25. An Estimation of the Measure of Linear Sets

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Let R be an abstract Riemann surface and suppose that a conformal metric is given on R, of which a line element ds is given by the local parameter t such that $ds=|\lambda(t)|dt$ and let O be a fixed point of R. Denote by D_ρ the domain bounded by the points having the distance ρ from O and suppose for $\rho<\infty$ that the domain D_ρ is compact and $\bigcup\limits_{\rho>0} D_\rho=R$. The boundary ∂D_ρ of D_ρ is composed of $n(\rho)$ components, $r_1, r_2, \cdots, r_n$. Denote by $\Lambda(\rho)$ the largest length of r_k $(k=1, 2, \cdots, n(\rho))$, that is,

$$l_k=\int\limits_{r_k} ds, \qquad \Lambda(\rho)=\max_k l_k.$$

Put $N(\rho)=\max\limits_{\rho'<\rho} n(\rho)$. *A. Pfluger proved that*

if
$$\limsup_{\rho=\infty}\left[4\pi\int_{\rho_0}^{\rho}\frac{d\rho}{\Lambda(\rho)}-\log N(\rho)\right]=\infty,$$ [1)]

then

$$R\in O_{AB}.$$

The condition of this theorem depends not only the minimum modulus but also on the number of components. In this article we give a condition depending only on the minimum modulus but our criterion is applicable only to a special type of Riemann surface, i.e. the Riemann surface which is planer and whose boundary is a closed set on a straight line. Let $\{R_n\}$ $(n=1, 2, \cdots)$ be the exhaustion of R with compact relative boundaries $\{\partial R_n\}$. The open set $R_{n+1}-R_n$ $(n\geqq 1)$ consists of a finite number of ring domains $G_{i_1,i_2,\cdots,i_n}$ $(i_1=1, 2, \cdots, j_1,\ i_2=1, 2, \cdots, j_2, \cdots,\ i_n=1, 2, \cdots, j_n)$. Let $\omega(z)$ be a harmonic function in $G_{i_1,i_2,\cdots,i_n}$ such that $\omega(z)=0$ on the outer boundary of $G_{i_1,i_2,\cdots,i_n}$ contained in ∂R_n and $\omega(z)=1$ on the inner boundary of $G_{i_1,i_2,\cdots,i_n}$ contained in ∂R_{n+1}. Let $D(\omega(z))$ be the Dirichlet's integral of $\omega(z)$ and put $\mathrm{mod}\,(G_{i_1,i_2,\cdots,i_n})=1/D(\omega(z))$. We call it the modulus of $G_{i_1,i_2,\cdots,i_n}$ and further put $\mathfrak{M}_n=\min\limits_{i_n}\ \mathrm{mod}\,(G_{i_1,i_2,\cdots,i_n})$. Then we can prove the following

Theorem. Let R be a planer domain and suppose that its ideal

1) A. Pfluger: Sur l'existence de fonctions non constants, analytiques, uniformes et borneés sur une surface de Riemann ouverte, C. R. Acad. Sci. Paris, 230 (1950).

boundary lies on the real axis. If every boundary component of $G_{i_1,i_2,\cdots,i_n}$ *is convex and*

$$\sum_{n=1}^{\infty} e^{-\frac{\pi^2}{\mathfrak{M}_n}} = \infty,$$

then the boundary F of R is a set of linear measure zero. In the other words this means $R \in O_{AB}$.

Let G be a simply connected domain whose boundary is the set E of the union of two closed intervals $[\infty, -1]$ and $[1, \infty]$ and let $\bigcup_i S_i$ be a closed set, on the real axis, consisting of a finite number of segments S_i $(i=1, 2, \cdots, n)$ in an open interval $(-1, 1)$. Denote by $(G, \bigcup_i S_i)$ the ring domain whose outer boundary is E and the inner boundary is $\bigcup_i S_i$. We call it A-type ring. Let $U(z)$ be a bounded positive harmonic function in $(G, \bigcup_i S_i)$ such that $U(z)=1$ on $\bigcup_i S_i$ and $U(z)=0$ on E. Let $G(z, p)$ be the Green's function of G with pole at p. To observe the behaviour of the normal derivative of $U(z)$ at $\bigcup_i S_i$, we consider the Riemann surface constructed as follows: let $\widetilde{(G, \bigcup_i S_i)}$ be the same ring as $(G, \bigcup_i S_i)$ and connect $\widetilde{(G, \bigcup_i S_i)}$ and $(G, \bigcup_i S_i)$ crosswise on $\bigcup_i S_i$. Then we obtain a two-sheeted Riemann surface R whose boundary components are Γ_1 and Γ_2 on E. Let $w(z)$ be a harmonic function in R such that $w(z)=0$ on Γ_1 and $w(z)=2$ on Γ_2. Then $w(z) \equiv U(z)$. Hence $\left|\frac{\partial U(z)}{\partial n}\right| < 0\left(\frac{1}{\sqrt{r}}\right)$ in the neighbourhood of end points of $\bigcup_i S_i$, where r is the distance between the set of end points of $\bigcup_i S_i$ and z. Therefore, we have by Green's formula

$$U(p) = \frac{1}{2\pi} \int_{\bigcup_i S_i} G(z, p) \frac{\partial U(z)}{\partial n} ds,$$

where the integration is taken over two sides of $\bigcup_i S_i$. Because $U(z) = 1$ on $\bigcup_i S_i$ and $\frac{\partial G(z, p)}{\partial n}$ is continuous and $\frac{\partial G(z, p)}{\partial n}$ has the same absolute values and opposite signature on two sides of $\bigcup_i S_i$.

In order to study the case when the measure of $\bigcup_i S_i$ of an A-type ring with given modulus is maximal, we consider rings as follows: let S_i^+ or S_i^- be the set of points contained in S_i and lying on the positive or negative real axis and denote by $m^+(z)$ or $m^-(z)$ the linear measure of $\bigcup_i S_i^+$ or $\bigcup_i S_i^-$ contained in the interval $(0, z)$ or $(z, 0)$. Put $m(z) = m^+(z)$ or $-m^-(z)$ according to $z \geqq 0$ or $0 < z$. Then $m(z)$ does not increase or decrease on the complementary set of $\bigcup_i S_i$ with respect to the interval $(-1, 1)$. Let S' be the image of $\bigcup_i S_i$ by $m(z)$ $(-1<z<1)$. Then S' is a closed interval in $(-1, 1)$. By definition, we have the following:

If $z_1 \geqq z_2 \geqq 0$, $m(z_1) \geqq m(z_2)$, $z_i \geqq m(z_i)$ $(i=1,2)$ and $|z_1-z_2| \geqq |m(z_1)-m(z_2)|$.

If $z_1 \leqq z_2 < 0$, $m(z_1) \leqq m(z_2)$, $z_i \leqq m(z_i)$ $(i=1,2)$ and $|z_1-z_2| \geqq |m(z_1)-m(z_2)|$.

If $z_1 \geqq 0 > z_2$, $m(z_1) \geqq m(z_2)$, $z_1 \geqq m(z_1) \geqq m(z_2) \geqq z_2$ and

$$|z_1-z_2| \geqq |m(z_1)-m(z_2)|.$$

Next, we consider the function $f(z)=\dfrac{z-\alpha}{-\alpha z+1}$ by which G is invariant and $f(\alpha)=0$. Then we have by brief computation

$$\left|\frac{z_1-z_2}{-z_1 z_2+1}\right| \geqq \left|\frac{m(z_1)-m(z_2)}{-m(z_1)m(z_2)+1}\right|.$$

Hence
$$G(z_1, z_2) \leqq G(m(z_1), m(z_2)) \tag{1}$$
for every pair of z_1 and z_2 in $(-1, 1)$.

We consider the ring domain (G, S') whose outer boundary is E and the inner boundary is S'. Since $\dfrac{\partial U(z)}{\partial n}$ (>0) is continuous on $\bigcup_i S_i$ except at end points of $\bigcup_i S_i$ where $\dfrac{\partial U(z)}{\partial n} < 0\left(\dfrac{1}{\sqrt{r}}\right)$, we can construct a positive harmonic function $\tilde{U}(z)$ in (G', S) such that $\tilde{U}(z)=0$ on E and $\dfrac{\partial \tilde{U}(m(z))}{\partial n}=\dfrac{\partial U(z)}{\partial n}$ on S'. Then we have by (1) and by Green's formula

$$U(p)=\frac{1}{2\pi}\int_{\bigcup_i S_i} G(z, p)\frac{\partial U(z)}{\partial n}\,ds \leqq \frac{1}{2\pi}\int_{S'} G(m(z), m(p))\frac{\partial \tilde{U}(m(z))}{\partial n}\,ds$$

$$=\tilde{U}(m(p)) \tag{2}$$

because $\tilde{U}(z+iy)=\tilde{U}(z-iy)$. Hence $H \supset S'$, where H is the domain in which $\tilde{U}(z) \geqq 1$. Now the Dirichlet's integrals are

$$\underset{G-\bigcup_i S_i}{D}(U(z))=\int_{\bigcup_i S_i}\frac{\partial U(z)}{\partial n}\,ds=\int_E \frac{\partial U(z)}{\partial n}\,ds=\int_{S'}\frac{\partial \tilde{U}(z)}{\partial n}\,ds=\int_E \frac{\partial \tilde{U}(z)}{\partial n}\,ds$$

and

$$\underset{G-H}{D}(\tilde{U}(z))=\int_E \frac{\partial \tilde{U}(z)}{\partial n}\,ds=\int_E \frac{\partial U(z)}{\partial n}\,ds=\underset{G-\bigcup_i S_i}{D}(U(z)).$$

On the other hand, let $\tilde{\tilde{U}}(z)$ be a harmonic function in (G, S') such that $\tilde{\tilde{U}}(z)=1$ on S' and $\tilde{\tilde{U}}(z)=0$ on E. Then by Dirichlet's principle

$$\underset{G-S'}{D}(\tilde{\tilde{U}}(z)) \leqq \underset{G-H}{D}(\tilde{U}(z)).$$

Translate S' to a closed interval S so that S lies symmetrically with respect to the origin. Then we obtain a ring domain (G, S) whose outer boundary is E and the inner boundary is S. We call it B-type domain. Let $\hat{U}(z)$ be a harmonic function in (G, S) such that $\hat{U}(z)=1$ on S and $\hat{U}(z)=0$ on E. Then we have also $D_{G-S}(\hat{U}(z)) \leqq D_{G-S'}(\tilde{\hat{U}}(z)) \leqq D_{G-\bigcup_i S_i}(U(z))$ as above. Since $\text{mes}\, S=\text{mes}\,(\bigcup_i S_i)$, we have the following

Lemma 1. Let $(G, \bigcup_i S_i)$ and (G, S) be ring domains of types A and B respectively such that $\text{mod}\,(G, \bigcup_i S_i)=\text{mod}\,(G, S)$. Then $\text{mes}\, S \geqq \text{mes}\,(\bigcup_i S_i)$.

It is clear that the ratio $\text{mes}\, S/2$ is a decreasing function of $\text{mod}\,(G, S)$. We denote it by $P(\text{mod}\,(G, S))$. Therefore $\text{mes}\,(\bigcup_i S_i)/2 \leqq \text{mes}\, S/2 = P(\text{mod}\,(G, S)) = P(\text{mod}\,(G, \bigcup_i S_i))$.

Let (G, S) be a ring of type B, where S is a closed interval $[-p, p]$ and consider the function $f(z)=\sqrt{\frac{1+p}{2p}}\left(\frac{z+p}{z+1}\right)$ mapping (G, S) to a ring domain $\left(\infty, 0, r, \frac{1}{r}\right)$ $\left(\text{where } r=\sqrt{\frac{2p}{1+p}}\right)$ whose outer boundary is the union of closed intervals $[\infty, 0]$ and $\left[\frac{1}{r}, \infty\right]$ and the inner boundary is $\left[0, \frac{1}{r}\right]$. Let D be a rectangle: $Im\, z \geqq 0$ with vertices $\infty, 0, r$ and $\frac{1}{r}$ and let $\text{mod}\,(D)$[2)] be its modulus. Then $\text{mod}\,(G, S) = 2\,\text{mod}(D)$. To estimate the ratio $\text{mes}\, S/2\,(=p)$, when $\text{mod}\,(G, S)\to 0$ in other words when $p\to 1$, we consider the behaviour of $\text{mod}\,(D)$ as $r\to 1$.

Lemma 2. $$\varlimsup_{r\to 1} \text{mod}\,(D) \log \frac{1}{1-r} \leqq \frac{\pi^2}{2}.$$

By Schwarz-Christoffel's transformation, we have

$$\text{mod}\,(D)=\pi \int_r^1 \frac{dt}{\sqrt{t(t-r)(1-rt)}} \Big/ \int_0^r \frac{dt}{\sqrt{t(r-t)(1-rt)}}. \tag{3}$$

We take $1-r$ and $\delta>0$ small enough and divide the integral in the denominator into two parts:

$$\int_0^r = \int_0^{r-\delta} + \int_{r-\delta}^r .$$

2) We map the rectangle D: $\left[\infty, 0, \frac{1}{r}, r\right]$ onto a ring $1 \leqq |\zeta| \leqq e^m$ of which $|\zeta|=e^m$, $|\zeta|=1$ and $1\leqq \zeta \leqq e^m$ correspond to $[0, r]$, $\left[\frac{1}{r}, \infty\right]$ and the union of $\left[r, \frac{1}{r}\right]$ and $[\infty, 0]$ respectively. In this case, we define $\text{mod}\, D$ by the modulus of this ring, i.e. by m.

Then

$$2\sqrt{\frac{r}{r-\delta}}=\frac{1}{\sqrt{r}}\int_0^{r-\delta}\frac{dt}{\sqrt{t}}<\int_0^{r-\delta}<\frac{1}{\delta\sqrt{r}}\int_0^{r-\delta}\frac{dt}{\sqrt{t}}=\frac{2}{\delta}\sqrt{\frac{r-\delta}{r}},$$

$$\frac{1}{\sqrt{r}}\int_{r-\delta}^{r}\frac{dt}{\sqrt{(r-t)(1-rt)}}<\int_{r-\delta}^{r}<\frac{1}{\sqrt{r-\delta}}\int_{r-\delta}^{r}\frac{dt}{\sqrt{(r-t)(1-rt)}}$$

$$\int_{r-\delta}^{r}\frac{dt}{\sqrt{(r-t)(1-rt)}}=\frac{2}{\sqrt{r}}\left[\log\left(\sqrt{\delta}+\sqrt{\frac{1}{r}-r+\delta}\right)-\log\sqrt{\frac{1}{r}-r}\right].$$

Hence $\frac{1}{r}\log\frac{1}{1-r}+m_1(r,\delta)<\int^{r}<\frac{1}{\sqrt{r(r-\delta)}}\log\frac{1}{1-r}+m_2(r,\delta)$ (4)

where $m_\nu(r,\delta)$ $(\nu=1,2)$ remains bounded for r and δ. On the other hand,

$$\frac{1}{\sqrt{r}}\int_r^1\frac{dt}{\sqrt{(t-r)\left(\frac{1}{r}-t\right)}}<\int_r^1<\frac{1}{r}\int_r^1\frac{dt}{\sqrt{(t-r)\left(\frac{1}{r}-t\right)}},$$

$$\int_r^1\frac{dt}{\sqrt{(t-r)\left(\frac{1}{r}-t\right)}}=-2\left[\text{arc tan}\sqrt{\frac{1/r-t}{t-r}}\right]_r^1=\pi-2\,\text{arc tan}\sqrt{\frac{1}{r}}.$$

Hence

$$\frac{1}{\sqrt{r}}\left(\pi-2\,\text{arc tan}\sqrt{\frac{1}{r}}\right)<\int_r^1<\frac{1}{r}\left(\pi-2\,\text{arc tan}\sqrt{\frac{1}{r}}\right). \tag{5}$$

Divide (5) by (4) and let r tend to 1, δ being fixed. Then

$$\frac{\sqrt{1-\delta}\,\pi^2}{2}\varliminf_{r\to1}\text{mod}\,(D)\log\frac{1}{1-r}\leqq\varlimsup_{r\to1}\text{mod}\,(D)\log\frac{1}{1-r}\leqq\frac{\pi^2}{2}.$$

Since δ was can be chosen arbitrarily small, our assertion follows when we let δ tend to 0.

Put $\mathfrak{M}=\text{mod}\,(G,\bigcup_i S_i)$. We consider $P(\mathfrak{M})$, when $\mathfrak{M}\to0$, i.e. $p\to1$; and $r\to1$. In this case, since $r=\sqrt{\frac{2p}{1+p}}$, and by Lemma 2, we have a brief computation

$$\frac{\text{mes}\,(\bigcup_i S_i)}{2}\leqq P(\mathfrak{M})=p=\frac{r^2}{1-r^2}\leqq\frac{(e^{-\frac{\pi^2}{\mathfrak{M}}}-1)^2}{2-(e^{-\frac{\pi^2}{\mathfrak{M}}}-1)^2}=1-\varepsilon_n,$$

where $\varepsilon_n\geqq e^{-\frac{\pi^2}{\mathfrak{M}}}$.

Proof of the theorem. Let $G_{i_1,i_2,\cdots,i_n}$ be one of ring domains which are the components of $R_{n+1}-R_n$ whose outer boundary is $\Gamma_{i_1,i_2,\cdots,i_n}$ and the inner boundary is $\bigcup_{i_{n+1}}\Gamma_{i_1,i_2,\cdots,i_n,i_{n+1}}$. Let $L_{i_1,i_2,\cdots,i_n}$ and $L_{i_1,i_2,\cdots,i_n,i_{n+1}}$ be segments on the real axis contained in $\Gamma_{i_1,i_2,\cdots,i_n}$ and $\Gamma_{i_1,i_2,\cdots,i_n,i_{n+1}}$ respectively. Let $L_{i_1,i_2,\cdots,i_n}$ be the complementary set of $L_{i_1,i_2,\cdots,i_n}$ with respect to the real axis. Then since every $\Gamma_{i_1,i_2,\cdots,i_n}$ is convex, by assumption, we have

$$\mathfrak{M}_n \leqq \mathrm{mod}\,(G_{i_1,i_2}\,_{\cdots,i_n}) \leqq \mathrm{mod}\,(\tilde{L}_{i_1,i_2,\cdots,i_n}, \bigcup_{i_{n+1}} L_{i_1,i_2,\cdots,i_n,i_{n+1}}),$$

where $(\tilde{L}_{i_1,i_2,\cdots,i_n}, \bigcup_{i_{n+1}} L_{i_1,i_2,\cdots,i_n,i_{n+1}})$ is the ring domain whose outer boundary is $\tilde{L}_{i_1,i_2,\cdots,i_n}$ and the inner boundary is $\bigcup_{i_{n+1}} L_{i_1,i_2,\cdots,i_n,i_{n+1}}$. Therefore by Lemma 1, $\mathrm{mes}\,(\bigcup_{i_{n+1}} L_{i_1,i_2,\cdots,i_n,i_{n+1}}) / \mathrm{mes}\, L_{i_1,i_2,\cdots,i_n} \leqq P(\mathfrak{M}_n)$ $(n\geqq 1)$. Let F be the boundary of the Riemann surface R. Then

$$F \subset \bigcup_{i_1} \bigcup_{i_2}, \cdots, \bigcup_{i_n} L_{i_1,i_2,\cdots,i_n}, \quad (n\geqq 1).$$

Hence

$$\mathrm{mes}\, F \leqq \mathrm{mes}\,(\bigcup_{i_1} \bigcup_{i_2}, \cdots, \bigcup_{i_n} L_{i_1,i_2,\cdots,i_n}), \quad (n\geqq 1).$$

On the other hand, since $\mathrm{mes}\,(\bigcup_{i_{n+1}} L_{i_1,i_2,\cdots,i_n,i_{n+1}}) \leqq P(\mathfrak{M}_n)\ \mathrm{mes}\, L_{i_1,i_2,\cdots,i_n}$, hence we have

$$\mathrm{mes}\, F \leqq \mathrm{mes}\,(\bigcup_{i_1} L_{i_1}) \prod_{i=1}^{\infty} P(\mathfrak{M}_i).$$

Therefore, if there are infinitely many $\mathfrak{M}_n$ such that $\mathfrak{M}_n \geqq \delta > 0$, then the theorem is clear. For there exists a positive number δ' $(0<\delta'<1)$ such that $P(\mathfrak{M}_n) \leqq 1-\delta'$, hence $\mathrm{mes}\, F=0$. If $\overline{\lim_{n=\infty}}\, \mathfrak{M}_n = 0$, for sufficiently small $\mathfrak{M}_n\,(n\geqq n_0)$ we have $P(\mathfrak{M}_n) \leqq 1-\varepsilon_n$ $(\varepsilon_n > e^{-\frac{\pi^2}{\mathfrak{M}_n}})$ and

$$\mathrm{mes}\, F \leqq \mathrm{mes}\,(\bigcup_{i_1} L_{i_1}) \prod_{m=1}^{n_0} P(\mathfrak{M}_m) \prod_{n=n_0+1}^{\infty} (1-\varepsilon_n).$$

From the assumption, $\sum \varepsilon_n$ is divergent, therefore we have the theorem.