6. On the Convergence Character of Fourier Series. II

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1. Let $f(x)$ be an integrable function with period 2π and $s_n(x)$ be the nth partial sum of its Fourier series. S. Izumi[1] has proved the following

Theorem I. *If $f(x)$ belongs to the Lip $\alpha(0<\alpha\leqq 1)$ class, then the series*

$$\sum_{n=2}^{\infty} | s_n(x)-f(x) |^2/n^\beta (\log n)^\gamma$$

converges uniformly, where $\beta=1-2\alpha$ and $\gamma>1$ or >2, according as $0<\alpha<1/2$ or $1/2\leqq\alpha\leqq 1$.

In a previous paper,[2] we have shown that Theorem I is still valid even if the restriction $\gamma>2$ is replaced by $\gamma>1$ for $\alpha=1/2$. The object of this paper is to show that the restriction $\gamma>2$ in Theorem I may be replaced by $\gamma>1$ for $\alpha\geqq 1/2$. In fact we prove

Theorem 1. *Let $1\geqq\alpha>0$ and $k>0$. If $f(x)$ belongs to the Lip α class, then the series*

$$\sum_{n=2}^{\infty} \frac{| s_n(x)-f(x) |^k}{n^\delta (\log n)^\gamma}$$

converges uniformly, where $\delta=1-k\alpha$ and $\gamma>1$.

Proof of Theorem 1.[3] we have

$$\begin{aligned} s_n(x)-f(x) &= \frac{1}{\pi}\int_0^\pi \varphi_x(t) \sin (n+1/2)t/\{2\sin t/2\}\, dt \\ &= \frac{1}{\pi}\int_0^\pi \varphi_x(t)p(t) \sin nt\, dt + \frac{1}{2\pi}\int_0^\pi \varphi_x(t) \cos nt\, dt, \\ &= P_n(x)+Q_n(x), \end{aligned}$$

where $\varphi_x(t)=\varphi(t)=f(x+t)+f(x-t)-2f(x)$ and $p(t)=\cos t/2/\{2\sin t/2\}$.

We may take a number p' such that $p'\geqq 2$, $p'\geqq k$ and $p'>1/\alpha$ for given α and k.

By the Hausdorff-Young inequality, we get[4]

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1) S. Izumi: Some trigonometrical series. III, Proc. Japan Acad., **31**, 257–260 (1955).

2) M. Kinukawa: On the convergence character of Fourier series, Proc. Japan Acad., **31**, 513–516 (1955).

3) M. Kinukawa: Some strong summability of Fourier series (to appear).

4) A denotes an absolute constant, which may be different in each occurrence, and p' denotes the conjugate number of p, that is, $1/p+1/p'=1$.

$$\left\{\sum_{n=1}^{\infty} | P_n(x) \sin nh |^{p'}\right\}^{p/p'} \leqq A \int_0^{\pi} | \varphi(t+h)p(t+h)-\varphi(t-h)p(t-h) |^p dt$$

$$\leqq A\left\{\int_0^{\pi} | \varphi(t+h)-\varphi(t-h) |^p \, | p(t+h) |^p dt\right.$$

$$\left. + \int_0^{\pi} | \varphi(t-h) |^p \, | p(t+h)-p(t-h) |^p dt\right\}$$

$$= A\{I(x)+J(x)\},$$

where

$$(1) \qquad I(x) \leqq h^{p\alpha} \int_0^{\pi} \frac{dt}{(t+h)^p} \leqq A h^{p\alpha-p+1}$$

We divide $J(x)$ into two parts such that

$$J(x) = \int_0^h + \int_h^{\pi} = J_1(x) + J_2(x),$$

where

$$J_1(x) \leqq A \int_0^h | \varphi(t) |^p \{| p(t+2h) |^p + | p(t) |^p\} dt \leqq \int_0^h t^{\alpha p-p} dt$$

$$(2) \qquad \leqq A\, h^{\alpha p-p+1}$$

since $\alpha p-p>-1$ by the assumption $\alpha>1/p'$, and

$$J_2(x) = \int_h^{\pi} | \varphi(t-h) |^p \, | p(t+h)-p(t-h) |^p dt$$

$$= \int_0^{\pi-h} | \varphi(t) |^p \, | p(t+2h)-p(t) |^p dt \leqq \int_0^h + \int_h^{\pi}$$

$$(3) \qquad \leqq A\, h^{\alpha p-p+1} + A\, h^p \int_h^{\pi} t^{\alpha p-2p} dt \leqq A\, h^{\alpha p-p+1}, \text{ for } 0<h<1,$$

since $\alpha p-2p<-1$.

Summing up the estimations (1), (2) and (3), we get

$$\left\{\sum_{n=1}^{\infty} | P_n(x) \sin nh |^{p'}\right\}^{p/p'} \leqq A\, h^{\alpha p-p+1}$$

Let $h=\pi/2^{(\lambda+1)}$, then we can easily see that

$$\left\{\sum_{n=2^{\lambda-1}+1}^{2^\lambda} | P_n(x) |^{p'}\right\}^{p/p'} \leqq A\, 2^{\lambda(p-1-\alpha p)}$$

Thus we have

$$(4) \qquad \sum_{n=2^{\lambda-1}+1}^{2^\lambda} |P_n(x) |^{p'} \leqq A\, 2^{\lambda(p-1-\alpha p)p'/p} \leqq A\, 2^{\lambda(1-p'\alpha)}.$$

We may consider the case $0<k<p'$. In this case we get by the Hölder inequality,

$$\sum_{n=2^{\lambda-1}+1}^{2^\lambda} | P_n(x) |^k \leqq 2^{\lambda/q}\left(\sum_{n=2^{\lambda-1}+1}^{2^\lambda} | P_n(x) |^{kq'}\right)^{1/q'},$$

where $kq'=p'$ and $q=p'/(p'-k)$. Hence, by (4),

$$(5) \qquad \sum_{n=2^{\lambda-1}+1}^{2^\lambda} | P_n(x) |^k \leqq A\, 2^{\lambda[1/q+(1-p'\alpha)/q']} \leqq A\, 2^{\lambda(1-k\alpha)}.$$

In the case $p=k$, we get also (5).

For the proof of the theorem, it is sufficient to show that the series

$$\sum_{n=2}^{\infty} | P_n(x) |^k / n^\delta (\log n)^\gamma$$

is convergent, since the corresponding series containing $Q_n(x)$ converges obviously.

$$\sum_{n=2}^{\infty} | P_n(x) |^k / n^\delta (\log n)^\gamma = \sum_{\lambda=1}^{\infty} \sum_{n=2^{\lambda-1}+1}^{2^\lambda} | P_n(x) |^k / n^\delta (\log n)^\gamma$$

$$\leqq A \sum_{\lambda=1}^{\infty} \frac{1}{2^{\lambda\delta}\lambda^\gamma} \sum_{n=2^{\lambda-1}+1}^{2^\lambda} | P_n(x) |^k \leqq A \sum_{\lambda=1}^{\infty} \frac{1}{\lambda^\gamma} < \infty .$$

Thus we have proved the theorem completely.

2. In this section we shall prove

Theorem 2. *Let $0<\alpha<1$ and $0<k$. If*

$$| f(x+t)-f(x) | \leqq A | t |^\alpha \Big/ \left(log \frac{1}{|t|} \right)^\gamma ,$$

uniformly, then the series

$$\sum_{n=1}^{\infty} | s_n(x) - f(x) |^k / n^\delta$$

converges uniformly, where $\delta=1-k\alpha$ and $\gamma>1/k$.

Proof of Theorem 2. Using the notation in §1, we have

$$\left\{ \sum_{n=1}^{\infty} | P_n(x) \sin nh |^{p'} \right\}^{p/p'} \leqq A \{ I(x) + J(x) \},$$

where

$$I(x) \leqq A\, h^{p\alpha-p+1} \Big/ \left(\log \frac{1}{h} \right)^{\gamma p}$$

and

$$J(x) \leqq A\, h^{p\alpha-p+1} \Big/ \left(\log \frac{1}{h} \right)^{\gamma p}$$

$$+ h^p \left\{ \int_h^{h^\mu} + \int_{h^\mu}^{\pi} \right\} t^{p\alpha-2p} \Big/ \left(\log \frac{1}{t} \right)^{\gamma p} dt \quad (0<\mu<1)$$

$$\leqq A\, h^{p\alpha-p+1} \Big/ \left(\log \frac{1}{h} \right)^{\gamma p} .$$

Thus we get, by the same way used in §1,

$$\sum_{n=2^{\lambda-1}+1}^{2^\lambda} | P_n(x) |^{p'} \leqq A\, 2^{\lambda(1-p'\alpha)} / \lambda^{\gamma p'} .$$

Hence, by the Hölder inequality,

$$\sum_{n=2^{\lambda-1}+1}^{2^\lambda} | P_n(x) |^k \leqq A\, 2^{\lambda(1-k\alpha)} / \lambda^{\gamma k}, \quad (\lambda=1, 2, \cdots).$$

Summing up these inequalities with respect to λ, we get easily the theorem.