155. A Characterisation of Regular Semi-group

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The concept of regular ring was introduced by J. v. Neumann [3]. Recently, L. Kovács [1] has given an interesting characterisation of regular ring. On the other hand, some writers studied regular semi-groups. In this short Note, we shall give a characterisation for regular semi-group which is similar to Theorem 1 of L. Kovács [1].

Following W. D. Munn and R. Penrose [2], we define a regular semi-group. A semi-group is said to be *regular* if for any given element a of S there is at least one element x of S such that $axa=a$. A non empty subset L of S is said to be a *left ideal* if $SL\subset L$. Similarly we define *right ideals*. Then we have the following

Theorem 1. Any semi-group S is regular if and only if

$$AB=A\frown B$$

for every right ideal A and every left ideal B of S.

Proof. Let S be a regular semi-group, and let $a\in A\frown B$, then there is an element x such that $axa=a$. Since B is a left ideal, $xa\in B$. Therefore $a=a(xa)\in AB$. This shows $AB\supset A\frown B$. Clearly $AB\subset A\frown B$. Hence $AB=A\frown B$.

To prove the converse, let a be an element of S. Then $\{ax\,|\,x\in S\}\smile a$ is the right ideal (a) of S generated by a. By the hypothesis,

$$(a)=(a)\frown S=(a)S=aS.$$

Therefore, we have $a\in aS$. Similarly $a\in Sa$. Hence

$$a\in aR\frown Ra=aR^2a,$$

and there is an element x such that $a=axa$.

Now, let us suppose that a given regular semi-group S is *commutative*, then, by Theorem 1, any ideal A in S is idempotent, i.e. $A^2=A$. Conversely, suppose that every ideal in a commutative semi-group S is idempotent. If A and B are ideals in S, then we have $A\frown B=(A\frown B)^2=(A\frown B)(A\frown B)\subset AB$. On the other hand, $A\frown B\supset AB$. Hence $A\frown B=AB$. By Theorem 1, S is regular, therefore we have the following

Theorem 2. A commutative semi-group is regular if and only if every ideal is idempotent.

From Theorem 2, it is easily seen that there is no non-zero nilpotent element in a commutative regular semi-group with 0.

Corollary. Any commutative regular semi-group with 0 *does not* *contain non-zero nilpotent element.*

Corollary follows from the identity $a^2x=a$ also.

References

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[3] J. v. Neumann: On regular rings, Proc. Nat. Acad. Sci. U. S. A., **22**, 707–713 (1936).