67. *Perturbation of Continuous Spectra by Trace Class Operators*

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1. Introduction. In a previous paper[1] the writer has shown, among others, that the absolutely continuous part of the spectrum of a self-adjoint operator is stable under a self-adjoint perturbation of finite rank. The purpose of the present note is to extend this result to a wider class of perturbations.

Let $\mathfrak{H}$ be a Hilbert space, $\mathbf{B}$ the algebra of all bounded linear operators on $\mathfrak{H}$ to $\mathfrak{H}$, $\mathbf{S}\subset\mathbf{B}$ the Schmidt class and $\mathbf{T}\subset\mathbf{S}$ the trace class.[2] We denote by $\|\ \|$, $\|\ \|_2$, $\|\ \|_1$ the ordinary norm, the Schmidt norm and the trace norm respectively. The subset of $\mathbf{T}$ consisting of all self-adjoint operators will be denoted by $\mathbf{T}_s$.

THEOREM 1. *Let H_0 be a (not necessarily bounded) self-adjoint operator and let $V\in\mathbf{T}_s$. Then $H_1=H_0+V$ is also self-adjoint. Let $\mathfrak{M}_0$ and $\mathfrak{M}_1$ be the absolutely continuous*[3] *parts of $\mathfrak{H}$ with respect to H_0 and H_1, and let P_0 and P_1 be the projections on $\mathfrak{M}_0$ and $\mathfrak{M}_1$, respectively. Then the strong limits*

(1.1) $$s-\lim_{t\to\pm\infty}\exp(itH_1)\exp(-itH_0)P_0=U_\pm$$

exist and are partially isometric operators with initial set $\mathfrak{M}_0$ and final set $\mathfrak{M}_1$. Their adjoints satisfy

(1.2) $$s-\lim_{t\to\pm\infty}\exp(itH_0)\exp(-itH_1)P_1=U_\pm^*.$$

The parts of H_0 and H_1 in $\mathfrak{M}_0$ and $\mathfrak{M}_1$ respectively are unitarily equivalent to each other, and the transformation of H_0P_0 into H_1P_1 is effected by either of $U_\pm$:[4]

(1.3) $$H_1P_1=U_\pm H_0P_0U_\pm^*,\qquad H_0P_0=U_\pm^*H_1P_1U_\pm.$$

The second theorem concerns itself with the properties of the mappings which assign to each pair H_0, H_1 the operators $U_\pm$ by (1.1). Let $\mathbf{H}$ be any one of the equivalence classes of self-adjoint operators

1) T. Kato: On finite-dimensional perturbations of self-adjoint operators, J. Math. Soc. Japan, **9**, 239–249 (1957). This paper is quoted as (F).

2) For the Schmidt and trace classes, see R. Schatten: A theory of cross spaces, Ann. Math. Studies, Princeton (1950).

3) For the terms "absolutely continuous" as applied to operators and vectors, see (F).

4) Theorem 1 contains as a special case a theorem by M. Rosenblum in his paper "Perturbation of the continuous spectrum and unitary equivalence", to be published in Pacific J. Math. The writer is indebted to Professor Rosenblum for having a chance of seeing his paper before formal publication.

modulo $\mathbf{T}_s$. Theorem 1 shows that all operators of $\mathbf{H}$ have absolutely continuous parts which are unitarily equivalent to one another. We consider the mappings $(H_0, H_1) \to U_\pm = U_\pm(H_1, H_0)$ from $\mathbf{H} \times \mathbf{H}$ into the set $\mathbf{U}$ of all partially isometric operators. In $\mathbf{H}$ we introduce a metric in which the distance $d(H_0, H_1)$ is equal to $\| H_1 - H_0 \|_1$. In $\mathbf{U}$ we consider either strong or weak topology induced by the corresponding one of $\mathbf{B} \supset \mathbf{U}$.

THEOREM 2. *The mappings defined above are transitive in the sense that (P_0 etc. are as in Theorem 1)*

(1.4) $U_\pm(H_2, H_0) = U_\pm(H_2, H_1) U_\pm(H_1, H_0)$, *in particular* $U_\pm(H_0, H_0) = P_0$. *For a fixed H_0 and variable H_1, these mappings $U_\pm(H_1, H_0)$ are strongly continuous. For a fixed H_1 and variable H_0, they are weakly continuous but not necessarily strongly continuous. For both variable H_0 and H_1, they are not necessarily weakly continuous.*

2. Inequalities. For the proof of these theorems we need some inequalities, which are deduced on the basis of (F). For the moment we assume that V is of finite rank. As has been proved in (F), then all assertions of Theorem 1 are true. Set

$$U_t = \exp(itH_1)\exp(-itH_0), \qquad -\infty < t < +\infty. \tag{2.1}$$

Then we have

$$(U_t - U_s)x = i\int_s^t \exp(itH_1) V \exp(-itH_0)x\, dt. \tag{2.2}$$

We now assume that $x \in \mathfrak{M}_0$ or $x = P_0 x$. Then $\lim U_t x = \lim U_t P_0 x = U_+ x$, $t \to +\infty$, by (1.1). Thus

$$(U_+ - U_s)x = i\int_s^{+\infty} \exp(itH_1) V \exp(-itH_0)x\, dt. \tag{2.3}$$

Since $\| U_+ x \| = \| x \|$ for $x \in \mathfrak{M}_0$ and $\| U_t x \| = \| x \|$, we have $\| (U_+ - U_s)x \|^2 = 2\,\mathrm{Re}\,((U_+ - U_s)x, U_+ x)$. Thus it follows from (2.3) that

$$\| (U_+ - U_s)x \|^2 = 2\,\mathrm{Re}\, i\int_s^{+\infty} (V\exp(-itH_0)x, \quad U_+ \exp(-itH_0)x)\, dt, \tag{2.4}$$

where we also used the fact that

$$\exp(-itH_1) U_\pm = U_\pm \exp(-itH_0) \tag{2.5}$$

which is a direct consequence of (1.1).

V can be decomposed in the form $V = W|V|^{1/2}|V|^{1/2}$, where $W = \mathrm{Sign}\, V$ is partially isometric. Then it follows from (2.4), by making use of the Schwartz inequalities, that

$$\| (U_+ - U_s)x \|^2 \leq 2\left[\int_s^{+\infty} \| |V|^{1/2} \exp(-itH_0)x \|^2 dt\right]^{1/2} \left[\int_s^{+\infty} \| |V|^{1/2} W^* U_+ \exp(-itH_0)x \|^2 dt\right]^{1/2}. \tag{2.6}$$

The integrals on the right are finite under certain additional assumptions on x, as is seen by the following lemma due essentially to Rosenblum.[4)]

LEMMA 2.1. *Let* $H=\int \lambda dE(\lambda)$ *be a self-adjoint operator and let* x *be absolutely continuous with respect to* H. *Furthermore let*

(2.7) $$d\|E(\lambda)x\|^2/d\lambda = d(E(\lambda)x, x)/d\lambda \le m^2 \quad a.e.$$

for some constant m^2. *Then we have for any* $A \in \mathbf{S}$

$$\int_{-\infty}^{+\infty} \|A\exp(-itH)x\|^2\, dt \le 2\pi m^2 \|A\|_2^2.$$

PROOF. As is easily seen, we may assume without loss of generality that H is absolutely continuous. Let $\{\varphi_n\}$ be any countable orthonormal set which spans the same subspace as the range of A (the latter being separable). Then

(2.8) $$\|A\exp(-itH)x\|^2 = \sum_n |(A\exp(-itH)x, \varphi_n)|^2 = \sum_n \left|\int \exp(-it\lambda) d(E(\lambda)x, A^*\varphi_n)\right|^2.$$

But

$$|d(E(\lambda)x, A^*\varphi_n)/d\lambda|^2 \le (d\|E(\lambda)x\|^2/d\lambda)(d\|E(\lambda)A^*\varphi_n\|^2/d\lambda) \le m^2 d\|E(\lambda)A^*\varphi_n\|^2/d\lambda,$$

so that $d(E(\lambda)x, A^*\varphi_n)/d\lambda$ belongs to $L_2(-\infty, +\infty)$ with the L_2 norm not exceeding $m\|A^*\varphi_n\|$. Since the expression in $|\ \ |$ in (2.8) is the Fourier transform of this function, the application of Parseval's formulas gives the desired result

$$\int_{-\infty}^{+\infty} \|A\exp(-itH)x\|^2 dt \le 2\pi m^2 \sum \|A^*\varphi_n\|^2 \le 2\pi m^2 \|A^*\|_2^2 = 2\pi m^2 \|A\|_2^2.$$

We now replace the second integral $\int_s^{+\infty}$ on the right side of (2.6) by $\int_{-\infty}^{+\infty}$ and apply Lemma 2.1 to this integral. Since $\|\,|V|^{1/2}W^*U_+\|_2 \le \|V\|_1^{1/2}$, we obtain, after taking the square root, the inequality

(2.9) $$\|(U_+ - U_s)x\| \le (8\pi m^2 \|V\|_1)^{1/4} \left[\int_s^{+\infty} \|\,|V|^{1/2}\exp(-itH_0)x\|^2 dt\right]^{1/4},$$

where x is subjected to the condition

(2.10) $$d(E_0(\lambda)x, x)/d\lambda \le m^2 \qquad \left(H_0 = \int \lambda dE_0(\lambda)\right).$$

From (2.9) and a similar inequality with U_s replaced by U_t, we obtain finally the inequality

(2.11) $$\|(U_s - U_t)x\| \le (8\pi m^2\|V\|_1)^{1/4}\left[\left(\int_s^{+\infty} \|\,|V|^{1/2}\exp(-itH_0)x\|^2 dt\right)^{1/4} + \left(\int_t^{+\infty} \cdots\right)^{1/4}\right].$$

So far V has been assumed to be of finite rank. But it is now easy to remove this assumption by a simple limiting procedure, and (2.11) is seen to be valid for any operator $V \in \mathbf{T}_s$.[5)]

5) We have based the proof of (2.11) on the result of (F). If (2.11) could be proved directly, the whole theory would be greatly simplified.

3. Proof of Theorem 1. It follows from (2.11) that $\lim U_t x$ exists for $t \to +\infty$ provided $x \in \mathfrak{M}_0$ and (2.10) is satisfied. But the set of such x is dense in $\mathfrak{M}_0$ if the number m^2 is varied over all positive numbers. Since U_t is uniformly bounded, $\lim U_t x$ exists for every $x \in \mathfrak{M}_0$, that is, $s-\lim U_t P_0 = U_+$ exists. Since $U_t P_0$ is partially isometric with initial set $\mathfrak{M}_0$, the same is true with U_+. Furthermore, (2.5) holds as before, and this implies that the part of H_1 in $U_+ \mathfrak{M}_0$ is unitarily equivalent to the part of H_0 in $\mathfrak{M}_0$. This part of H_1 is therefore absolutely continuous, so that $U_+ \mathfrak{M}_0 \subset \mathfrak{M}_1$.

By symmetry we conclude in the same way that $s-\lim U_t^* P_1 = U_+'$ exists, that U_+' is partially isometric with initial set $\mathfrak{M}_1$, that the part of H_0 in $U_+' \mathfrak{M}_1$ is unitarily equivalent to the part of H_1 in $\mathfrak{M}_1$ and that $U_+' \mathfrak{M}_1 \subset \mathfrak{M}_0$. Furthermore, we have

(3.1) $U_+' U_+ = s-\lim U_t^* P_1 U_t P_0 = s-\lim \exp(itH_0) P_1 \exp(-itH_0) P_0.$

But $\| U_+' U_+ x \| = \| P_1 U_+ x \| = \| U_+ x \| = \| P_0 x \|$, since $U_+ \mathfrak{H} = U_+ \mathfrak{M}_0 \subset \mathfrak{M}_1$. Hence we can eliminate the factor P_1 on the right side of (3.1) by an argument used in § 5 of (F). Then (3.1) reduces to $U_+' U_+ = P_0$, and it is easily seen that $U_+' \mathfrak{M}_1 = \mathfrak{M}_0$, and similarly $U_+ \mathfrak{M}_0 = \mathfrak{M}_1$. These results show that the final set of the partially isometric operator U_+ is exactly $\mathfrak{M}_1$ and that U_+' coincides with U_+^*. Since the operator U_- can be dealt with similarly, the proof of Theorem 1 is complete.

It will be noted that, since the existence of U_+ has been established, all results of § 2 are true not only for V of finite rank but also for any $V \in \mathbf{T}_s$.

4. Proof of Theorem 2. The transitivity of the mappings $U_\pm(H_1, H_0)$ can be proved quite in the same way as we have proved $U_+' U_+ = P_0$ at the end of § 3.

To prove the continuity of these mappings for fixed H_0 or fixed H_1, it is sufficient to show that, given a sequence $V_n \in \mathbf{T}_s$ such that $\lim \| V_n - V \|_1 = 0$, we have for $n \to \infty$

(4.1) $$s-\lim U_\pm(H_0 + V_n, H_0) = U_\pm(H_0 + V, H_0)$$

and

(4.2) $$w-\lim U_\pm(H_0, H_0 + V_n) = U_\pm(H_0, H_0 + V).$$

But since $U_\pm(H_0, H_1) = U_\pm(H_1, H_0)^*$, (4.2) is a consequence of (4.1). Thus it is sufficient to prove (4.1).

By the transitivity (1.4) already proved, it is sufficient to prove (4.1) for the special case $V=0$. For simplicity we write $U_\pm^{(n)} = U_\pm(H_0 + V_n, H_0)$ and $U_t^{(n)} = \exp[it(H_0 + V_n)] \exp(-itH_0)$. Then (2.9) is true with U_+, U_s, V replaced by $U_+^{(n)}, U_s^{(n)}, V_n$ respectively (see the end of § 3), provided $x \in \mathfrak{M}_0$ and (2.10) is satisfied. Applying Lemma 2.1 again to the integral on the right side of this inequality and setting $s=0$, we obtain the inequality $\| (U_+^{(n)} - 1)x \| \leq 2m(\pi \| V_n \|_1)^{1/2}$, whence follows $\lim U_+^{(n)} x = x$. Since this is true for all x of a set dense in $\mathfrak{M}_0$, we conclude that $s-\lim U_+^{(n)}=s-\lim U_+^{(n)}P_0=P_0$, which is identical with (4.1) for $V=0$ because $U_+(H_0, H_0)=P_0$. U_- can be treated similarly.

The remaining negative assertions of Theorem 2 are verified by a simple example. According to Friedrichs,[6] there is an H_0 and a V of rank 2 such that $H^{(n)}=H_0+n^{-1}V$ is absolutely continuous for each $n=1, 2, \cdots$ but H_0 is not. This implies that $U_\pm(H^{(n)}, H^{(n)})=P^{(n)}=1\neq P_0=U_\pm(H_0, H_0)$. But if U_+ were weakly continuous in both variables jointly, we should have $w-\lim U_+(H^{(n)}, H^{(n)})=U_+(H_0, H_0)$ contrary to the above example. Again, if the same mapping were strongly continuous in the second variable for fixed first variable, we should have $s-\lim U_+(H_0, H^{(n)})=U_+(H_0, H_0)=P_0$. Combined with $s-\lim U_+(H^{(n)}, H_0)=P_0$ already proved, this would give $s-\lim U_+(H^{(n)}, H^{(n)})=s-\lim U_+(H^{(n)}, H_0)U_+(H_0, H^{(n)})=P_0$, again a contradiction.

6) K. O. Friedrichs: On the perturbation of continuous spectra, Comm. Pure Appl. Math., **1**, 361–406 (1948). The cited example is given in § 6 of this paper, where it is not stated explicitly that the perturbation under consideration is of rank 2, but this can be verified easily.