For given nbds U_1, U_2,

$$f_{U_1}(x, y)-f_{U_2}(x, y)\in U_1+U_2.$$

The idea of the proof is due to W. Maak [3].

Proof. For U_1, U_2, there are y_1', y_2' such that

$$f_{U_1}(x, y)=f(x, y_1'), \qquad f_{U_2}(x, y)=f(x, y_2').$$

By Theorem 13, for any nbd U, we can find x' such that

$$\begin{aligned} f(x, y_1')-f(x, y_2')&=\{f(x, y_1')-f(x'yy_1')\}\\ &+\{f(x'yy_1')-f(x')\}+\{f(x')-f(x'yy_2')\}+\{f(x'yy_2')-f(xy_2')\}\\ &\in U+U+\{f(x'yy_1')-f(x')\}+\{f(x')-f(x'yy_2')\}.\end{aligned}$$

Therefore, since $f(x'yy_1')-f(x')\in U_1$, $f(x')-f(x'yy_2')\in U_2$, we obtain

$$f_{U_1}(x, y)-f_{U_2}(x, y)\in U_1+U_2. \qquad \text{Q.E.D.}$$

Any metrisable and complete locally convex vector space is called (F)-space. The excellent treaties of (F)-space is in A. Grothendieck ([6], pp. 155–165).

Especially, if E is (F)-space, by Theorem 3 and Lemma, there is the limit of $f_U(x, y)$ relative to U. Let $f(x, y)$ be the limit function $f_U(x, y)$ for $U\to 0$. Then $f(x, y)$ is almost periodic of x for every y of G.

Theorem 14. For every almost periodic function $f(x)$ on G to a (F)-space E, there exists the Maak function $f(x, y)$ of the function $f(x)$ on G. Such a function $f(x, y)$ is unique.

Proof. We shall show that the function $f(x, y)$ constructed above for a given almost periodic function $f(x)$ is a Maak function on G. It is clear that $f(x, y)$ satisfies the condition (18). To prove that $f(x, 1)\equiv f(x)$ for every x, let $f(x, 1)=f(x, y')$ for nbd U, then

$$f_U(x, 1)-f(x)=f(xy')-f(x)\in U.$$

For $U\to 0$, we have $f(x, 1)=f(x)$. This shows the condition (19). We must prove $f(xa, ya)=f(x, y)$. The idea of the proof is due to W. Maak [3]. For a nbd U, let

$$f_U(xa, ya)=f(xa(ya)'), \qquad f_U(x, y)=f(xy'),$$

then, we have, by Theorem 13

$$\begin{aligned} & f(xa(ya)')-f(xy') \\ &= \{f(xa(ya)')-f(x'ya(ya)')\} + \{f(x'ya(ya)')-f(x')\} \\ &\quad + \{f(x')-f(x'yy')\} + \{f(x'yy')-f(xy')\} \\ &\in U+U+\{f(x'ya(ya)')-f(x')\} + \{f(x')-f(x'yy')\} \\ &\in U+U+U+U. \end{aligned}$$

Hence, we have

$$f_U(xa, ya)-f_U(x, y) \in U+U+U+U.$$

This shows $f(xa, ya)=f(x, y)$. On the uniqueness, let $f_1(x, y)$, $f_2(x, y)$ be two Maak functions for $f(x)$, then

$$f_i(xy, y)=f_i(x, 1)=f_i(x) \quad (i=1, 2). \tag{21}$$

It is easily seen from (21) that $f_1(x, y)=f_2(x, y)$.

References

[6] A. Grothendieck: Théorie des espaces vectoriels topologiques, San Paulo (1954).
[7] K. Iséki: Vector-space valued functions on semi-groups. II, Proc. Japan Acad., **31**, 152–155 (1955).