85. *Polarized Varieties, the Fields of Moduli and Generalized Kummer Varieties of Abelian Varieties*[1)]

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(Comm. by Z. SUETUNA, M.J.A., June 12, 1956)

1. Let V be a complete variety non-singular in co-dimension 1 and F be an algebraic family of positive V-divisors. We shall say that F is a *total* family if for every divisor Z algebraically equivalent to 0 on V, there is a divisor X in F such that

$$Z \sim X - X_0$$

with a fixed X_0 in F. F is called a *maximal* family, if there is no algebraic family containing F as a sub-family. In particular F is called a *complete* family if every positive divisor which is algebraically equivalent to a divisor in F is already contained in F and if every divisor in F determines the complete linear system of the same dimension. A linear system on V is called *ample* if it determines a projective imbedding of V, i.e., an everywhere biregular birational transformation of V into a projective space. When a linear system is ample, it is clear that the complete linear system determined by it is ample. Let X be a V-divisor. We shall say that X is *linearly effective* if the complete linear system determined by X is ample. We shall say that X is *algebraically effective*, if every divisor which is algebraically equivalent to X is linearly effective. Finally we shall say that X is *numerically effective*, if every divisor Y such that mY is algebraically equivalent to mX for a convenient integer m, is linearly effective.

When V is a projective variety, there is a finite number of maximal algebraic family containing the given divisor X, and in fact, the set of positive V-divisors of the given degree forms a finite number of maximal families (Chow-v.d. Waerden [2]). Also in this case, there is a total family on V and when X is any divisor on V and C is a hyperplane section of V, there is a total family which is a set of positive divisors algebraically equivalent to $X+mC$ for large m (Matsusaka [3, 4]). In this paper, we need the following theorem on maximal families on non-singular projective varieties.

Theorem 1. Let V be a non-singular variety in a projective space

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1) This research was supported by the National Science Foundation. We follow the convention and terminology in Weil's book (Weil [6]). The writer got various suggestions and advices from Weil, to whom he wishes to express his deepest thanks.

and $G_n(V)$, $G_a(V)$ *be respectively the groups of divisors numerically equivalent to* 0 *and the group of divisors algebraically equivalent to* 0. (*i*) *Then* $G_n(V)/G_a(V)$ *is a finite group* (*Matsusaka* [*5*]). (*ii*) *When* X *is linearly effective, there is a positive integer* m_0 *such that whenever* $m \geqq m_0$ *any divisor in* $mX+G_n(V)$ *is algebraically equivalent to a positive divisor, and when* $Y_1, \cdots, Y_t$ *are the complete set of representatives of* $mX+G_n(V)$, *modulo* $G_a(V)$, Y_i-Y_1 *form a complete set of representatives of* $G_n(V) \bmod G_a(V)$ (*Matsusaka* [*5*]). (*iii*) *Let* Y *be any* V*-divisor and* X *is a* V*-divisor which is linearly effective. Then there is a positive integer* m_0 *such that whenever* $m \geqq m_0$, $Y+mX$ *is numerically effective, and any positive* V*-divisor numerically equivalent to it belongs to a complete total family* (*Matsusaka* [*4*, *5*]).

Let A be an Abelian variety and X be a positive A-divisor. We shall say that X is non-degenerate if number of points a on A such that $X_a \sim X$ is finite. In the case of Abelian varieties we have the following finer theorem.

Theorem 2. Let X *be a positive non-degenerate divisor on an Abelian variety. There is a positive integer* m_0 *such that whenever* $m \geqq m_0$, mX *is linearly effective. When that is so,* mX *is also algebraically effective* (*Weil* [*7*]).

Let us return to the general case where V is a complete variety non-singular in co-dimension 1 and X_0 be a positive V-divisor. Let $\mathfrak{X}$ be the set of positive divisors X such that

$$mX \equiv m'X_0 (mod\, G_a(V))$$

for convenient m, m'. $\mathfrak{X}$ is uniquely determined when one of the divisors contained in it is given. We shall say that $\mathfrak{X}$ defines a structure of *polarization* on V when $\mathfrak{X}$ contains a linearly effective divisor. When we consider V the variety with a structure of polarization, we shall say that V is a *polarized variety*. Therefore, a polarized variety is the variety with a set of divisors with the property described above on it. The variety without structure shall be called the *underlying* variety of the polarized variety.

From now on, let us assume that *underlying varieties are non-singular varieties and classes which define polarizations contain algebraically effective divisors*. By Theorem 1 we can define on any non-singular projective variety a natural polarization, that is, the polarization defined by the class of divisors determined by hyperplane sections. From now on, let us assume that *every non-singular projective variety is polarized by its natural polarization*. When the given variety V is an Abelian variety with the origin O, we emphasize here that it is a variety with an additional structure, i.e., a structure obtained by putting on it a point O.

2. Let V be a polarized variety polarized by the class of divisors $\mathfrak{X}$ and let X be a linearly effective divisor in $\mathfrak{X}$. The complete linear system determined by X defines a projective imbedding f_X of V, which is determined, up to a projective transformation, by X. Let $P(V, \mathfrak{X})$ be the set of varieties of the form $f_X(V)$ where X runs over all algebraically effective divisors in $\mathfrak{X}$ and f_X runs over all the set of projective imbedding determined by the complete linear system $\mathfrak{L}(X)$ (we do not include here such projective imbeddings f'_X which imbed V into lower dimensional spaces than the general f_X). Let $S(V, F)$ be the set of varieties of the form $f_X(V)$ where X runs over all divisors contained in a complete family F containing algebraically effective divisors. Every variety in $P(V, \mathfrak{X})$ is, as already mentioned above, supposed to be polarized by its natural polarization.

Theorem 3. Let F be any complete family containing algebraically effective divisors in $\mathfrak{X}$. Then $S(V, F)$ is contained in $P(V, \mathfrak{X})$ and when V is an Abelian variety, $P(V, \mathfrak{X})$ is the join of such $S(V, F)$.[2] *$S(V, F)$ is such that its closure is an algebraic family, i.e., the closure of the set of Chow-points of varieties contained in it forms an algebraic variety whenever F is complete, and contains algebraically effective divisors.*

The first statement is an immediate consequence of the definition. As to the latter, we can parametrize the set of complete linear systems contained in F by a subvariety of the Picard variety of V. Since the closure of the set of varieties projectively equivalent to V in a projective space is an algebraic family, we get our theorem.

Theorem 4. Every variety in $P(V, \mathfrak{X})$ is birationally equivalent to each other by an everywhere biregular birational correspondence. In particular, when V is an Abelian variety, underlying varieties of polarized Abelian varieties in $S(V, F)$ are projectively equivalent to each other.

The first statement is clear from the definition. There is a birational transformation f between Abelian varieties A, A' in $S(V, F)$ such that $f(O)=O'$ where O and O' are origins of A, A', and that f is the birational transformation determined by a divisor on A which is algebraically equivalent to hyperplane sections. Applying a suitable translation on A', we can get a birational transformation f' of A onto A' such that hyperplane sections are transformed by f' to hyperplane sections. This implies that f' is a projective transformation.

Theorem 5. Let A be a polarized Abelian variety and G be the

2) In general $P(V, \mathfrak{X}) - \cup S(V, F)$ is the set of varieties $f_X(V)$ such that even though X is algebraically effective, there is a positive divisor X' algebraically equivalent to X with $l(X) \neq l(X')$.

group of automorphisms of it. Then G is a finite group.

According to Theorem 4, any underlying varieties of polarized Abelian varieties A, A' in $S(V, F)$ are projectively equivalent. On the other hand, the set of projective transformations which transform the underlying variety A onto itself forms a finite number of algebraic families. According to Chow's theorem (Chow [1]), there is no algebraic family of Abelian subvarieties on the given Abelian variety. From this we see that the set of projective transformations which transform the underlying A onto itself forms a finite group. Our theorem is an easy consequence of this.

Theorem 6. There is the smallest field K over which one of the $S(V, F)$ is defined for the given $\mathfrak{X}$ on V, whenever the characteristic is 0, and K is the intersection of the smallest fields of definitions of varieties contained in $P(V, \mathfrak{X})$. When V is a complete non-singular curve, the above statement is true even in the case of arbitrary characteristic. In the former case, every other $S(V, F')$ is defined over an algebraic extension of K of a finite degree. In the latter case, every other $S(V, F')$ is also defined over K.

Since F is uniquely determined by one of its members X, let us write the smallest field of definition of $S(V, F)$ as k_X. By Th. 1, mX is again algebraically effective and belongs to the complete total family. We see that k_{mX} is contained in k_X and k_X is an algebraic extension of degree at most equal to $G_n(V)/G_a(V)$. Hence there is an integer m_0 such that whenever m and m' are multiples of m_0, $k_{mX}=k_{m'X}$. Call this field K. Then it is easy to see that the smallest field of definition of $S(V, F')$ contains K and is an algebraic extension of a finite degree. Let U be any variety in $P(V, \mathfrak{X})$ and K' be the smallest field of definition of U. If we observe that K' contains k_{mnX} when m is sufficiently large, where n is a certain integer, our theorem follows easily.

The field defined in the above theorem shall be called the field of moduli of the polarized variety.

3. Let U be a complete variety and G be a group of everywhere biregular birational transformations of U onto itself. Let us assume that G is a finite group consisting of f_i $(i=1,\cdots,m)$. Let W be a variety and g be a rational mapping of U onto W such that

(i) g is defined everywhere on U;

(ii) $g(u)=g(f_i(u))$ for every u on U;

(iii) when W' and g' are another variety and a rational mapping from U onto W', satisfying generically (i) and (ii), there is a rational mapping h from W onto W' such that $g'=h\cdot g$ and that h is defined at $g(u')$ whenever g' is defined at u'.[3]

The variety W satisfying (i), (ii), (iii) is defined to be a *quotient variety* of U with respect to G and g is defined to be the *canonical mapping* of U onto W. When W and g exist, W and g are determined uniquely up to everywhere biregular birational transformations.

When U is an Abelian variety polarized by the class $\mathfrak{X}$ and when G is the group of automorphisms of it, the quotient variety W is defined to be a *generalized Kummer variety* of it. When G is the symmetric group on n letters operating on $U \times \cdots \times U$ of the product of n factors equal to U in the obvious manner, the quotient variety of it by G is called the *symmetric product of U of degree n.*

Theorem 7. Let U be an absolutely normal projective variety defined over a field k (without polarization) and G be a finite group of everywhere biregular birational transformations of U onto itself. There exists a quotient variety of U with respect to G. Moreover, when every element of G is separably algebraic over k and its conjugate over k is also an element of G, there are a quotient variety and a canonical mapping both defined over k. And the quotient variety is absolutely normal.

As a special case of this, we see that the symmetric product of an absolutely normal projective variety defined over k is defined over k together with a canonical mapping and is absolutely normal.

Now we have the following theorem, which is a consequence of Weil's results (Weil [8]) on the field of definition of the moving varieties and Th. 5, Th. 6, Th. 7 of this paper.

Theorem 8. Let us assume either that the characteristic of our universal domain is zero or V is a curve and that the group G of everywhere biregular birational transformations of V onto itself is a finite group. Let K be the field of moduli of V. There is a quotient variety W of V with respect to G defined over K such that when U is contained in $P(V, \mathfrak{X})$ and is defined over a field K' containing K, there is a canonical mapping g of U onto W defined over K'. When U' is another variety in $P(V, \mathfrak{X})$ and f is an everywhere biregular birational transformation of U onto U', and when g' is a canonical mapping of U' onto W, then

$$g = g' \cdot f.$$

3) Generally speaking, the quotient variety of the variety U with respect to a group operating on it should be defined in terms of quotient rings. But since in our case the matter is very simple, the writer prefers this definition.

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