128. *Ideal Theory of Semiring*

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Quite recently, some writers have considered a non-commutative lattice which is a generalisation of the notion of lattices and have shown that the theory of non-commutative lattices are very useful for the theoretic physics. On the other hand, any semirings we shall develop are considered as a extensive generalisation of a non-commutative case for distributive lattices. In this paper, we shall develop the ideal theory of a semiring[1] and consider a structure space of a semiring.

Let R be a semiring. Unless otherwise stated, the word *ideal* shall mean two-sided ideal.

Definition 1. An ideal P is *prime*, if and only if $AB\subset P$ for any two ideals A, B implies $A\subset P$ or $B\subset P$.

Definition 2. An ideal I is *irreducible*, if and only if $A\frown B=I$ for two ideals A, B implies $A=I$ or $B=I$.

Definition 3. An ideal S is *strongly irreducible*, if and only if $A\frown B\subset S$ for any two ideals A, B implies $A\subset S$ or $B\subset S$.

A notion of strongly irreducible ideals was introduced by L. Fuchs [5] who calls *primitive*. In his paper [2], R. L. Blair used a terminology strongly irreducible. We shall follow his terminology.

From $AB\subset A\frown B$ for any two ideals A, B, any prime ideals are strongly irreducible and any strongly irreducible ideals are irreducible.

Theorem 1. The following conditions are equivalent.

(1) *P is a prime ideal.*

(2) *If (a), (b) are principal ideals[2] and $(a)(b)\subset P$, then $a\in P$ or $b\in P$.*

(3) *$aRb\subset P$ implies $a\in P$ or $b\in P$.*

(4) *If I_1, I_2 are right ideals and $I_1I_2\subset P$, then $I_1\subset P$ or $I_2\subset P$.*

(5) *If I_1, I_2 are left ideals and $J_1J_2\subset P$, then $J_1\subset P$ or $J_2\subset P$.*

Theorem 1 was proved by N. H. McCoy [10] for the case of rings.

Proof. It is clear that (1) implies (2). To prove that (2) implies (3), let $aRb\subset P$, then $RaRbR\subset P$, and hence we have $(a)^2(b)^3\subset P$. This implies $a\in P$ or $b\in P$.

To prove that (3) implies (4), let $I_1I_2\subset P$ for right ideals I_1, I_2

1) For the detail of a semiring, see K. Iséki and Y. Miyanaga [8].

2) (a) denotes the principal two-sided ideal generated by a.

and suppose $I_1 \not\subset P$. There is an element a of I_1 not in P. Then, for every element b of I_2,

$$aRb \subset I_1 \cdot I_2 \subset P.$$

Hence, from (3), $b \in P$ and this shows $I_2 \subset P$. Similarly, we can prove that (3) implies (5). It is trivial that (4) or (5) implies (1).

Following N. H. McCoy, we shall define m-system as follows: A subset M of R is an *m-system*, if and only if $a, b \in M$ implies that there is an element x of R such that $axb \in M$.

Then we have

Corollary 1. An ideal P is prime if and only if the set complement of P in R is an m-system.

Proof. Let P be a prime ideal, and let P' be the set complement of P. Suppose that $axb \in P'$ for some a, b of P' and every element x of R. By Theorem 1, (3), we have $a \in P$ or $b \in P$, which is a contradiction. Hence P' is an m-system. Conversely, let M be an m-system, and let $axb \in R-M$ for every element x of R. Suppose that $a, b \in M$, then, since M is an m-system, there is an element x such that $axb \in M$. Hence $a \in R-M$ or $b \in R-M$.

Following R. L. Blair [2], we shall define an i-system. A set M of R is an *i-system* if and only if $a, b \in M$ implies that $(a) \frown (b) \frown M$ is not empty.

By an argument of Theorem 1 and Corollary 1, or the technique of R. L. Blair [2], we can prove the following

Theorem 2. The following conditions are equivalent for an ideal S.

(1) *S is a strongly irreducible ideal.*

(2) *$(a) \frown (b) \subset S$ implies $a \in S$ or $b \in S$.*

(3) *The set complement of S in R is an i-system.*

The following term for rings was introduced by L. Fuchs [5].

Definition 4. A semiring R is said to be *arithmetic*, if, for ideals A, B and C,

$$A \smile (B \frown C) = (A \smile B) \frown (A \smile C).$$

The identity $A \smile (B \frown C) = (A \smile B) \frown (A \smile C)$ is equivalent to $A \frown (B \smile C) = (A \frown B) \smile (A \frown C)$. Then we have the following

Theorem 3. In any arithmetic semiring R, an ideal of R is irreducible, if and only if, it is strongly irreducible.

Proof. Let A, B and C be ideals of R and suppose that $A \frown B \subset C$. Then, for $C_1 = C \smile A$, $C_2 = C \smile B$, we have

$$C_1 \frown C_2 = (C \smile A) \frown (C \smile B) = C \smile (A \frown B) = C.$$

If C is irreducible, then $C_1 = C$ or $C_2 = C$. Hence $A \subset C$ or $B \subset C$. Therefore C is a strongly irreducible ideal.

Conversely, in a semiring R any strongly irreducible ideals are irreducible. This completes the proof.

In particular, we have the following

Theorem 4. In a distributive lattice, an ideal is irreducible, if and only if, it is strongly irreducible.

By a theorem of G. Birkhoff and O. Frink [1] (see also K. Iséki [6, 7]), we have the following

Theorem 5. In a distributive lattice, prime ideals, irreducible ideals and strongly irreducible ideals are same.[3]

For any semiring R, we shall prove the following

Theorem 6. Any ideal is the intersection of all irreducible ideals containing it.

Proof. Let A be an ideal of R, and let $\{A_\alpha\}$ be the set of all irreducible ideals containing A. Since R is an irreducible ideal, $\{A_\alpha\}$ is a non-empty family. Then it is clear that $A \subset \bigcap_\alpha A_\alpha$. To prove that $A \supset \bigcap_\alpha A_\alpha$, it is sufficient to show the following

Lemma. If a is a non-zero element of R, and A is an ideal not containing a, then there is an irreducible ideal B containing A but not a.

To prove Lemma, we shall use the transfinite induction or Zorn's lemma. Let $\{B_\alpha\}$ be the set of all ideals containing A but not a. Since the family $\{B_\alpha\}$ contains A, it is non-empty. By Zorn's lemma, we can find an ideal B which is maximal with respect to the conditions: B contains A and B does not contain the element a. Then the ideal B is irreducible. Suppose that $B = C_1 \frown C_2$, then, since B does not contain a, at least one of C_1, C_2 does not contain a. If $C_1 \not\ni a$, then, the construction of B and $B \subset C_1$, we have $B = C_1$. Therefore B is an irreducible ideal and the proof of Lemma is complete. This shows that Theorem 6 holds true.

Theorem 7. If any irreducible ideal of a semiring R is strongly irreducible, then R is arithmetic.

Proof. Let A, B and C be ideals of R. Then we have

$$A \smile (B \frown C) \subset (A \smile B) \frown (A \frown C).$$

If I is any irreducible ideal containing $A \smile (B \frown C)$, then we have $A \subset I$ and $B \frown C \subset I$. By the assumption, I is strongly irreducible and hence $B \subset I$ or $C \subset I$. Therefore $A \smile B \subset I$ or $A \smile C \subset I$, and we have $(A \smile B) \frown (A \frown C) \subset I$. By Theorem 6, $(A \smile B) \frown (A \frown C) \subset A \smile (B \frown C)$. The proof is complete.

Further, we have the following

Corollary 2. In an arithmetic semiring, any ideal is the intersection of all strongly irreducible ideals containing it.

In our papers [8, 9] we considered the structure spaces $\mathfrak{M}$ and $\mathfrak{P}$ of a commutative semiring with a unit 1. In the next section, we shall consider a structure space $\mathfrak{S}$ of all strongly irreducible ideals of a commutative semiring with 1.

3) For the ideal theory in distributive lattices, see A. A. Monteiro [11].

In our previous discussion, the commutativity is not essential. However for brief we shall assume the commutativity of R. Clearly $\mathfrak{M}\subset\mathfrak{P}\subset\mathfrak{S}$. For the theory of structure spaces for narings and Boolean algebras, see E. A. Behrens [3, 4] and C. Pauc [12].

Let $\mathfrak{S}$ be the set of all strongly irreducible ideals of R. To give a *topology* σ on $\mathfrak{S}$, we shall take $\sigma_x=\{S\mid x\bar{\in}S,\ S\in\mathfrak{S}\}$ for every x of R as an open base of $\mathfrak{S}$. First of all, we shall show the following

Theorem 8. Let $\mathfrak{A}$ *be a subset of* $\mathfrak{S}$, *then we have*

$$\bar{\mathfrak{A}}=\{S'\mid \bigcap_{S\in\mathfrak{A}} S\subset S' \text{ and } S'\in\mathfrak{S}\}$$

where $\bar{\mathfrak{A}}$ *is the closure of* $\mathfrak{A}$ *by* σ.

Proof. Let $\mathfrak{B}$ be $\{S'\mid \bigcap_{S\in\mathfrak{A}} S\subset S' \text{ and } S'\in\mathfrak{S}\}$ and let $S'\in\mathfrak{B}$. Let σ_x be an open base of S', then, by the definition of the topology σ, $x\bar{\in}S'$. Hence we have $x\bar{\in}\bigcap_{S\in\mathfrak{A}} S$. It follows from this that there is a strongly irreducible ideal S of $\mathfrak{A}$ such that x is not contained in S. Hence $\sigma_x\ni S$. Therefore $S'\in\bar{\mathfrak{A}}$ and $\mathfrak{B}\subset\bar{\mathfrak{A}}$.

To prove $\mathfrak{B}\supset\bar{\mathfrak{A}}$, take a strongly irreducible ideal S' such that $S'\bar{\in}\mathfrak{B}$. Then $\bigcap_{S\in\mathfrak{A}} S-S'$ is not empty. For an element x of $\bigcap_{S\in\mathfrak{A}} S-S'$, we have $x\in S$ $(S\in\mathfrak{A})$ and $x\bar{\in}S'$. Hence $\sigma_x\ni S'$ and $\sigma_x\bar{\ni}S$ for all S of $\mathfrak{A}$. Therefore $\mathfrak{A}\frown\sigma_x=0$ and then we have $S'\bar{\in}\bar{\mathfrak{A}}$. Hence $\mathfrak{B}\supset\bar{\mathfrak{A}}$. The proof of Theorem 8 is complete.

Now we shall prove that the topological space $\mathfrak{S}$ for the topology σ is a compact T_0-space.

To prove that $\mathfrak{S}$ is a T_0-space, it is sufficient to verify the following conditions:

(1) $\mathfrak{A}\subseteq\bar{\mathfrak{A}}$.

(2) $\bar{\mathfrak{A}}=\bar{\bar{\mathfrak{A}}}$.

(3) $\overline{\mathfrak{A}\smile\mathfrak{B}}=\bar{\mathfrak{A}}\smile\bar{\mathfrak{B}}$.

(4) $\bar{S}_1=\bar{S}_2$ implies $S_1=S_2$.

The conditions (1) and (2) are clear, and $\mathfrak{A}\smile\mathfrak{B}$ implies $\bar{\mathfrak{A}}\subset\bar{\mathfrak{B}}$. From this relation, we have $\bar{\mathfrak{A}}\smile\bar{\mathfrak{B}}\subset\overline{\mathfrak{A}\smile\mathfrak{B}}$. For some element S of $\overline{\mathfrak{A}\smile\mathfrak{B}}$, suppose that $S\bar{\in}\bar{\mathfrak{A}}$ and $\bar{S}\bar{\in}\mathfrak{B}$. From Theorem 8, we have

$$S\not\supset\bigcap_{S'\in\mathfrak{A}} S'=S_{\mathfrak{A}}$$

and

$$S\not\supset\bigcap_{S'\in\mathfrak{B}} S'=S_{\mathfrak{B}}.$$

$S_{\mathfrak{A}}$ and $S_{\mathfrak{B}}$ are ideals. If $S_{\mathfrak{A}}\frown S_{\mathfrak{B}}\subset S$, by the definition of S, $S_{\mathfrak{A}}\subset S$ or $S_{\mathfrak{B}}\subset S$. Hence $S\not\supset S_{\mathfrak{A}}\frown S_{\mathfrak{B}}=S_{\mathfrak{A}\smile\mathfrak{B}}$. This shows $S\bar{\in}\overline{\mathfrak{A}\smile\mathfrak{B}}$.

To prove that $\bar{S}_1=\bar{S}_2$ implies $S_1=S_2$, we shall use the condition (1). Then $\bar{S}_1\ni S_2$ and by the definition of closure operation, we have

$S_1 \subset S_2$. Similarly we have $S_1 \supset S_2$ and $S_1 = S_2$. Therefore we complete the proof that $\mathfrak{S}$ is a T_0-space.

We shall prove that $\mathfrak{S}$ is a compact space. Let $\mathfrak{A}_\lambda$ be a family of closed sets with empty intersection. Let $S_{\mathfrak{A}_\lambda} = \bigcap_{S \in \mathfrak{A}_\lambda} S$, suppose that $\sum_\lambda S_{\mathfrak{A}_\lambda} \neq S$, then there is a maximal ideal M containing the ideal $\sum_\lambda S_{\mathfrak{A}_\lambda}$. Therefore we have $S_{\mathfrak{A}_\lambda} \subset M$ for every λ. By the definition of $S_{\mathfrak{A}}$, $\mathfrak{A}_\lambda \ni M$ for every λ. Hence $\bigcap_\lambda \mathfrak{A}_\lambda \ni M$, which contradicts our hypothesis of $\mathfrak{A}_\lambda$. Therefore $\sum_\lambda S_{\mathfrak{A}_\lambda} = R$. Hence the unit 1 of R can be expressed by the sum of elements a_i of some $S_{\mathfrak{A}_{\lambda_i}}$ $(i=1,2,\cdots,n)$: $1 = \sum_{i=1}^n a_i (a_i \in S_{\mathfrak{A}_{\lambda_i}})$. Hence we have $R = \sum_{i=1}^n S_{\mathfrak{A}_{\lambda_i}}$. If $\bigcap_{i=1}^n \mathfrak{A}_{\lambda_i}$ is non-empty, for every strongly irreducible ideal S of $\bigcap_{i=1}^n \mathfrak{A}_{\lambda_i}$, $S \supset S_{\mathfrak{A}_{\lambda_i}}$ $(i=1,2,\cdots,n)$ and $S \supset \sum_{i=1}^n S_{\mathfrak{A}_{\lambda_i}}$. If $\bigcap_{i=1}^n \mathfrak{A}_{\lambda_i} = R$, we can prove easily that $\mathfrak{S}$ is a compact space. If $\bigcap_{i=1}^n \mathfrak{A}_{\lambda_i}$ contains a proper strongly irreducible ideal S, we have $S \supset \sum_{i=1}^n S_{\mathfrak{A}_{\lambda_i}}$, which is a contradiction to $R = \sum_{i=1}^n S_{\mathfrak{A}_{\lambda_i}}$. Therefore $\bigcap_{i=1}^n \mathfrak{A}_{\lambda_i} = 0$. Hence $\mathfrak{S}$ is a compact space.

Theorem 9. The structure space $\mathfrak{S}$ with the topology σ is compact T_0-space.

By the representation theory of a semiring, we shall prove the converse of Corollary 2. It is sufficient to show that the lattice of ideals of R is isomorphic with the lattice of some closed sets of $\mathfrak{S}$. Since each ideal A is the intersection of all strongly irreducible ideals A_α containing A, the correspondence $A \to \{A_\alpha\}$ is one-to-one, and by the definition of the topology σ, the set $\{A_\alpha\}$ is closed in $\mathfrak{S}$. Hence, the mapping $A \to \{A_\alpha\}$ gives a lattice isomorphism between the lattice of ideals of R and a lattice of some closed sets of $\mathfrak{S}$. Therefore we have

Theorem 10. The lattice of ideals of R is distributive, if and only if each ideal is the intersection of all strongly irreducible ideals containing it.

In my paper [9], we introduced the notions of the $\mathfrak{M}$-radical and the $\mathfrak{P}$-radical of a semiring. By a similar way, we shall define $\mathfrak{S}$-radical of a semiring.

Definition 4. By the $\mathfrak{S}$*-radical* $r(\mathfrak{S})$ of a semiring, we mean the intersection of all strongly irreducible ideals of it, i.e. $\bigcap_{S \in \mathfrak{S}} S$.

From $\mathfrak{M} \subset \mathfrak{P} \subset \mathfrak{S}$, we have $r(\mathfrak{M}) \supset r(\mathfrak{P}) \supset r(\mathfrak{S})$.

Theorem 11. The subset $\mathfrak{P}$ of $\mathfrak{S}$ is dense in $\mathfrak{S}$, if and only if $r(\mathfrak{P}) = r(\mathfrak{S})$.

Proof. Let $\overline{\mathfrak{P}}=\mathfrak{S}$ for the topology σ, then we have

$$\{S \mid \bigcap_{P\in\mathfrak{P}} P\subset S\}=\mathfrak{S}.$$

Hence, we have

$$r(\mathfrak{P})=\bigcap_{P\in\mathfrak{P}} P\subset \bigcap_{S\in\mathfrak{S}} S=r(\mathfrak{S}).$$

On the other hand, $r(\mathfrak{P})\supset r(\mathfrak{S})$. This shows $r(\mathfrak{S})=r(\mathfrak{P})$.

Conversely, suppose that $\mathfrak{S}-\overline{\mathfrak{P}}$ is non-empty, then there is a strongly irreducible ideal S such that $S\bar{\in}\overline{\mathfrak{P}}$ and $S\in\mathfrak{S}$. Therefore there is a neighbourhood σ_x of S which does not meet $\mathfrak{P}$. Hence $r(\mathfrak{S})=\bigcap_{S\in\mathfrak{S}} S$ is a proper subset of $\bigcap_{P\in\mathfrak{P}} P$, and we have $r(\mathfrak{S})\neq r(\mathfrak{P})$.

Corollary 3. The subset $\mathfrak{M}$ *of* $\mathfrak{S}$ *is dense in* $\mathfrak{S}$, *if and only if* $r(\mathfrak{M})=r(\mathfrak{S})$.

Corollary 4. Let R *be a semiring with* 0. *If* 0 *is the zero ideal* (0) *and* R *is* $\mathfrak{M}$*-semisimple,* $\mathfrak{M}$ *and* $\mathfrak{P}$ *are dense in* $\mathfrak{S}$.

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