46. Some Operations on the Ranked Spaces. I

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Prof. K. Kunugi introduced the notion of the ranked spaces in the Note "Sur les espaces complets et régulièrement complets. I".[1] It is the purpose of this Note to study the product spaces of ranked spaces and the function spaces $F(E, G)$ which denotes the totality of functions of a fixed set E into a fixed ranked space G.[2]

1. In [I] it is required that the system of neighbourhoods satisfies F. Hausdorff's axiom (C).[3] We shall attempt to exclude this hypothesis.

Let R be a space whose topology is given by a system of neighbourhoods which satisfies F. Hausdorff's axiom (A).[3] Then we can calculate the depth[4] of R and introduce the notion of the ranked spaces according to [I]. We shall conform ourselves, without contrary indication, to the notions and the terminology of [I]. But it is necessary to modify some notions as follows.

Definition 1. A ranked space is called to be *complete*[6] if, for every fundamental sequence $v_\alpha(p_\alpha)$, $0 \leqq \alpha \leqq \omega_\mu$, the following conditions (1), (2) are satisfied:

$$(1) \qquad \bigcap_\alpha v_\alpha(p_\alpha) \neq 0.$$

$$(2) \qquad \bigcap_\alpha I\{v_\alpha(p_\alpha)\}^{5)} \neq 0 \quad \text{if } \omega_\mu < \omega_\nu.$$

Definition 2. A set E is called to be *non-dense*[7] if, for every point p of R and every neighbourhood $v(p)$ of p, $I\{v(p)\} \not\subseteq \bar{A}$. A set F is called to be of the first category if it is a union of an ω_ν-sequence of non-dense sets: $F = \bigcup_{0 \leq \alpha < \omega_\nu} F_\alpha$ where every F_α is non-dense.

Then we can prove Baire's theorem:

Theorem 1. *In the complete ranked spaces any non-empty open set is not of the first category.*

Proof. For proving the theorem it is sufficient to show that, if G is a non-empty open set and $E_{2\alpha}$, $0 \leq \alpha < \omega_\nu$, are non-dense sets, then $G \neq \bigcup_\alpha E_{2\alpha}$.

Since E_0 is non-dense, $G \not\subseteq \bar{E}_0$. Therefore there exist a point p_0, a rank γ_0 and $v_0(p_0)$ of rank γ_0 such that $v_0(p_0) \subseteq G$ and $v_0(p_0) \cap E_0 = 0$. Suppose that we have already defined p_β, γ_β and $v_\beta(p_\beta)$ for all β such that $0 \leq \beta < \alpha$ where $0 < \alpha < \omega_\nu$ and they satisfy the following conditions (3) and (4):

$$(3) \qquad v_0(p_0) \supseteq v_1(p_1) \supseteq \cdots, \quad \gamma_0 \leq \gamma_1 \leq \cdots, \quad v_\beta(p_\beta) \in \mathfrak{B}_{\gamma_\beta}.$$

(4) For every even number β, $p_\beta = p_{\beta+1}$, $\gamma_\beta < \gamma_{\beta+1}$ and $v_\beta(p_\beta) \cap E_\beta = 0$.
Suppose, first, α is an even and isolated number. Since E_α is non-dense, $I\{v_{\alpha-1}(p_{\alpha-1})\} \not\subseteq \overline{E}_\alpha$. Then, owing to the condition (a),[8] there exist a point p_α, a rank γ_α and $v_\alpha(p_\alpha)$ of rank γ_α such that $v_\alpha(p_\alpha) \subseteq v_{\alpha-1}(p_{\alpha-1})$, $v_\alpha(p_\alpha) \cap E_\alpha = 0$ and $\gamma_\alpha > \gamma_{\alpha-1}$. Suppose, next, α is a limiting number. Since R is complete and $\alpha < \omega_\nu$, $\bigcap_\beta I\{v_\beta(p_\beta)\} \neq 0$. On the other hand $\alpha < \omega(R)$ implies $\bigcap_\beta I\{v_\beta(p_\beta)\} = I\{\bigcap_\beta v_\beta(p_\beta)\}$. So we can choose a point q and $v(q)$ such that $v(q) \subseteq \bigcap_\beta v_\beta(p_\beta)$. Since E_α is non-dense, $I\{v(q)\} \not\subseteq \overline{E}_\alpha$. Then, by the condition (a), there exist a point p_α, a rank γ_α and $v_\alpha(p_\alpha)$ of rank α such that $v_\alpha(p_\alpha) \subseteq \bigcap_\beta v_\beta(p_\beta)$, $v_\alpha(p_\alpha) \cap E_\alpha = 0$ and $\gamma_\alpha > \sup_\beta \gamma_\beta$. If α is odd number, put $p_\alpha = p_{\alpha-1}$ and choose, by (a), a rank γ_α and $v_\alpha(p_\alpha)$ of rank γ_α such that $\gamma_\alpha > \gamma_{\alpha-1}$ and $v_\alpha(p_\alpha) \subseteq v_{\alpha-1}(p_{\alpha-1})$.

Thus we have a fundamental sequence $v_\alpha(p_\alpha)$, $0 \leq \alpha < \omega_\nu$, such that $\bigcap_\alpha v_\alpha(p_\alpha) \subseteq G$ and $\bigcap_\alpha v_\alpha(p_\alpha) \cap (\bigcup_\alpha E_{2\alpha}) = 0$. Since R is complete, $\bigcap_\alpha v_\alpha(p_\alpha) \neq 0$. Therefore $G \not\subseteq \bigcup_\alpha E_{2\alpha}$. q.e.d.

Examples of complete ranked spaces not satisfying Hausdorff's axiom (C) will be given in Section 4.

2. *On the product spaces.* Let I be a non-empty set of indices. And let R_i, $i \in I$, be the ranked spaces satisfying the following conditions (5) and (6):

(5) The ranks of R_i, $i \in I$, are given by the same ordinal number ω_ν.[9]

(6) Each of R_i satisfies the condition (a*):

(a*) For every neighbourhood $v(p)$ of a point p there exists α such that $0 \leq \alpha < \omega_\nu$ and for all ordinal numbers β, $\alpha \leq \beta < \omega_\nu$, there exists a neighbourhood $u(p)$ of rank β included in $v(p)$.

The cartesian product of R_i, $i \in I$, is denoted by $P_{i\in I}R_i$ and π_i denotes the projection of $P_{i\in I}R_i$ into the i-th coordinate space R_i.

Now we shall give a topology and a rank to $P_{i\in I}R_i$ as follows: The system of neighbourhoods of $p=(p_i)$[10] is the totality of

$$\bigcap_{i\in A} \pi_i^{-1}(v(p_i)) \tag{7}$$

where $v(p_i)$ is a neighbourhood of p_i and A is a subset of I whose power is $< \aleph_\nu$. Then $\omega(P_{i\in I}R_i) \geq \omega_\nu$. So we take as $\mathfrak{V}_\alpha$, $0 \leq \alpha < \omega_\nu$, the totality of (7) where every $v(p_i)$ is of rank α in R_i. Then the condition (a) is satisfied.

Definition 3. The cartesian product thus ranked is called the product (ranked) space of R_i, $i \in I$, and denoted simply by $P_{i\in I}R_i$.

Theorem 2. *The product space of ranked spaces is complete if and only if each coordinate space is complete.*

Proof. Suppose that R_i is a complete ranked space for each i of

I. Let $p^\alpha=(p_i^\alpha)$ and $v^\alpha(p^\alpha)=\bigcap_{i\in A_\alpha}\pi_i^{-1}(v^\alpha(p_i^\alpha))$, $0\leq\alpha<\omega_\mu$, be a fundamental sequence in $P_{i\in I}R_i$. Since $v^0(p^0)\supseteq v^1(p^1)\supseteq\cdots$, then $A_0\subseteq A_1\subseteq\cdots$. Hence, for each i of $\bigcup_\alpha A_\alpha$, there exists an ordinal number $\alpha(i)$ such that $v^\alpha(p_i^\alpha)$, $\alpha(i)\leq\alpha<\omega_\mu$, is a fundamental sequence in R_i. Since R_i is complete, then $\bigcap_\alpha v^\alpha(p_i^\alpha)\neq 0$. Therefore $0\neq\bigcap_{i\in\bigcup_\alpha A_\alpha}\{\pi_i^{-1}(\bigcap_{\alpha(i)\leq\alpha<\omega_\mu}v^\alpha(p_i^\alpha))\}$ $\subseteq\bigcap_{i\in\bigcup_\alpha A_\alpha}\{\bigcap_{\alpha(i)\leq\alpha<\omega_\mu}\pi_i^{-1}(v^\alpha(p_i^\alpha))\}\subseteq\bigcap_\alpha\{\bigcap_{i\in A_\alpha}\pi_i^{-1}(v^\alpha(p_i^\alpha))\}=\bigcap v^\alpha(p^\alpha)$ and consequently (1) is satisfied in the product space. As to (2) it is sufficient to show that, if $\omega_\mu<\omega_\nu$, $I\{\bigcap_{i\in A_\alpha}\pi_i^{-1}(v^\alpha(p_i^\alpha))\}\supseteq\bigcap_{i\in A_\alpha}\pi_i^{-1}(I\{v^\alpha(p_i^\alpha)\})$ because it implies that $\bigcap_\alpha I\{v^\alpha(p^\alpha)\}\supseteq\bigcap_{i\in\bigcup_\alpha A_\alpha}[\pi_i^{-1}(\bigcap_{\alpha(i)\leq\alpha<\omega_\mu}I\{v^\alpha(p_i^\alpha)\})]$. And it results clearly from the definition. Thus the product space is complete. The proof of the converse is obvious.

3. *On the function spaces*

Definition 4. A set G is called a right (left) ranked group if the following conditions (8), (9) and (10) are satisfied:

(8) G is a group.

(9) G is a ranked space.

(10) $\mathfrak{V}_\alpha$ is the totality of $v(e)p$[11] ($pv(e)$) where $p\in G$ and $v(e)$ is a neighbourhood of the unit element e of G such that $v(e)\in\mathfrak{V}_\alpha$.

Let $F(E,G)$ denote the family of all functions of a fixed set E into a fixed group G. Then $F(E,G)$ is a group with the natural group structure: if $f,g\in F(E,G)$, $(f\cdot g)(x)=f(x)\cdot g(x)$ and $(f^{-1})(x)=\{f(x)\}^{-1}$ for all x of E; the unit element of $F(E,G)$ is the function f_e whose value is constantly the unit element e of G.

Moreover, let G be a right (left) ranked group where the rank is given by ω_ν and Γ be a family of subsets of E satisfying the condition (11):

(11) Γ contains at least one non-empty set of E and if A_α, $0\leq\alpha<\gamma<\omega_\nu$, belong to Γ, there exists a member A of Γ such that $A\supseteq\bigcup_\alpha A_\alpha$.

Then we shall construct the rank of $F(E,G)$ as follows. Let $W_{\{A,v(e)\}}(f)$, where $f\in F(E,G)$, $A\in\Gamma$ and $v(e)$ is a neighbourhood of e, denote the set of all functions g such that, for every x of A, $(g\cdot f^{-1})(x)((f^{-1}\cdot g)(x))\in v(e)$. We shall regard the family of all sets of this form where f is fixed as the system of neighbourhoods of f. Then $\omega(F(E,G))\geq\omega_\nu$. So we take as $\mathfrak{V}_\alpha$, $0\leq\alpha<\omega_\nu$, the family of neighbourhoods where $v(e)$ is of rank α in G. Then the condition (a) is satisfied and $F(E,G)$ is a right (left) ranked group.

Definition 5. The rank of $F(E,G)$ described above is called the rank defined by Γ and $F(E,G)$ thus ranked is denoted by $F_\Gamma(E,G)$.

Theorem 3. *If G satisfies the following condition* (C_0), $F_\Gamma(E,G)$

is complete if and only if G is complete.

(C_0) If $v_1(e)$ and $v_2(e)$ are two neighbourhoods of e such that $v_1(e)\in\mathfrak{B}_{\gamma_1}$, $v_2(e)\in\mathfrak{B}_{\gamma_2}$, $\gamma_1<\gamma_2$ and $v_1(e)\supseteq v_2(e)$, then there exists a neighbourhood $u(e)$ of e and, for every point q of $I\{v_2(e)\}$, $u(e)q(qu(e))\in v_1(e)$.

Proof. Suppose that G is complete. Let

(12) $$W_{\{A_0,v_0(e)\}}(f_0)\supseteq\cdots\supseteq W_{\{A_\alpha,v_\alpha(e)\}}(f_\alpha)\supseteq\cdots,\ 0\leq\alpha<\omega_\mu$$

be a fundamental sequence of $F_\Gamma(E,G)$. Since $A_0\subseteq A_1\subseteq\cdots$, then, for each x of $\bigcup_\alpha A_\alpha$, there exists an ordinal number $\alpha(x)$ such that, if $\alpha\geq\alpha(x)$, $x\in A_\alpha$. So, for such an x,

(13) $$v_{\alpha(x)}(e)f_{\alpha(x)}(x)\supseteq\cdots\supseteq v_\alpha(e)f_\alpha(x)\supseteq\cdots,\ \alpha(x)\leq\alpha<\omega_\mu$$

is a fundamental sequence of G. Since G is complete, there exists a point contained in $\bigcap_{\alpha(x)\leq\alpha<\omega_\mu} v_\alpha(e)f_\alpha(x)$, say, p_x. Now let f denote the element of $F_\Gamma(E,G)$ such that $f(x)=p_x$ for $x\in\bigcup_\alpha A_\alpha$ and $f(x)=e$ for $x\notin\bigcup_\alpha A_\alpha$. Then $f\in\bigcap_\alpha W_{\{A_\alpha,v_\alpha(e)\}}(f_\alpha)$. Thus (1) is satisfied for (12).

Suppose, moreover, $\omega_\mu<\omega_\nu$ for (12). Then we can take p_x in $\bigcap_{\alpha(x)\leq\alpha<\omega_\mu} I\{v_\alpha(e)f_\alpha(x)\}$. Now we shall show that

(14) $$f\in\bigcap_\alpha I\{W_{\{A_\alpha,v_\alpha(e)\}}f(x)\}.$$

For any α, $0\leq\alpha<\omega_\mu$, let x be an arbitrary point contained in A_α and let β be an even ordinal number such that $\alpha\leq\beta<\omega_\mu$. Then $p_x\in I\{v_{\beta+1}(e)f_{\beta+1}(x)\}$, so $p_xf_{\beta+1}(x)^{-1}\in I\{v_{\beta+1}(e)\}$. From the definition of the fundamental sequence, $v_\beta(e)\supseteq v_{\beta+1}(e)$ and the rank of $v_\beta(e)<$ the rank of $v_{\beta+1}(e)$. Therefore, from the hypothesis (C_0), there exists a neighbourhood $u(e)$ of e such that, for every x of A_α, $u(e)p_xf_\beta(x)^{-1}\subseteq v_\beta(e)$,[12] that is, $u(e)p_x\in v_\beta(e)f_\beta(x)$. As $\alpha\leq\beta$, then $u(e)p_x\in v_\alpha(e)f_\alpha(x)$ and consequently $W_{\{A_\alpha,u(e)\}}(f)\subseteq W_{\{A_\alpha,v_\alpha(e)\}}(f_\alpha)$. It implies (14) and so $F_\Gamma(E,G)$ is complete.

To prove the converse of the proposition suppose that $F_\Gamma(E,G)$ is complete. Let $v_\alpha(e)p_\alpha$, $0\leq\alpha<\omega_\mu$, be a fundamental sequence of G and A be a non-empty set such that $A\in\Gamma$. And let f_p denote the function such that $f_p(x)\equiv p$ for all x of E. Then $W_{\{A,v_\alpha(e)\}}(f_{p_\alpha})$, $0\leq\alpha<\omega_\mu$, is a fundamental sequence of $F_\Gamma(E,G)$. Since $F_\Gamma(E,G)$ is complete, there exists an element f of $F_\Gamma(E,G)$ contained in $\bigcap_\alpha W_{\{A,v_\alpha(e)\}}(f_{p_\alpha})$. Then $f(A)\subseteq\bigcap_\alpha v_\alpha(e)p_\alpha$. If $\omega_\nu>\omega_\mu$, we can take f in $\bigcap_\alpha I\{W_{\{A,v_\alpha(e)\}}(f_{p_\alpha})\}$. Then $f(A)\subseteq\bigcap_\alpha I\{v_\alpha(e)p_\alpha\}$ and consequently G is complete. q.e.d.

We note that in the proof of Theorem 3 the hypothesis (C_0) was concerned only with (2). Hence we obtain

Theorem 4. *If $\omega_\nu=\omega_0$, $F_\Gamma(E,G)$ is complete if and only if G is complete.*

4. *Examples*

Example 1.[13] Let R be the set of all pairs $p=(x,y)$ of real numbers x, y and let l_0 be a fixed positive number. We shall topologize R as follows: Let n be a non-negative integer and l be a positive number such that $l>l_0$. And let $V_{n,l}$ denote the set of all points $p=(x,y)$ such that $0\leq x<l$, $0\leq y<l$ and $xy<\frac{1}{n+1}$. The system of neighbourhoods of the origin $0=(0,0)$ is the totality of $V_{n,l}$ and the neighbourhoods of another point is given by the translation.

Then we have $\omega(R)=\omega_0$ and we can put $\mathfrak{V}_n$=the family of all neighbourhoods $V_{n,l}$, of all points. R is a complete ranked space not satisfying Hausdorff's axiom (C).

Example 2. Let G be the ranked group PU of Example 3 of [II], E be an arbitrary infinite set and Γ be composed by only one element E. Then, from Theorem 4, $F_\Gamma(E,G)$ is complete. Hausdorff's (C) is satisfied in G but not in $F_\Gamma(E,G)$.

References

1) K. Kunugi: Sur les espaces complets et régulièrement complets. I–III, Proc. Japan Acad., **30**, 553–556, 912–916 (1954); **31**, 49–53 (1955), cited here as [I], [II] and [III].
2) In this Note we treat the case where G is a (ranked) group.
3) F. Hausdorff: Grundzüge der Mengenlehre 1914 p. 213; T. Shirai gave another method for excluding this axiom, see Proc. Japan Acad., **33**, p. 139.
4) Profondeur: See [I, Définition 1].
5) If A is a set of a space, $I\{A\}$ denotes the interior of A: $p\in I\{A\}$ if and only if there exists a neighbourhood $v(p)$ of p such that $v(p)\subseteq A$. And $\bar{A}$ denotes the closure of A.
6) If Hausdorff's (C) is satisfied, this definition coincides with the definition given in [I].
7) If Hausdorff's (C) is satisfied, this definition coincides with the classical one. Cf. for example, C. Kuratowski: Topologie I.
8) See [I, Définition 2].
9) Idem quod 8).
10) p_i denotes the i-th coordinate of p.
11) $v(e)p$ denotes the set of all points qp where $q\in v(e)$.
12) Note that $f_\beta=f_{\beta+1}$.
13) Cf. the paper of T. Shirai cited in 3), p. 142.